Chapter 1

Exercise 1.1

1. (i) coefficient of $x^2 = 3$
(ii) coefficient of $x = -9$
(iii) independent term $= 5$.

2. (i) degree 2
(ii) degree 3
(iii) degree 4.

3. $\frac{-4}{x} = -4x^{-1}$, -1 is not a positive power;
$x^{\frac{3}{2}}$, $\frac{3}{2}$ is not an integer.

4. (i) $3x^2 - 6x + 7 + 5x^2 + 2x - 9 = 8x^2 - 4x - 2$
(ii) $x^3 - 4x^2 - 5x + 3x^3 + 6x^2 - x = 4x^3 + 2x^2 - 6x$
(iii) $x(x + 4) + 3x(2x - 3) = x^2 + 4x + 6x^2 - 9x$
$= 7x^2 - 5x$
(iv) $3(x^2 - 7) + 2x(3x - 1) - 7x + 2 = 3x^2 - 21 + 6x^2 - 2x - 7x + 2$
$= 9x^2 - 9x - 19$

5. (i) $3x^2(4x + 2) + 5x^2(2x - 5) = 12x^3 + 6x^2 + 10x^3 - 25x^2$
$= 22x^3 - 19x^2$
(ii) $x^3(x - 2) + 4x^3(2x - 6) = x^4 - 2x^3 + 8x^4 - 24x^3$
$= 9x^4 - 26x^3$
(iii) $x(x^3 + 4x^2 - 7x) + 3x^2(2x^2 - 3x + 4) = x^4 + 4x^3 - 7x^2 + 6x^4 - 9x^3 + 12x^2$
$= 7x^4 - 5x^3 + 5x^2$
(iv) $3x(x^2 - 7x + 1) + 2x^2(6x - 5) = 3x^3 - 21x^2 + 3x + 12x^3 - 10x^2$
$= 15x^3 - 31x^2 + 3x$

6. (i) $(x + 4)(2x + 5) = 2x^2 + 5x + 8x + 20 = 2x^2 + 13x + 20$
(ii) $(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6 = 2x^2 - x - 6$
(iii) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6 = 3x^2 + 7x - 6$
(iv) $(3x - 2)(4x - 1) = 12x^2 - 3x - 8x + 2 = 12x^2 - 11x + 2$
(v) $(3x - 1)(2x + 5) = 6x^2 + 15x - 2x - 5 = 6x^2 + 13x - 5$
(vi) $(4x + 1)(2x - 6) = 8x^2 - 24x + 2x - 6 = 8x^2 - 22x - 6$
(vii) $(x - 2)(x + 2) = x^2 + 2x - 2x - 4 = x^2 - 4$
(viii) $(2x + 5)(2x - 5) = 4x^2 - 10x + 10x - 25 = 4x^2 - 25$
(ix) $(ax - by)(ax + by) = a^2x^2 + abxy - abxy - b^2y^2 = a^2x^2 - b^2y^2$

7. (i) $(x + 2)^2 = x^2 + 4x + 4$
(ii) $(x - 3)^2 = x^2 - 6x + 9$
(iii) $(x + 5)^2 = x^2 + 10x + 25$
(iv) $(a + b)^2 = a^2 + 2ab + b^2$
(v) $(x - y)^2 = x^2 - 2xy + y^2$
(vi) $(a + 2b)^2 = a^2 + 4ab + 4b^2$
(vii) $(3x - y)^2 = 9x^2 - 6xy + y^2$
(viii) $(x - 5y)^2 = x^2 - 10xy + 25y^2$
(ix) $(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$

8. (i) $\left(x+\frac{1}{2}\right)^2 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = x^2 + x + \frac{1}{4}$

(ii) $8\left(x-\frac{1}{4}\right)^2 = 8\left(x^2 - 2(x)\left(\frac{1}{4}\right) + \left(\frac{-1}{4}\right)^2\right) = 8\left(x^2 - \frac{x}{2} + \frac{1}{16}\right)$

$= 8x^2 - 4x + \frac{1}{2}$

(iii) $-(1-x)^2 = -(1-2x+x^2) = -1+2x-x^2$

9. (i) $x^2 + 5x + 25$; No, cannot be written in the form $(x+k)^2$

(ii) $9x^2 - 6x - 1$; No, cannot be written in the form $(x+k)^2$

(iii) $4 + 12x + 9x^2 = (2+3x)^2$; YES.

10. $px^2 + 4x + 1 = (ax+1)^2$

$px^2 + 4x + 1$ can be written in the form $(ax+1)^2 = a^2x^2 + 2ax + 1$

$\therefore 2a = 4 \Rightarrow a = 2$

$\therefore (ax+1)^2 = (2x+1)^2 = 4x^2 + 4x + 1$

$\therefore p = 4.$

11. $25x^2 + tx + 4 = (5x+2)^2$ $\quad$ or $\quad$ $25x^2 + tx + 4 = (5x-2)^2$

$= 25x^2 + 20x + 4$ $\qquad$ $= 25x^2 - 20x + 4$

$\Rightarrow \quad t = 20$ $\qquad$ $\Rightarrow \quad t = -20$

12. $9x^2 + 24x + s = (3x+a)^2 = 9x^2 + 6ax + a^2$

$\Rightarrow 6a = 24$

$a = 4.$

$\therefore 9x^2 + 24x + s = (3x+4)^2$

$= 9x^2 + 24x + 16$

$\Rightarrow s = 16.$

13. (i) $(x+2)(x^2+2x+6) = x^3 + 2x^2 + 6x + 2x^2 + 4x + 12$

$= x^3 + 4x^2 + 10x + 12$

(ii) $(x-4)(2x^2+3x-1) = 2x^3 + 3x^2 - x - 8x^2 - 12x + 4$

$= 2x^3 - 5x^2 - 13x + 4$

(iii) $(2x+3)(x^2-3x+2) = 2x^3 - 6x^2 + 4x + 3x^2 - 9x + 6$

$= 2x^3 - 3x^2 - 5x + 6$

(iv) $(3x-2)(2x^2-4x+2) = 6x^3 - 12x^2 + 6x - 4x^2 + 8x - 4$

$= 6x^3 - 16x^2 + 14x - 4.$

14. $(x+y)(x^2-xy+y^2) = x^3 - \cancel{x^2y} + \cancel{xy^2} + \cancel{x^2y} - \cancel{xy^2} + y^3$

$= x^3 + y^3$

15. $(x-y)(x^2+xy+y^2) = x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$

$= x^3 - y^3$

16. $(2x-3)(3x^2-2x+4) = 6x^3 - 4x^2 + 8x - 9x^2 + 6x - 12$

$= 6x^3 - 13x^2 + 14x - 12$

$\Rightarrow$ coefficient of $x = 14$.

17. $(x+3)(x-4)(2x+1) = (x+3)(2x^2 + x - 8x - 4)$

$= (x+3)(2x^2 - 7x - 4)$

$= 2x^3 - 7x^2 - 4x + 6x^2 - 21x - 12$

$= 2x^3 - x^2 - 25x - 12.$

18. $(x^2-3x-2)(2x^2-4x+1) = 2x^4 - 4x^3 + x^2 - 6x^3 + 12x^2 - 3x$

$- 4x^2 + 8x - 2$

$= 2x^4 - 10x^3 + 9x^2 + 5x - 2.$

19. $(3x^2 + 5x - 1)(2x^2 - 6x - 5)$; x^2 coefficients include $-15x^2 - 30x^2 - 2x^2$

$$= -47x^2$$

coefficient of $x^2 = -47$.

20. (i) $\dfrac{3x + 6}{3} = x + 2$

(ii) $\dfrac{x^2 + 2x}{x} = x + 2$

(iii) $\dfrac{3x^3 - 6x^2}{3x} = \dfrac{3x^2(x - 2)}{3x} = x(x - 2) = x^2 - 2x$

(iv) $\dfrac{15x^2y - 10xy^2}{5xy} = 3x - 2y$

21. (i) $\dfrac{6x^2y + 9xy^2 - 3xy}{3xy} = x + 3y - 1$

(ii) $\dfrac{6x^4 - 9x^3 + 12x^2}{3x^2} = \dfrac{3x^2(x^2 - 3x + 4)}{3x^2}$

$$= x^2 - 3x + 4$$

22. (i) $\dfrac{\overset{4}{\cancel{12}}a^{\cancel{2}}\cancel{b}}{\cancel{3}\cancel{a}\cancel{b}} = 4a$

(ii) $\dfrac{\overset{4}{\cancel{12}}a^{\cancel{2}}b\cancel{c}}{\cancel{3}\cancel{a}\cancel{c}} = 4ab$

(iii) $\dfrac{\overset{2}{\cancel{4}}\cancel{x}y^{\cancel{2}}z}{\cancel{2}\cancel{x}\cancel{y}} = 2yz$

(iv) $\dfrac{3xy}{2} \cdot \dfrac{4}{6x^2} = \dfrac{\cancel{12}\cancel{x}y}{\cancel{12}x^2} = \dfrac{y}{x}$

23. (i) $\dfrac{2x^2 + 5x - 3}{2x - 1} = \dfrac{\cancel{(2x - 1)}(x + 3)}{\cancel{2x - 1}} = x + 3$

(ii) $\dfrac{2x^2 - 2x - 12}{x - 3} = \dfrac{2(x^2 - x - 6)}{x - 3} = \dfrac{2\cancel{(x - 3)}(x + 2)}{\cancel{x - 3}}$

$$= 2(x + 2) = 2x + 4$$

(iii) $\dfrac{8x^2 + 8x - 6}{4x - 2} = \dfrac{2(4x^2 + 4x - 3)}{2(2x - 1)} = \dfrac{\cancel{2}\cancel{(2x - 1)}(2x + 3)}{\cancel{2}\cancel{(2x - 1)}}$

$$= 2x + 3$$

24. (i)

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ \hline x - 1 & x^3 - 8x^2 + 19x - 12 \\ & \underline{x^3 - x^2} \\ & \quad -7x^2 + 19x - 12 \\ & \quad \underline{-7x^2 + 7x} \\ & \qquad\qquad 12x - 12 \\ & \qquad\qquad \underline{12x - 12} \end{array}$$

(ii)

$$\begin{array}{r|l} & x^2 - 1 \\ \hline 2x - 1 & 2x^3 - x^2 - 2x + 1 \\ & \underline{2x^3 - x^2} \\ & \qquad\quad -2x + 1 \\ & \qquad\quad \underline{-2x + 1} \end{array}$$

(iii)

$$\begin{array}{r|l} & x^2 - 1 \\ \hline 3x - 4 & 3x^3 - 4x^2 - 3x + 4 \\ & \underline{3x^3 - 4x^2} \\ & \qquad\quad -3x + 4 \\ & \qquad\quad \underline{-3x + 4} \end{array}$$

(iv)
$$\begin{array}{r|l} & 4x^2+5x-6 \\ \hline x-3 & 4x^3-7x^2-21x+18 \\ & \underline{4x^3-12x^2} \\ & \quad +5x^2-21x+18 \\ & \quad \underline{+5x^2-15x} \\ & \qquad\quad -6x+18 \\ & \qquad\quad \underline{-6x+18} \end{array}$$

(v)
$$\begin{array}{r|l} & x^2-5x+3 \\ \hline x+5 & x^3 \qquad\quad -22x+15 \\ & \underline{x^3+5x^2} \\ & \quad -5x^2-22x+15 \\ & \quad \underline{-5x^2-25x} \\ & \qquad\quad +3x+15 \\ & \qquad\quad \underline{+3x+15} \end{array}$$

(vi)
$$\begin{array}{r|l} & 2x^2+3x+6 \\ \hline x-2 & 2x^3-x^2 \qquad\quad -12 \\ & \underline{2x^3-4x^2} \\ & \qquad 3x^2 \qquad\quad -12 \\ & \qquad \underline{3x^2-6x} \\ & \qquad\qquad\quad 6x-12 \\ & \qquad\qquad\quad 6x-12 \end{array}$$

25. (i)
$$\begin{array}{r|l} & x-2 \\ \hline x^2+2 & x^3-2x^2+2x-4 \\ & \underline{x^3 \qquad\quad +2x} \\ & \quad -2x^2 \qquad -4 \\ & \quad \underline{-2x^2 \qquad -4} \end{array}$$

(ii)
$$\begin{array}{r|l} & x-3 \\ \hline x^2-6x+9 & x^3-9x^2+27x-27 \\ & \underline{x^3-6x^2+9x} \\ & \quad -3x^2+18x-27 \\ & \quad \underline{-3x^2+18x-27} \end{array}$$

(iii)
$$\begin{array}{r|l} & 3x-1 \\ \hline x^2+x-2 & 3x^3+2x^2-7x+2 \\ & \underline{3x^3+3x^2-6x} \\ & \qquad -x^2-x+2 \\ & \qquad \underline{-x^2-x+2} \end{array}$$

(iv)
$$\begin{array}{r|l} & x+2 \\ \hline 5x^2+4x-1 & 5x^3+14x^2+7x-2 \\ & \underline{5x^3+4x^2-x} \\ & \quad +10x^2+8x-2 \\ & \quad \underline{+10x^2+8x-2} \end{array}$$

26. (i)
$$\begin{array}{r|l} & x^2+2x+4 \\ \hline x-2 & x^3 \qquad\qquad -8 \\ & \underline{x^3-2x^2} \\ & \quad 2x^2 \qquad -8 \\ & \quad \underline{2x^2-4x} \\ & \qquad\quad 4x-8 \\ & \qquad\quad 4x-8 \end{array}$$

(ii)

$$\begin{array}{r|l} & 4x^2 + 6xy + 9y^2 \\ \hline 2x - 3y & 8x^3 \qquad\qquad\qquad -27y^3 \\ & 8x^3 - 12x^2y \\ \hline & 12x^2y \qquad\qquad -27y^3 \\ & 12x^2y - 18xy^2 \\ \hline & 18xy^2 - 27y^3 \\ & 18xy^2 - 27y^3 \end{array}$$

Exercise 1.2

1. x cm length of smaller side,

$(x + 4)$ cm length of longer side.

(i) $A(x) = x(x + 4) = (x^2 + 4x)$ cm^2

(ii) $P(x) = 2[x + (x + 4)] = (4x + 8)$ cm

2. (i) Area = length × width

$$\Rightarrow \text{width} = \frac{\text{Area}}{\text{length}} = \frac{6x^2 + 4x - 2}{3x - 1}$$

$$= \frac{(3x - 1)(2x + 2)}{3x - 1} = 2x + 2$$

(ii) Perimeter = 2(length + width)

$= 2((3x - 1) + (2x + 2))$

$P(x) = 10x + 2$

3. (a) $V(x) = (2x + 3)(x)(x + 1)$

$= (2x + 3)(x^2 + x)$

$= 2x^3 + 2x^2 + 3x^2 + 3x$

$= 2x^3 + 5x^2 + 3x$

(b) $S(x) = (x)(2x + 3) + 2(x)(x + 1) + 2(2x + 3)(x + 1)$

$= 2x^2 + 3x + 2x^2 + 2x + 4x^2 + 4x + 6x + 6$

$= 8x^2 + 15x + 6$

(c) (i) $V(5) = 2(5)^3 + 5(5)^2 + 3(5) = 390$ cm^3

(ii) $S(5) = 8(5)^2 + 15(5) + 6 = 281$ cm^2

4. $f(x) = 2x^3 - x^2 - 5x - 4$

(a) $f(0) = 2(0)^3 - (0)^2 - 5(0) - 4 = -4$

(b) $f(1) = 2(1)^3 - (1)^2 - 5(1) - 4 = -8$

(c) $f(-2) = 2(-2)^3 - (-2)^2 - 5(-2) - 4 = -14$

(d) $f(3a) = 2(3a)^3 - (3a)^2 - 5(3a) - 4 = 54a^3 - 9a^2 - 15a - 4$

5. $f(x) = x^2 - 3x + 6$

(a) $f(0) = (0)^2 - 3(0) + 6 = 6$

(b) $f(-5) = (-5)^2 - 3(-5) + 6 = 46$

(c) $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 6 = \frac{31}{4} = 7.75$

(d) $f\left(\frac{a}{4}\right) = \left(\frac{a}{4}\right)^2 - 3\left(\frac{a}{4}\right) + 6 = \frac{a^2}{16} - \frac{3a}{4} + 6$

6. Length = $(x - y)$.

Width = $(2x + 3y)$.

(a) Area = $(x - y)(2x + 3y) = 2x^2 + 3xy - 2xy - 3y^2$

$= 2x^2 + xy - 3y^2$

(b) Perimeter = $2[(x - y) + (2x + 3y)] = 2[3x + 2y] = 6x + 4y$

7. Length $= x$ cm

Width $= (x - 5)$ cm

Height $= 2x$ cm.

(a) Volume $=$ Length $\times$ Width $\times$ Height

$= (x)(x - 5)(2x) = 2x^3 - 10x^2$

(b) Surface area $= 2(x)(x - 5) + 4(2x)(x - 5) + 4(2x)(x)$

$= 2x^2 - 10x + 8x^2 - 40x + 8x^2$

$= 18x^2 - 50x.$

8. (i) $d(4) =$ the number of diagonals in a 4-sided polygon

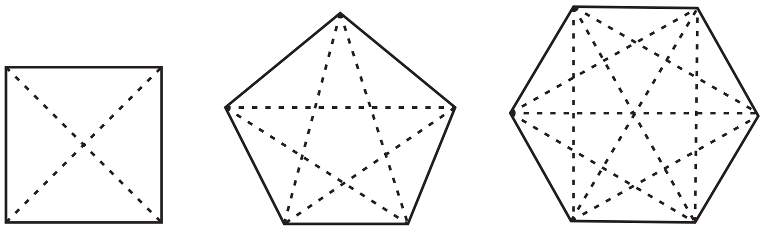
(ii) $d(5) =$ the number of diagonals in a 5-sided polygon.

$$d(4) = \frac{(4)^2}{2} - \frac{3(4)}{2} = 8 - 6 = 2.$$

$$d(5) = \frac{(5)^2}{2} - \frac{3(5)}{2} = \frac{25}{2} - \frac{15}{2} = \frac{10}{2} = 5.$$

$$d(6) = \frac{(6)^2}{2} - \frac{3(6)}{2} = \frac{36}{2} - \frac{18}{2} = \frac{18}{2} = 9.$$

$d(3) = 0$ because a triangle has no diagonal.



9. If $f(x) = x + 5$,

$f(a^2) - 3f(a) + 2 = a^2 + 5 - 3(a + 5) + 2$

$= a^2 + 5 - 3a - 15 + 2$

$= a^2 - 3a - 8.$

10. $f(x) = x^2 - 3x + 6.$

(i) $f(-2t) = (-2t)^2 - 3(-2t) + 6 = 4t^2 + 6t + 6.$

(ii) $f(t^2) = (t^2)^2 - 3(t^2) + 6 = t^4 - 3t^2 + 6.$

(iii) $f(t - 2) = (t - 2)^2 - 3(t - 2) + 6 = t^2 - 4t + 4 - 3t + 6 + 6.$

$= t^2 - 7t + 16.$

(i) $4t^2 + 6t + 6$ is of degree 2.

(ii) $t^4 - 3t^2 + 6$ is of degree 4.

(iii) $t^2 - 7t + 16$ is of degree 2.

11. $V(r, h) = \frac{1}{3}\pi r^2 h.$

(i) $V(r = 14, h = 21) = \frac{1}{3}\pi \cdot 14^2 \cdot 21 = 1372\pi \text{ cm}^3$

(ii) $V(r, h = r) = \frac{1}{3}\pi \cdot r^2 \cdot r = \frac{1}{3}\pi r^3$

(iii) $V(r = 2h, h) = \frac{1}{3}\pi(2h)^2 \cdot h = \frac{4}{3}\pi h^3$

12. The dimensions of the base of the box are $24 - 2x$ and $18 - 2x$, in cm.

The height of the box is x, in cm.

Thus the volume, V, of the box, in cm^3, is

$V = x(24 - 2x)(18 - 2x)$

$V = x(432 - 48x - 36x + 4x^2)$

$V = x(4x^2 - 84x + 432)$

$V = 4x^3 - 84x^2 + 432x$

The smallest possible value of x is 0. The largest possible value of x is given by

$$18 - 2x = 0$$
$$9 = x$$

Hence $0 < x < 9$, and so $a = 0$ and $b = 9$.

13. $T = 2\pi\sqrt{\dfrac{l}{g}}$

$\therefore\ T^2 = 4\pi^2\dfrac{l}{g}$

$\therefore\ \dfrac{gT^2}{4\pi^2} = l$

$\therefore\ l = \dfrac{gT^2}{4\pi^2}$

$\Rightarrow$ when $T = 4$ and $g = 10$, $l = \dfrac{10.4^2}{4\pi^2} = \dfrac{40}{\pi^2}$ m.

14. $V = \dfrac{4}{3}\pi r^3$

$\Rightarrow \dfrac{4}{3}\pi r^3 = V$

$\Rightarrow \quad r^3 = \dfrac{3V}{4\pi}$

$\Rightarrow \quad r = \sqrt[3]{\dfrac{3V}{4\pi}}$

when $V = \dfrac{792}{7}$ and $\pi = \dfrac{22}{7}$, $r = \sqrt[3]{\dfrac{3 \times \overset{198}{\cancel{792}} \times \cancel{7}}{\cancel{4} \times \cancel{7} \times 22}} = \sqrt[3]{27}$

$= 3$ m

15. $H(x) = \dfrac{x}{2}(x - 1)$, $x =$ number of students.

(i) $x = 5 \Rightarrow H(5) = \dfrac{5}{2}(5 - 1) = 10$

(ii) $x = 6 \Rightarrow H(6) = \dfrac{6}{2}(6 - 1) = 15$

(iii) $x = 10 \Rightarrow H(10) = \dfrac{10}{2}(10 - 1) = 45$

(iv) $H(x) = 136 = \dfrac{x}{2}(x - 1)$

$272 = x(x - 1)$

$\Rightarrow$ The product of two consecutive numbers $= 272$.

$\Rightarrow$ if $x = 16$, $x - 1 = 15 \quad \therefore 16 \times 15 = 240$.

if $x = 17$, $x - 1 = 16 \quad \therefore 17 \times 16 = 272$.

$\therefore x = 17$.

or

$272 = x^2 - x$

$\Rightarrow x^2 - x - 272 = 0$

$(x - 17)(x + 16) = 0$

$\therefore x - 17 = 0 \Rightarrow x = 17$

or $x + 16 = 0 \Rightarrow x = -16$ which is invalid

since x stands for the number of students

$\therefore x = 17$.

Exercise 1.3

1. $5x^2 - 10x = 5x(x - 2)$

2. $6ab - 12bc = 6b(a - 2c)$

3. $3x^2 - 6xy = 3x(x - 2y)$

4. $2x^2y - 6x^2z = 2x^2(y - 3z)$

5. $2a^3 - 4a^2 + 8a = 2a(a^2 - 2a + 4)$

6. $5xy^2 - 20x^2y = 5xy(y - 4x)$

7. $2a^2b - 4ab^2 + 12abc = 2ab(a - 2b + 6c)$

8. $3x^2y - 9xy^2 + 15xyz = 3xy(x - 3y + 5z)$

9. $4\pi r^2 + 6\pi rh = 2\pi r(2r + 3h)$

10. $3a(2b - c) - 4(2b - c) = (2b - c)(3a - 4)$

11. $x^2 - ax + 3x - 3a = x(x - a) + 3(x - a)$
$= (x - a)(x + 3)$

12. $2c^2 - 4cd + c - 2d = 2c(c - 2d) + c - 2d$
$= (c - 2d)(2c + 1)$

13. $8ax + 4ay - 6bx - 3by = 4a(2x + y) - 3b(2x + y)$
$= (2x + y)(4a - 3b)$

14. $7y^2 - 21by + 2ay - 6ab = 7y(y - 3b) + 2a(y - 3b)$
$= (y - 3b)(7y + 2a)$

15. $6xy + 12yz - 8xz - 9y^2 = 6xy - 9y^2 + 12yz - 8xz$
$= 3y(2x - 3y) + 4z(3y - 2x)$
$= 3y(2x - 3y) - 4z(2x - 3y)$
$= (2x - 3y)(3y - 4z)$

16. $6x^2 - 3y(3x - 2a) - 4ax = 6x^2 - 4ax - 3y(3x - 2a)$
$= 2x(3x - 2a) - 3y(3x - 2a)$
$= (3x - 2a)(2x - 3y)$

17. $3ax^2 - 3ay^2 - 4bx^2 + 4by^2 = 3a(x^2 - y^2) - 4b(x^2 - y^2)$
$= (x^2 - y^2)(3a - 4b)$
$= (x - y)(x + y)(3a - 4b)$

18. $a^2 - b^2 = (a - b)(a + b)$

19. $x^2 - 4y^2 = (x - 2y)(x + 2y)$

20. $9x^2 - y^2 = (3x - y)(3x + y)$

21. $16x^2 - 25y^2 = (4x)^2 - (5y)^2 = (4x - 5y)(4x + 5y)$

22. $36x^2 - 25 = (6x - 5)(6x + 5)$

23. $1 - 36x^2 = (1 - 6x)(1 + 6x)$

24. $49a^2 - 4b^2 = (7a)^2 - (2b)^2 = (7a - 2b)(7a + 2b)$

25. $x^2y^2 - 1 = (xy - 1)(xy + 1)$

26. $4a^2b^2 - 16c^2 = (2ab)^2 - (4c)^2 = (2ab - 4c)(2ab + 4c)$

27. $3x^2 - 27y^2 = 3(x^2 - 9y^2)$
$= 3(x - 3y)(x + 3y)$

28. $45 - 5x^2 = 5(9 - x^2)$
$= 5(3 - x)(3 + x)$

29. $45a^2 - 20 = 5(9a^2 - 4)$
$= 5(3a - 2)(3a + 2)$

30. $(2x + y)^2 - 4 = (2x + y - 2)(2x + y + 2)$

31. $(3a - 2b)^2 - 9 = (3a - 2b - 3)(3a - 2b + 3)$

32. $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 - b^2)(a^2 + b^2)$
$= (a - b)(a + b)(a^2 + b^2)$

33. $x^2 + 9x + 14 = (x + 2)(x + 7)$

34. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

35. $2x^2 + 11x + 14 = (2x + 7)(x + 2)$

36. $x^2 - 9x + 14 = (x - 2)(x - 7)$

37. $x^2 - 11x + 28 = (x - 7)(x - 4)$

38. $2x^2 - 7x + 3 = (2x - 1)(x - 3)$

39. $3x^2 - 17x + 20 = (3x - 5)(x - 4)$

40. $7x^2 - 18x + 8 = (7x - 4)(x - 2)$

41. $2x^2 - 7x - 15 = (2x + 3)(x - 5)$

42. $3x^2 + 11x - 20 = (3x - 4)(x + 5)$

43. $12x^2 - 11x - 5 = (4x - 5)(3x + 1)$

44. $6x^2 + x - 15 = (3x + 5)(2x - 3)$

45. $3x^2 + 13x - 10 = (3x - 2)(x + 5)$

46. $6x^2 - 11x + 3 = (3x - 1)(2x - 3)$

47. $36x^2 - 7x - 4 = (9x - 4)(4x + 1)$

48. $15x^2 - 14x - 8 = (5x + 2)(3x - 4)$

49. $6y^2 + 11y - 35 = (3y - 5)(2y + 7)$

50. $12x^2 + 17xy - 5y^2 = (4x - y)(3x + 5y)$

51. (i) $x^2 + 3\sqrt{3}x + 6$.

$\Rightarrow a = 1, b = 3\sqrt{3}, c = 6.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3\sqrt{3} \pm \sqrt{27 - 4(1)(6)}}{2(1)}$$

$$= \frac{-3\sqrt{3} \pm \sqrt{3}}{2}$$

$$\therefore \quad x = \frac{-3\sqrt{3} + \sqrt{3}}{2} \quad \text{or} \quad x = \frac{-3\sqrt{3} - \sqrt{3}}{2}$$

$\therefore \quad x = -\sqrt{3} \quad \text{or} \quad -2\sqrt{3}$

$\therefore$ Factors are $(x + \sqrt{3})$ and $(x + 2\sqrt{3})$

(ii) $x^2 + 2\sqrt{5}x - 15$.

$\Rightarrow a = 1, b = 2\sqrt{5}, c = -15.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\sqrt{5} \pm \sqrt{20 - 4(1)(-15)}}{2(1)}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{80}}{2}$$

$$= \frac{-2\sqrt{5} \pm 4\sqrt{5}}{2}$$

$$\therefore \quad x = \frac{-2\sqrt{5} + 4\sqrt{5}}{2} \quad \text{or} \quad x = \frac{-2\sqrt{5} - 4\sqrt{5}}{2}$$

$\therefore \quad x = \sqrt{5} \quad \text{or} \quad -3\sqrt{5}$

$\therefore$ Factors are $(x - \sqrt{5})$ and $(x + 3\sqrt{5})$

(iii) $2x^2 - 5\sqrt{2}x - 6.$

$\Rightarrow a = 2, b = -5\sqrt{2}, c = -6.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5\sqrt{2} \pm \sqrt{50 - 4(2)(-6)}}{2(2)}$$

$$= \frac{5\sqrt{2} \pm \sqrt{98}}{4}$$

$$= \frac{5\sqrt{2} \pm 7\sqrt{2}}{4}$$

$$\therefore \quad x = \frac{5\sqrt{2} + 7\sqrt{2}}{4} \quad \text{or} \quad x = \frac{5\sqrt{2} - 7\sqrt{2}}{4}$$

$$\therefore \quad x = 3\sqrt{2} \quad \text{or} \quad \frac{-\sqrt{2}}{2}$$

$\therefore$ Factors are $(x - 3\sqrt{2})$ and $\left(x + \frac{\sqrt{2}}{2}\right)$

But since coefficient of x^2 is 2, one of the factors must

contain $2x$. $\quad \therefore x + \frac{\sqrt{2}}{2} = 0$

$\Rightarrow 2x + \sqrt{2} = 0$

$\therefore$ Factors are $(x - 3\sqrt{2})$ and $(2x + \sqrt{2})$

52. (i) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(ii) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(iii) $8x^3 + y^3 = ((2x)^3 + y^3) = (2x + y)(4x^2 - 2xy + y^2)$

53. (i) $27x^3 - y^3 = (3x)^3 - y^3 = (3x - y)(9x^2 + 3xy + y^2)$

(ii) $x^3 - 64 = x^3 - 4^3 = (x - 4)(x^2 + 4x + 16)$

(iii) $8x^3 - 27y^3 = (2x)^3 - (3y)^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$

54. (i) $8 + 27k^3 = (2)^3 + (3k)^3 = (2 + 3k)(4 - 6k + 9k^2)$

(ii) $64 - 125a^3 = (4)^3 - (5a)^3 = (4 - 5a)(16 + 20a + 25a^2)$

(iii) $27a^3 + 64b^3 = (3a)^3 + (4b)^3 = (3a + 4b)(9a^2 - 12ab + 16b^2)$

55. (i) $a^3 - 8b^3c^3 = a^3 - (2bc)^3 = (a - 2bc)(a^2 + 2abc + 4b^2c^2)$

(ii) $5x^3 + 40y^3 = 5(x^3 + 8y^3)$

$= 5(x^3 + (2y)^3)$

$= 5(x + 2y)(x^2 - 2xy + 4y^2)$

(iii) $(x + y)^3 - z^3 = (x + y - z)[(x + y)^2 + (x + y)z + z^2]$

Exercise 1.4

1. (i) $\frac{\cancel{8}^{4}\cancel{y}}{\cancel{2}y^{\cancel{3}2}} = \frac{4}{y^2}$ $\qquad$ (ii) $\frac{\cancel{7}\,\cancel{a^6}^{a}\,\cancel{b^3}}{\cancel{14}_{2}\,\cancel{a^5}\,\cancel{b^4}_{b}} = \frac{a}{2b}$

(iii) $\frac{(2x)^2}{4x} = \frac{\cancel{4}\cancel{x^2}^{x}}{\cancel{4x}} = x$

(iv) $\frac{7y + 2y^2}{7y} = \frac{\cancel{y}(7 + 2y)}{7\cancel{y}} = \frac{7 + 2y}{7}$

(v) $\frac{5ax}{15a + 10a^2} = \frac{\cancel{5}\cancel{a}x}{\cancel{5}\cancel{a}(3 + 2a)} = \frac{x}{3 + 2a}$

2. (a) $\frac{2x}{5} + \frac{4x}{3} = \frac{6x}{15} + \frac{20x}{15} = \frac{26x}{15}$

(b) $\frac{3x}{5} - \frac{x}{2} = \frac{6x}{10} - \frac{5x}{10} = \frac{x}{10}$

(c) $\frac{2x + 3}{4} + \frac{x}{3} = \frac{6x + 9}{12} + \frac{4x}{12} = \frac{10x + 9}{12}$

(d) $\frac{x+1}{4}+\frac{2x-1}{5}=\frac{5x+5}{20}+\frac{8x-4}{20}=\frac{13x+1}{20}$

(e) $\frac{3x-4}{6}-\frac{2x+1}{3}=\frac{3x-4}{6}-\frac{4x+2}{6}=\frac{-x-6}{6}$

(f) $\frac{3x-2}{6}-\frac{x-3}{4}=\frac{6x-4}{12}-\frac{3x-9}{12}=\frac{3x+5}{12}$

(g) $\frac{5x-1}{4}-\frac{2x-4}{5}=\frac{25x-5}{20}-\frac{8x-16}{20}=\frac{17x+11}{20}$

(h) $\frac{3x+5}{6}-\frac{2x+3}{4}-\frac{1}{12}=\frac{6x+10}{12}-\frac{6x+9}{12}-\frac{1}{12}$

$=\frac{0}{12}=0$

(i) $\frac{3x-2}{4}+\frac{3}{5}-\frac{2x-1}{10}=\frac{15x-10}{20}+\frac{12}{20}-\frac{4x-2}{20}$

$=\frac{11x+4}{20}$

(j) $\frac{1}{3x}+\frac{1}{5x}=\frac{5}{15x}+\frac{3}{15x}=\frac{8}{15x}$

(k) $\frac{3}{4x}-\frac{5}{8x}=\frac{6}{8x}-\frac{5}{8x}=\frac{1}{8x}$

(l) $\frac{1}{x}+\frac{1}{x+3}=\frac{x+3+x}{x(x+3)}=\frac{2x+3}{x(x+3)}$

(m) $\frac{2}{x+2}+\frac{3}{x+4}=\frac{2(x+4)+3(x+2)}{(x+2)(x+4)}=\frac{5x+14}{(x+2)(x+4)}$

(n) $\frac{2}{x-2}+\frac{3}{2x-1}=\frac{2(2x-1)+3(x-2)}{(x-2)(2x-1)}=\frac{7x-8}{(x-2)(2x-1)}$

(o) $\frac{5}{3x-1}-\frac{2}{x+3}=\frac{5(x+3)-2(3x-1)}{(3x-1)(x+3)}=\frac{-x+17}{(3x-1)(x+3)}$

(p) $\frac{3}{2x-7}-\frac{1}{5x+2}=\frac{3(5x+2)-(2x-7)}{(2x-7)(5x+2)}=\frac{13x+13}{(2x-7)(5x+2)}$

(q) $\frac{2}{3x-5}-\frac{1}{4}=\frac{8-(3x-5)}{4(3x-5)}=\frac{13-3x}{4(3x-5)}$

(r) $\frac{5}{2x-1}-\frac{3}{x-2}=\frac{5(x-2)-3(2x-1)}{(2x-1)(x-2)}=\frac{-x-7}{(2x-1)(x-2)}$

(s) $\frac{x}{x-y}-\frac{y}{x+y}=\frac{x(x+y)-y(x-y)}{(x-y)(x+y)}=\frac{x^2+\cancel{xy}+\cancel{xy}+y^2}{(x-y)(x+y)}$

$=\frac{x^2+y^2}{x^2-y^2}$

(t) $\frac{3}{x}+\frac{4}{3y}-\frac{2}{3xy}=\frac{3(3y)+4(x)-2}{3xy}=\frac{9y+4x-2}{3xy}$

(u) $\frac{3}{x}-\frac{2}{x-1}-\frac{4}{x(x-1)}=\frac{3(x-1)-2(x)-4}{x(x-1)}=\frac{x-7}{x(x-1)}$

3. (i) $\frac{2z^2-4z}{2z^2-10z}=\frac{\cancel{2z}(z-2)}{\cancel{2z}(z-5)}=\frac{z-2}{z-5}$

(ii) $\frac{y^2+7y+10}{y^2-25}=\frac{\cancel{(y+5)}(y+2)}{\cancel{(y+5)}(y-5)}=\frac{y+2}{y-5}$

(iii) $\frac{t^2+3t-4}{t^2-3t+2}=\frac{(t+4)\cancel{(t-1)}}{(t-2)\cancel{(t-1)}}=\frac{t+4}{t-2}$

(iv) $\frac{x}{x^2-4}-\frac{1}{x+2}=\frac{x}{(x+2)(x-2)}-\frac{1}{x+2}$

$=\frac{x-(x-2)}{(x+2)(x-2)}=\frac{2}{(x+2)(x-2)}$

(v) $\dfrac{2}{a+3} - \dfrac{a+2}{a^2-9} = \dfrac{2}{a+3} - \dfrac{a+2}{(a+3)(a-3)}$

$= \dfrac{2(a-3)-(a+2)}{(a+3)(a-3)} = \dfrac{a-8}{(a+3)(a-3)}$

(vi) $\dfrac{x-1}{x^2-4} + \dfrac{1}{x-2} = \dfrac{x-1}{(x-2)(x+2)} + \dfrac{1}{x-2}$

$= \dfrac{x-1+x+2}{(x-2)(x+2)} = \dfrac{2x+1}{(x-2)(x+2)}$

4. (i) $\dfrac{10}{2x^2-3x-2} - \dfrac{2}{x-2} = \dfrac{10}{(2x+1)(x-2)} - \dfrac{2}{(x-2)}$

$= \dfrac{10-2(2x+1)}{(2x+1)(x-2)}$

$= \dfrac{8-4x}{(2x+1)(x-2)} = \dfrac{4(2-x)}{(2x+1)\,(x-2)}$

$= \dfrac{-4\cancel{(x-2)}}{(2x+1)\cancel{(x-2)}}$

(ii) $\dfrac{x+2}{2x^2-x-1} - \dfrac{1}{x-1} = \dfrac{x+2}{(2x+1)(x-1)} - \dfrac{1}{x-1}$

$= \dfrac{x+2-(2x+1)}{(2x+1)(x-1)}$

$= \dfrac{-x+1}{(2x+1)(x-1)} = \dfrac{\cancel{-(x-1)}}{(2x+1)\cancel{(x-1)}} = \dfrac{-1}{2x+1}$

5. (i) $\dfrac{1}{x^2-9} - \dfrac{2}{x^2-x-6} = \dfrac{1}{(x-3)(x+3)} - \dfrac{2}{(x-3)(x+2)}$

$= \dfrac{x+2-2(x+3)}{(x-3)(x+3)(x+2)}$

$= \dfrac{-x-4}{(x-3)(x+3)(x+2)}$

(ii) $\dfrac{3}{x^2+x-2} - \dfrac{2}{x^2+3x+2} = \dfrac{3}{(x+2)(x-1)} - \dfrac{2}{(x+2)(x+1)}$

$= \dfrac{3(x+1)-2(x-1)}{(x+2)(x-1)(x+1)}$

$= \dfrac{x+5}{(x+2)(x-1)(x+1)}$

(iii) $\dfrac{2}{6x^2-5x-4} - \dfrac{3}{9x^2-16} = \dfrac{2}{(3x-4)(2x+1)} - \dfrac{3}{(3x-4)(3x+4)}$

$= \dfrac{2(3x+4)-3(2x+1)}{(3x-4)(2x+1)(3x+4)}$

$= \dfrac{\cancel{6x}+8-\cancel{6x}-3}{(3x-4)(2x+1)(3x+4)} = \dfrac{5}{(3x-4)(2x+1)(3x+4)}$

(iv) $\dfrac{1}{xy-x^2} - \dfrac{1}{y^2-xy} = \dfrac{1}{x(y-x)} - \dfrac{1}{y(y-x)}$

$= \dfrac{y-x}{x(y-x)\,y}$

$= \dfrac{\cancel{(y-x)}}{\cancel{(y-x)}\,xy} = \dfrac{1}{xy}$

6. (i) $$\frac{\frac12+\frac34}{\frac14} = \frac{\frac24+\frac34}{\frac14} = \frac{\frac54}{\frac14} = \frac54\times\frac41 = 5$$

(ii) $$\frac{\frac23+\frac56}{\frac38} = \frac{\frac46+\frac56}{\frac38} = \frac{\frac96}{\frac38} = \frac{\cancel{9}^{\cancel{3}\,1}}{\cancel{6}_{\cancel{2}\,1}} \times \frac{\cancel{8}^{4}}{\cancel{3}_{1}} = 4$$

(iii) $$\frac{x-\frac1x}{1+\frac1x} = \frac{\frac{x^2-1}{x}}{\frac{x+1}{x}} = \frac{x^2-1}{x+1} = \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}} = x-1$$

7. (i) $$\frac{\frac1x+1}{\frac1x-1} = \frac{\frac{1+x}{x}}{\frac{1-x}{x}} = \frac{(1+x)}{\cancel{x}}\cdot\frac{\cancel{x}}{(1-x)} = \frac{1+x}{1-x}$$

(ii) $$\frac{\frac{1}{x^2}-4}{\frac1x-2} = \frac{\frac{1-4x^2}{x^2}}{\frac{1-2x}{x}} = \frac{(1-4x^2)}{\cancel{x^2}}\cdot\frac{\cancel{x}^{1}}{(1-2x)}$$

$$= \frac{\cancel{(1-2x)}(1+2x)}{x\cancel{(1-2x)}} = \frac{1+2x}{x}$$

(iii) $$\frac{x+y}{\frac1x+\frac1y} = \frac{x+y}{\frac{y+x}{xy}} = \frac{\cancel{(x+y)}}{1}\cdot\frac{xy}{\cancel{(x+y)}} = xy$$

8. (i) $$\frac{4y-\frac32}{2} = \frac{\frac{8y-3}{2}}{2} = \frac{8y-3}{2}\times\frac12 = \frac{8y-3}{4}$$

(ii) $$\frac{2-\frac1x}{2} = \frac{\frac{2x-1}{x}}{2} = \frac{2x-1}{x}\cdot\frac12 = \frac{2x-1}{2x}$$

(iii) $$\frac{3x+\frac1x}{2} = \frac{\frac{3x^2+1}{x}}{2} = \frac{3x^2+1}{x}\cdot\frac12 = \frac{3x^2+1}{2x}$$

(iv) $$\frac{y+\frac14}{\frac12} = \frac{\frac{4y+1}{4}}{\frac12} = \frac{4y+1}{\cancel{4}_{2}}\cdot\frac{\cancel{2}^{1}}{1} = \frac{4y+1}{2}$$

9. (i) $$\frac{z-\frac13}{z-\frac12} = \frac{\frac{3z-1}{3}}{\frac{2z-1}{2}} = \frac{3z-1}{3}\cdot\frac{2}{2z-1} = \frac{6z-2}{6z-3}$$

(ii) $$\frac{2x+\frac12}{x+\frac14} = \frac{\frac{4x+1}{2}}{\frac{4x+1}{4}} = \frac{\cancel{(4x+1)}}{\cancel{2}_{1}}\cdot\frac{\cancel{4}^{2}}{\cancel{(4x+1)}} = 2$$

(iii) $$\frac{z-\frac{1}{2z}}{z-\frac{1}{3z}} = \frac{\frac{2z^2-1}{2z}}{\frac{3z^2-1}{3z}} = \frac{(2z^2-1)}{2\cancel{z}}\cdot\frac{3\cancel{z}}{(3z^2-1)}$$

$$= \frac{6z^2-3}{6z^2-2}$$

(iv) $$\frac{x-\frac{1}{x+1}}{x-1} = \frac{\frac{x(x+1)-1}{x+1}}{x-1} = \frac{x^2+x-1}{x+1}\cdot\frac{1}{x-1}$$

$$= \frac{x^2+x-1}{x^2-1}$$

10. (i) $$\frac{1+\frac2x}{\frac{x+2}{x-2}} = \frac{\frac{x+2}{x}}{\frac{x+2}{x-2}} = \frac{\cancel{(x+2)}}{x}\cdot\frac{x-2}{\cancel{(x+2)}} = \frac{x-2}{x}$$

(ii) $$\frac{2+\frac1x}{2x^2+x} = \frac{\frac{2x+1}{x}}{x(2x+1)} = \frac{\cancel{(2x+1)}}{x}\cdot\frac{1}{x\cancel{(2x+1)}} = \frac{1}{x^2}$$

(iii) $\dfrac{x+\frac{2x}{x-2}}{1+\frac{4}{(x+2)(x-2)}} = \dfrac{\frac{x(x-2)+2x}{(x-2)}}{\frac{(x+2)(x-2)+4}{(x+2)(x-2)}}$

$= \dfrac{x^2-\cancel{2}x+\cancel{2}x}{(x-2)} \cdot \dfrac{(x+2)(x-2)}{x^2-\cancel{4}+\cancel{4}}$

$= \dfrac{\cancel{x^2}}{\cancel{(x-2)}} \cdot \dfrac{(x+2)\cancel{(x-2)}}{\cancel{x^2}} = x+2$

11. (i) $\dfrac{\left(\frac{a+b}{a-b}\right)-\left(\frac{a-b}{a+b}\right)}{1+\left(\frac{a-b}{a+b}\right)} = \dfrac{\frac{(a+b)(a+b)-(a-b)(a-b)}{(a-b)+(a+b)}}{\frac{\cancel{(a+b)}+\cancel{(a-b)}}{(a+b)}}$

$= \dfrac{a^2+2ab+b^2-(a^2-2ab+b^2)}{(a-b)(a+b)} \cdot \dfrac{(a+b)}{2a}$

$= \dfrac{(\cancel{a^2}+2ab+\cancel{b^2}-\cancel{a^2}+2ab-\cancel{b^2})}{(a-b)\,(a+b)} \cdot \dfrac{(a+b)}{2a}$

$= \dfrac{{}^2\cancel{4}\cancel{a}b\cancel{(a+b)}}{(a-b)\cancel{(a+b)}\cancel{2}\cancel{a}} = \dfrac{2b}{a-b}$

(ii) $\dfrac{x+\frac{3}{x}}{x-\frac{9}{x^3}} = \dfrac{\frac{x^2+3}{x}}{\frac{x^4-9}{x^3}} = \dfrac{(x^2+3)}{\cancel{x}} \cdot \dfrac{x^{\cancel{3}2}}{(x^4-9)}$

$= \dfrac{\cancel{(x^2+3)}(x^2)}{(x^2-3)\cancel{(x^2+3)}} = \dfrac{x^2}{x^2-3}$

(iii) $\dfrac{9-\frac{1}{y^2}}{9+\frac{6}{y}+\frac{1}{y^2}} = \dfrac{\frac{9y^2-1}{y^2}}{\frac{9y^2+6y+1}{y^2}}$

$= \dfrac{9y^2-1}{\cancel{y^2}} \cdot \dfrac{\cancel{y^2}}{9y^2+6y+1}$

$= \dfrac{(3y-1)\,\cancel{(3y+1)}}{(3y+1)\,\cancel{(3y+1)}} = \dfrac{3y-1}{3y+1}$

12. $\dfrac{3x-5}{x-2}+\dfrac{1}{2-x} = \dfrac{(3x-5)(2-x)+(x-2)}{(x-2)(2-x)}$

$= \dfrac{6x-3x^2-10+5x+x-2}{(x-2)\,(2-x)}$

$= \dfrac{-3x^2+12x-12}{(x-2)(2-x)}$

$= \dfrac{-3(x^2-4x+4)}{(x-2)(2-x)}$

$= \dfrac{-3\,\cancel{(x-2)}\,\cancel{(x-2)}}{-\cancel{(x-2)}\,\cancel{(x-2)}} = 3.$

Exercises 1.5

1. By calculator:

(i) $\binom{7}{4} = 35$ (ii) $\binom{6}{2} = 15$ (iii) $\binom{6}{4} = 15$

(iv) $\binom{15}{4} = 1365$ (v) $\binom{10}{9} = 10$

2. (i) $\binom{9}{0} = 1$ (ii) $\binom{10}{1} = 10$ (iii) $\binom{13}{13} = 1$

(iv) $\binom{30}{0} = 1$ (v) $\binom{18}{17} = \binom{18}{1} = 18$

3. $\binom{12}{3} = \binom{12}{12-3} = \binom{12}{9}$. Hence $k = 9$.

4. $\binom{16}{12} = \binom{16}{16-12} = \binom{16}{4}$. Hence $k = 4$.

5. $(a + 2b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3(2b) + \binom{4}{2}a^2(2b)^2 + \binom{4}{3}a(2b)^3 + \binom{4}{4}(2b)^4$

$= (1)a^4 + (4)a^3(2b) + (6)a^2(4b^2) + (4)a(8b^3) + (1)(16b^4)$

$= a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$

6. The coefficients in the 6th row of Pascal's triangle are 1 6 15 20 15 6 1.
Thus
$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

7. The fifth row of Pascal's triangle is 1 5 10 10 ...
The coefficient of the 4th term is thus 10. The 4th term is then
$10(1)^2(2x)^3 = 10(8x^3) = 80x^3$
Thus the coefficient is 80.

8. (i) $(a - 2b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3(-2b) + \binom{4}{2}a^2(-2b)^2 + \binom{4}{3}a(-2b)^3 + \binom{4}{4}(-2b)^4$

$= (1)a^4 + (4)a^3(-2b) + (6)a^2(4b^2) + (4)a(-8b^3) + (1)(16b^4)$

$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$

(ii) $(2x - y)^3 = \binom{3}{0}(2x)^3 + \binom{3}{1}(2x)^2(-y) + \binom{3}{2}(2x)(-y)^2 + \binom{3}{3}(-y)^3$

$= (1)(8x^3) + (3)(4x^2)(-y) + (3)(2x)(y^2) + (1)(-y^3)$

$= 8x^3 - 12x^2y + 6xy^2 - y^3$

(iii) $(p + 3q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3(3q) + \binom{4}{2}p^2(3q)^2 + \binom{4}{3}p(3q)^3 + \binom{4}{4}(3q)^4$

$= (1)p^4 + 4p^3(3q) + (6)p^2(9q^2) + (4)p(27q^3) + (1)(81q^4)$

$= p^4 + 12p^3q + 54p^2q^2 + 108pq^3 + 81q^4$

(iv) $(1 + 2y)^5 = \binom{5}{0}(1)^5 + \binom{5}{1}(1)^4(2y) + \binom{5}{2}(1)^3(2y)^2 + \binom{5}{3}(1)^2(2y)^3$

$+ \binom{5}{4}(1)(2y)^4 + \binom{5}{5}(2y)^5$

$= (1)(1) + (5)(1)(2y) + (10)(1)(4y^2) + (10)(1)(8y^3) + (5)(1)(16y^4) + (1)(32y^5)$

$= 1 + 10y + 40y^2 + 80y^3 + 80y^4 + 32y^5$

9. (i) $(2 + 3p)^6 = \binom{6}{0}2^6 + \binom{6}{1}2^5(3p) + \binom{6}{2}2^4(3p)^2 + \binom{6}{3}2^3(3p)^3 + \binom{6}{4}2^2(3p)^4$

$+ \binom{6}{5}2(3p)^5 + \binom{6}{6}(3p)^6$

$= (1)(64) + (6)(32)(3p) + (15)(16)(9p^2) + (20)(8)(27p^3) + (15)(4)(81p^4)$

$+ (6)(2)(243p^5) + (1)(729p^6)$

$= 64 + 576p + 2160p^2 + 4320p^3 + 4860p^4 + 2916p^5 + 729p^6$

(ii) $(1 - b)^7 = \binom{7}{0}1^7 + \binom{7}{1}1^6(-b) + \binom{7}{2}1^5(-b)^2 + \binom{7}{3}1^4(-b)^3 + \binom{7}{4}1^3(-b)^4$

$+ \binom{7}{5}1^2(-b)^5 + \binom{7}{6}(1)(-b)^6 + \binom{7}{7}(-b)^7$

$= (1)(1) + (7)(1)(-b) + (21)(1)(b^2) + (35)(1)(-b^3) + (35)(1)(b^4)$

$+ (21)(1)(-b^5) + (7)(1)(b^6) + (1)(-b^7)$

$= 1 - 7b + 21b^2 - 35b^3 + 35b^4 - 21b^5 + 7b^6 - b^7$

(iii) $(p-4q)^5 = \binom{5}{0}p^5 + \binom{5}{1}p^4(-4q) + \binom{5}{2}p^3(-4q)^2 + \binom{5}{3}p^2(-4q)^3$
$+ \binom{5}{4}p(-4q)^4 + \binom{5}{5}(-4q)^5$
$= (1)p^5 + (5)p^4(-4q) + (10)p^3(16q^2) + (10)p^2(-64q^3) + (5)p(256q^4)$
$+ (1)(-1024q^5)$
$= p^5 - 20p^4q + 160p^3q^2 - 640p^2q^3 + 1280pq^4 - 1024q^5$

10. The coefficients in the 7th row are then
1   1 + 6   6 + 15   15 + 20   20 + 15   15 + 6   6 + 1   1
or 1   7   21   35   35   21   7   1.

11. General term $= \binom{8}{r}x^{8-r}y^r$. For the 5th term, $r = 4$. Thus the 5th term is
$\binom{8}{4}x^{8-4}y^4 = 70x^4y^4$.

12. General term $= \binom{9}{r}x^{9-r}(-y)^r$. For the 4th term, $r = 3$. Thus the 4th term is
$\binom{9}{3}x^{9-3}(-y)^3 = (84)x^6(-y^3) = -84x^6y^3$

13. General term $= \binom{10}{r}(2x)^{10-r}y^r$. For the 6th term, $r = 5$. Thus the 6th term is
$\binom{10}{5}(2x)^5y^5 = (252)(32x^5)y^5 = 8064x^5y^5$

14. The number of terms in $(p + 2q)^6$ is $6 + 1 = 7$.
The middle term is then the 4th term (3 before and 3 after)
The general term is $\binom{6}{r}p^{6-r}(2q)^r$. For the 4th term, $r = 3$. Thus the middle term is
$\binom{6}{3}p^{6-3}(2q)^3 = (20)p^3(8q^3) = 160p^3q^3$.
The coefficient of this term is 160.

15. (i) $(2x-y)^8 = \binom{8}{0}(2x)^8 + \binom{8}{1}(2x)^7(-y) + \binom{8}{2}(2x)^6(-y)^2 + \binom{8}{3}(2x)^5(-y)^3$
$+ \binom{8}{4}(2x)^4(-y)^4 + \binom{8}{5}(2x)^3(-y)^5 + \binom{8}{6}(2x)^2(-y)^6$
$+ \binom{8}{7}(2x)(-y)^7 + \binom{8}{8}(-y)^8$
$= (1)(256x^8) + (8)(128x^7)(-y) + (28)(64x^6)(y^2) + (56)(32x^5)(-y^3)$
$+ (70)(16x^4)(y^4) + (56)(8x^3)(-y^5) + (28)(4x^2)(y^6)$
$+ (8)(2x)(-y^7) + (1)(y^8)$
$= 256x^8 - 1024x^7y + 1792x^6y^2 - 1792x^5y^3 + 1120x^4y^4 - 448x^3y^5$
$+ 112x^2y^6 - 16xy^7 + y^8$

(ii) $(a+2b)^9 = \binom{9}{0}a^9 + \binom{9}{1}a^8(2b) + \binom{9}{2}a^7(2b)^2 + \binom{9}{3}a^6(2b)^3 + \binom{9}{4}a^5(2b)^4$
$+ \binom{9}{5}a^4(2b)^5 + \binom{9}{6}a^3(2b)^6 + \binom{9}{7}a^2(2b)^7 + \binom{9}{8}a(2b)^8 + \binom{9}{9}(2b)^9$
$= (1)a^9 + (9)a^8(2b) + (36)a^7(4b^2) + (84)a^6(8b^3) + (126)a^5(16b^4)$
$+ (126)a^4(32b^5) + (84)a^3(64b^6) + (36)a^2(128b^7)$
$+ (9)a(256b^8) + (1)(512b^9)$
$= a^9 + 18a^8b + 144a^7b^2 + 672a^6b^3 + 2016a^5b^4 + 4032a^4b^5 + 5376a^3b^6$
$+ 4608a^2b^7 + 2304ab^8 + 512b^9$

16. The general term in the expansion of $(5x+1)^{10}$ is

$$\binom{10}{r}(5x)^{10-r}(1)^r.$$

For the 3rd term, $r = 2$. Thus the 3rd term is

$$\binom{10}{2}(5x)^8(1)^2 = (45)(390625x^8) = 17578125x^8$$

17. The general term in the expansion of $\left(x - \dfrac{3y}{2}\right)^9$ is

$$\binom{9}{r}x^{9-r}\left(\frac{-3y}{2}\right)^r$$

For the 4th term, $r = 3$. Thus the 4th term is

$$\binom{9}{3}x^6\left(\frac{-3y}{2}\right)^3 = (84)x^6\left(\frac{-27y^3}{8}\right) = -\frac{567}{2}x^6y^3$$

The coefficient is $-\dfrac{567}{2} = -283.5$.

Exercise 1.6

1. $ax^2 + bx + c = (2x-3)(3x+4)$ for all x

$$= 6x^2 + 8x - 9x - 12$$
$$= 6x^2 - x - 12$$

$\therefore a = 6, b = -1, c = -12$

2. $(3x-2)(x+5) = 3x^2 + px + q$ for all x

$3x^2 + 15x - 2x - 10 = 3x^2 + px + q$

$3x^2 + 13x - 10 = 3x^2 + px + q$

$\therefore p = 13, q = -10$

3. $x^2 + 6x + 16 = (x+a)^2 + b$ for all x

$x^2 + 6x + 16 = x^2 + 2ax + a^2 + b$

$\therefore \quad 2a = 6 \Rightarrow a = 3$

and $a^2 + b = 16$

$\therefore \quad 9 + b = 16 \Rightarrow b = 7$

4. $x^2 + 4x - 6 = (x+a)^2 + b$ for all x

$x^2 + 4x - 6 = x^2 + 2ax + a^2 + b$

$\therefore \; 2a = 4 \Rightarrow a = 2$

and $a^2 + b = -6$

$\therefore \quad 4 + b = -6 \Rightarrow b = -10$

5. $2x^2 + 5x + 6 = p(x+q)^2 + r$ for all x

$$= p(x^2 + 2xq + q^2) + r$$
$$= px^2 + 2pqx + pq^2 + r$$

$\therefore p = 2$

and $2pq = 5$

$\therefore 2(2)q = 5 \Rightarrow q = \dfrac{5}{4}$

and $pq^2 + r = 6$

$\therefore 2\left(\dfrac{5}{4}\right)^2 + r = 6$

$$\frac{25}{8} + r = 6 \Rightarrow r = 6 - \frac{25}{8}$$
$$= \frac{48-25}{8} = \frac{23}{8}$$

6. $(2x + a)^2 = 4x^2 + 12x + b$ for all values of x

$4x^2 + 4ax + a^2 = 4x^2 + 12x + b$

$\therefore\ 4a = 12 \Rightarrow a = 3$

and $a^2 = b \Rightarrow 9 = b$

7. $x^2 - 4x - 5 = (x - n)^2 - m$ for all values of x

$x^2 - 4x - 5 = x^2 - 2nx + n^2 - m$

$\therefore\quad -4 = -2n \Rightarrow n = 2$

and $-5 = n^2 - m$

$\therefore\quad -5 = 4 - m \Rightarrow m = 9$

8. (i) $V(x) = ax^3 + bx^2 + cx + d = (x + 5)(x + 3)(x + 2)$ for all x

$$= (x^2 + 8x + 15)(x + 2)$$

$$= x^3 + 2x^2 + 8x^2 + 16x + 15x + 30$$

$$= x^3 + 10x^2 + 31x + 30$$

$\therefore a = 1, b = 10, c = 31, d = 30.$

(ii) $S(x) = px^2 + qx + r = 2(x + 3)(x + 2) + 2(x + 5)(x + 3) + (x + 5)(x + 2)$

$$= 2(x^2 + 5x + 6) + 2(x^2 + 8x + 15) + (x^2 + 7x + 10)$$

$$= 5x^2 + 33x + 52$$

$\therefore\quad p = 5, q = 33, r = 52$

9. $3(x - p)^2 + q = 3x^2 - 12x + 7$ for all x

$\therefore\ 3(x^2 - 2px + p^2) + q = 3x^2 - 12x + 7$

$\therefore\ 3x^2 - 6px + 3p^2 + q = 3x^2 - 12x + 7$

$\therefore\quad -6p = -12 \Rightarrow p = 2$

and $3p^2 + q = 7$

$\therefore\quad 3(2)^2 + q = 7 \Rightarrow q = -5$

10. $V(x) = x^3 + 12x^2 + bx + 30 = (x^2 + cx + 4)(x + a)$

$$= x^3 + ax^2 + cx^2 + acx + 4x + 4a$$

$$= x^3 + x^2(a + c) + x(ac + 4) + 4a$$

$\therefore\quad a + c = 12$

and $b = ac + 4$

and $4a = 30 \Rightarrow a = \frac{30}{4} = \frac{15}{2}.$

$\therefore\quad a + c = 12 \Rightarrow c = 12 - a$

$$c = 12 - \frac{15}{2} = \frac{9}{2}$$

$\therefore\quad b = ac + 4 \Rightarrow b = \frac{15}{2} \cdot \frac{9}{2} + 4 = \frac{135 + 16}{4} = 37\frac{3}{4}$

11. $(x - 4)^3 = x^3 + px^2 + qx - 64$ for all x

$= (x - 4)(x - 4)(x - 4)$

$= (x^2 - 8x + 16)(x - 4)$

$= x^3 - 4x^2 - 8x^2 + 32x + 16x - 64$

$= x^3 - 12x^2 + 48x - 64$

$\therefore p = -12, q = 48$

12. $(x + a)(x^2 + bx + 2) = x^3 - 2x^2 - x - 6$ for all x

$= x^3 + bx^2 + 2x + ax^2 + abx + 2a$

$= x^3 + x^2(b + a) + x(2 + ab) + 2a$

$\therefore\quad b + a = -2$

and $2 + ab = -1$

and $2a = -6 \Rightarrow a = -3$

$\therefore\quad b + a = -2 \Rightarrow b - 3 = -2 \Rightarrow b = 1$

13. $(x-2)(x^2+bx+c) = x^3+2x^2-5x-6$ for all x

$= x^3+bx^2+cx-2x^2-2bx-2c$

$= x^3+x^2(b-2)+x(c-2b)-2c$

$\therefore \quad b-2=2 \Rightarrow b=4$

and $c-2b=-5 \Rightarrow c-2(4)=-5 \Rightarrow c=3$

14. $(5a-b)x+b+2c=0$ for all x

$\therefore (5a-b)x+b+2c = 0.x+0$

$\therefore 5a-b=0 \Rightarrow b=5a$

and $\quad b+2c=0 \Rightarrow b=-2c$

$\therefore 5a=-2c$

$a = \dfrac{-2c}{5}$

15. $(4x+r)(x^2+s) = 4x^3+px^2+qx+2$ for all x

$= 4x^3+4xs+rx^2+rs$

$= 4x^3+rx^2+4xs+rs$

$\therefore \quad r=p$

$\left.\begin{array}{l}\text{and } 4s=q\\ \text{and } rs=2\end{array}\right\} \Rightarrow pq = r.4s = 4rs$

$\therefore \quad pq = 4rs = 4(2) = 8$

16. $(x+s)(x-s)(ax+t) = ax^3+bx^2+cx+d$ for all x

$= (x^2-s^2)(ax+t)$

$= ax^3+tx^2-as^2x-ts^2$

$\therefore \quad t=b$

$\left.\begin{array}{l}\text{and } -as^2=c\\ \text{and } -ts^2=d\end{array}\right\} \therefore \dfrac{-a\cancel{s^2}}{-t\cancel{s^2}} = \dfrac{c}{d}$

$\Rightarrow -ad=-ct$

but $t=b$ $\qquad \therefore -ad=-cb$

$\Rightarrow ad=cb$

17. $\dfrac{1}{(x+1)(x-1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)}$ for all x

$\Rightarrow \dfrac{1}{(x+1)(x-1)} = \dfrac{A(x-1)+B(x+1)}{(x+1)(x-1)}$

$\Rightarrow \quad 1 = A(x-1)+B(x+1)$

$\Rightarrow \quad 1 = Ax-A+Bx+B$

$\Rightarrow \quad 1 = x(A+B)-A+B$

$\therefore \quad 0.x+1 = (A+B)x-A+B$

$\therefore \quad A+B=0$

and $\underline{-A+B}=1$

adding: $\quad 2B=1$

$B=\dfrac{1}{2}$

since $A+B=0 \Rightarrow A=-B=-\dfrac{1}{2}$.

18. $\dfrac{1}{(x+2)(x-3)} = \dfrac{C}{(x+2)} + \dfrac{D}{(x-3)}$

$\dfrac{1}{(x+2)(x-3)} = \dfrac{C(x-3)+D(x+2)}{(x+2)(x-3)}$

$\Rightarrow \quad 1 = C(x-3)+D(x+2)$

$1 = Cx-3C+Dx+2D$

$0.x+1 = x(C+D)-3C+2D$

$$\begin{aligned} &\Rightarrow & C + D &= 0 \\ \times 3 \quad &\text{and} & -3C + 2D &= 1 \\ & & \underline{3C + 3D} &= 0 \\ &\therefore \text{adding:} & 5D &= 1 \Rightarrow \\ & & D &= \frac{1}{5} \end{aligned}$$

since $C + D = 0 \Rightarrow C = -D = -\frac{1}{5}$.

19. $\dfrac{1}{(x+1)(x+4)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+4)}$

$\Rightarrow \dfrac{1}{(x+1)(x+4)} = \dfrac{A(x+4) + B(x+1)}{(x+1)(x+4)}$

$\Rightarrow 1 = Ax + 4A + Bx + B$

$0.x + 1 = x(A + B) + 4A + B$

$\therefore \quad A + B = 0$

and $\quad \underline{4A + B = 1}$

subtracting: $\quad -3A = -1 \quad \Rightarrow A = \frac{1}{3}$

since $\quad A + B = 0 \quad \Rightarrow B = -A = -\frac{1}{3}$.

20. $(x - 3)^2$ is a factor of $x^3 + ax + b$

$\Rightarrow (x - 3)^2 (x + k) = x^3 + ax + b$

$\therefore (x^2 - 6x + 9)(x + k) = x^3 + ax + b$

$\therefore x^3 + kx^2 - 6x^2 - 6kx + 9x + 9k = x^3 + ax + b$

$\therefore x^3 + x^2(k - 6) + x(-6k + 9) + 9k = x^3 + ax + b = x^3 + 0.x^2 + ax + b$

$\therefore k - 6 = 0 \Rightarrow k = 6$

also $-6k + 9 = a \quad \therefore -6(6) + 9 = a \Rightarrow a = -27$

also $\quad 9k \quad = b \quad \therefore 9(6) = b \Rightarrow b = 54$.

21. $(x - 2)^2$ is a factor of $x^3 + px + q$

$\Rightarrow (x - 2)^2 (x - k) = x^3 + px + q$

$\therefore \; (x^2 - 4x + 4)(x - k) = x^3 + 0.x^2 + px + q$

$\therefore \; x^3 - kx^2 - 4x^2 + 4kx + 4x - 4k = x^3 + 0.x^2 + px + q$

$\therefore \; x^3 + x^2(-k - 4) + x(4k + 4) - 4k =$

$\therefore \; -k - 4 = 0$

$\Rightarrow \; k = -4$

also $p = 4k + 4 = 4(-4) + 4 = -12$

$\quad q = -4k = -4(-4) = +16$.

22. $(x^2 - 4)$ is a factor of $x^3 + cx^2 + dx - 12$

$\therefore (x^2 - 4)(x + k) = x^3 + cx^2 + dx - 12$

$\therefore x^3 + kx^2 - 4x - 4k = x^3 + cx^2 + dx - 12$

$\therefore \quad k = c$

also $\quad d = -4$

and $-4k = -12 \Rightarrow k = 3 = c$

$\therefore \; (x^2 - 4)(x + 3) = x^3 + 3x^2 - 4x - 12$

$\Rightarrow (x - 2)(x + 2)(x + 3) = x^3 + 3x^2 - 4x - 12$.

23. $(x^2 + b)$ is a factor of $x^3 - 3x^2 + bx - 15$

$\Rightarrow (x^2 + b)(x + k) = x^3 - 3x^2 + bx - 15$

$\therefore x^3 + kx^2 + bx + bk = x^3 - 3x^2 + bx - 15$

$\Rightarrow \quad k = -3$

also $bk = -15 \Rightarrow b = \dfrac{-15}{k} = \dfrac{-15}{-3} = 5$.

24. $x^2 - px + 9$ is a factor of $x^3 + ax + b$

$\therefore (x^2 - px + 9)(x + k) = x^3 + 0.x^2 + ax + b$

$\therefore x^3 + kx^2 - px^2 - pkx + 9x + 9k = x^3 + 0.x^2 + ax + b$

$\therefore x^3 + x^2(k - p) + x(-pk + 9) + 9k = x^3 + 0.x^2 + ax + b$

$\therefore \quad k - p = 0 \Rightarrow k = p$

$-pk + 9 = a \Rightarrow -p(p) + 9 = a$

$\therefore a = 9 - p^2$

also $b = 9k \Rightarrow b = 9p$

$a + b = 17$

$\Rightarrow \quad 9 - p^2 + 9p = 17$

$-p^2 + 9p - 8 = 0$

$p^2 - 9p + 8 = 0$

$(p - 8)(p - 1) = 0$

$\therefore p = 8, 1$

25. $x^2 - kx + 1$ is a factor of $ax^3 + bx + c$

$$\begin{array}{r|l}
 & ax + ak \\ \hline
x^2 - kx + 1 & \cancel{ax^3} + \qquad + bx + c \\
 & \underline{\cancel{ax^3} - akx^2 + ax} \\
 & \qquad akx^2 + bx - ax + c \\
 & \qquad \cancel{akx^2} + (b - a)x + c \\
 & \qquad \underline{\cancel{akx^2} - ak^2x \quad + ak} \\
 & \qquad\qquad (b - a)x + ak^2x + c - ak \\
 & \qquad\qquad (b - a + ak^2)x + c - ak \quad \text{(remainder)}
\end{array}$$

Since $x^2 - kx + 1$ is a factor, there can be no remainder.

$\therefore \; (b - a + ak^2)x^2 + c - ak = 0 \quad$ for all x

$\Rightarrow b - a + ak^2 = 0$

and $c - ak = 0 \Rightarrow k = \dfrac{c}{a}$

$\therefore \; b - a + a\left(\dfrac{c}{a}\right)^2 = 0$

$\therefore \; b - a + \dfrac{c^2}{a} = 0 \Rightarrow ab - a^2 + c^2 = 0$

$\Rightarrow c^2 = a^2 - ab = a(a - b)$

26. $(x - a)^2$ is a factor of $x^3 + 3px + c$

i.e. $x^2 - 2ax + a^2$ is a factor of $x^3 + 3px + c$.

$$\begin{array}{r|l}
 & x + 2a \\ \hline
\therefore x^2 - 2ax + a^2 & \cancel{x^3} \qquad\quad + 3px + c \\
 & \underline{\cancel{x^3} - 2ax^2 + a^2x} \\
 & \qquad 2ax^2 + 3px - a^2x + c \\
 & \qquad \cancel{2ax^2} + (3p - a^2)x + c \\
 & \qquad \underline{\cancel{2ax^2} - 4a^2x + 2a^3} \\
 & \qquad\qquad (3p - a^2)x + 4a^2\,x + c - 2a^3 \text{ (remainder)}
\end{array}$$

Since $x^2 - 2ax + a^2$ is a factor, there can be no remainder.

$\therefore (3p - a^2 + 4a^2)x + c - 2a^3 = 0 \quad$ for all x

$\therefore 3p - a^2 + 4a^2 = 0 \Rightarrow 3p = -3a^2 \Rightarrow p = -a^2$

also, $c - 2a^3 = 0 \Rightarrow c = 2a^3$.

27. $x^2 + ax + b$ is a factor of $x^3 - k$.

$$\begin{array}{r|l} & \quad x - a \\ \hline \therefore x^2 + ax + b & \cancel{x^3} \qquad\qquad -k \\ & \underline{\cancel{x^3} + ax^2 + bx} \\ & \quad -\cancel{ax^2} - bx - k \\ & \quad \underline{-\cancel{ax^2} - a^2 x - b} \\ & \qquad\quad - bx + a^2 x - k + b \\ & \qquad\quad x(-b + a^2) - k + ba \ \text{(remainder)} \end{array}$$

Since $x^2 + ax + b$ is a factor, there can be no remainder.

$\therefore (-b + a^2) = 0 \Rightarrow b = a^2$

(i) also, $-k + ba = 0 \Rightarrow k = ab = a.a^2 = a^3$

(ii) Since $b = a^2 \Rightarrow b^3 = (a^2)^3 = a^6 = k^2$.

28.

$$\begin{array}{r|l} & \quad 2x - 1 \\ \hline 2x - \sqrt{3} & \cancel{4x^2} - 2(1 + \sqrt{3})x + \sqrt{3} \\ & \underline{\cancel{4x^2} - 2\sqrt{3}x} \\ & \quad -2(1 + \sqrt{3})x + 2\sqrt{3}x + \sqrt{3} \\ & \quad -2x - \cancel{2\sqrt{3}x} + \cancel{2\sqrt{3}x} + \sqrt{3} \\ & \quad -\cancel{2x} + \cancel{\sqrt{3}} \\ & \quad \underline{-\cancel{2x} + \cancel{\sqrt{3}}} \\ & \qquad\quad 0 \end{array}$$

$\therefore$ $2x - \sqrt{3}$ is a factor

and the second factor is $2x - 1$.

29. $5x + 3 = Ax(x + 3) + Bx(x - 1) + C(x - 1)(x + 3)$ for all x

$$\begin{aligned} 5x + 3 &= Ax^2 + 3Ax + Bx^2 - Bx + C(x^2 + 3x - x - 3) \\ &= Ax^2 + 3Ax + Bx^2 - Bx + Cx^2 + 3Cx - Cx - 3C \\ &= Ax^2 + 3Ax + Bx^2 - Bx + Cx^2 + 2Cx - 3C \\ &= (A + B + C)x^2 + (3A - B + 2C)x - 3C \end{aligned}$$

$\therefore A + B + C = 0$

also $3A - B + 2C = 5$

and $\quad -3C = 3 \Rightarrow C = -1$

$\therefore \quad A + B = 1$

$\quad \underline{3A - B = 7}$

adding: $\quad 4A = 8 \Rightarrow A = 2$

since $\quad A + B = 1 \Rightarrow B = 1 - A = 1 - 2 = -1$.

Exercise 1.7

1. (i) $3x - 2y = 4$

$$\begin{aligned} 3x &= 4 + 2y \\ x &= \frac{4 + 2y}{3} \end{aligned}$$

(ii) $2x - b = 4c$

$$\begin{aligned} 2x &= 4c + b \\ x &= \frac{4c + b}{2} \end{aligned}$$

(iii) $5x - 4 = \dfrac{y}{2}$

$$\begin{aligned} 5x &= \frac{y}{2} + 4 \\ x &= \frac{\frac{y}{2} + 4}{5} = \frac{y + 8}{10} \end{aligned}$$

(iv) $5(x-3) = 2y$

$$x - 3 = \frac{2y}{5}$$

$$x = \frac{2y}{5} + 3 = \frac{2y+15}{5}$$

(v) $3y = \frac{x}{3} - 2$

$$\frac{-x}{3} = -3y - 2$$

$$\frac{x}{3} = 3y + 2$$

$$x = 9y + 6$$

(vi) $xy = xz + yz$

$$xy - xz = yz$$

$$x(y-z) = yz$$

$$x = \frac{yz}{y-z}$$

2. (i) $2x - \frac{y}{3} = \frac{1}{3}$

$$2x = \frac{1}{3} + \frac{y}{3} = \frac{y+1}{3}$$

$$x = \frac{y+1}{6}$$

(ii) $z = \frac{y-2x}{3}$

$$3z = y - 2x$$

$$2x = y - 3z$$

$$x = \frac{y-3z}{2}$$

(iii) $\frac{a}{x} - b = c$

$$a - bx = cx$$

$$-cx - bx = -a$$

$$cx + bx = a$$

$$x(c+b) = a$$

$$x = \frac{a}{b+c}$$

3. (a) $V = \pi r^2 h$

$$\Rightarrow \pi r^2 h = V$$

$$\Rightarrow r^2 = \frac{V}{\pi h}$$

$$r = \sqrt{\frac{V}{\pi h}}$$

(b) $A = 2\pi rh$

$$\Rightarrow 2\pi rh = A$$

$$r = \frac{A}{2\pi h}$$

(c) $r = \sqrt{\frac{V}{\pi h}}$ and $r = \frac{A}{2\pi h}$

$$\Rightarrow \sqrt{\frac{V}{\pi h}} = \frac{A}{2\pi h}$$

$$\therefore \frac{V}{\pi h} = \frac{A^2}{4\pi^2 h^2}$$

$$\Rightarrow A^2 = \frac{4\pi^2 h^2 V}{\cancel{\pi}\cancel{h}} = 4\pi h V$$

4. (a) $A_{\text{circle}} = \pi r^2$

(b) The side of the square $= 2r$

$\Rightarrow A_{\text{square}} = l^2 = (2r)^2 = 4r^2$

(c) $A_{\text{corners}} = 4r^2 - \pi r^2 = (4 - \pi)r^2$

(d) area of new square $= (4r)^2 = 16r^2$

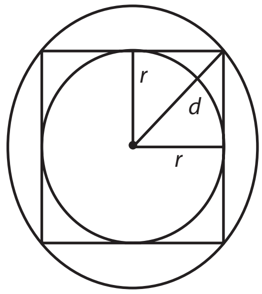
area of new circle $= \pi\left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}$

$\Rightarrow$ area of new shaded section $= 16r^2 - \frac{\pi r^2}{4}$

$$= r^2\left(16 - \frac{\pi}{4}\right)$$

$$= \frac{r^2}{2}(64 - \pi)$$

(e) To find the radius of the outer circle we need to find the distance from the centre of the circle to a corner (vertex) of the square.



$d = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r.$

$\Rightarrow$ Area of outer circle $= \pi d^2$

$= \pi(\sqrt{2}r)^2$

$= 2\pi r^2$

$=$ twice the area of the inner circle.

5. (i)

$$f' = \frac{fc}{c-u}$$

$$\Rightarrow f'(c-u) = fc$$

$$c - u = \frac{fc}{f'}$$

$$-u = \frac{fc}{f'} - c$$

$$u = c - \frac{fc}{f'} = \frac{f'c - fc}{f'} = \frac{c(f' - f)}{f'}$$

(ii)

$$f' = \frac{fc}{c-u}$$

$$f'(c-u) = fc$$

$$f'c - f'u = fc$$

$$f'c - fc = f'u$$

$$c(f' - f) = f'u$$

$$c = \frac{f'u}{f' - f}$$

6. (i) $T = 2\pi\sqrt{\frac{l}{g}}$

$T^2 = 4\pi^2\frac{l}{g}$

$l = \frac{gT^2}{4\pi^2}$

(ii) $T = 3, g = 10. \quad \Rightarrow \quad l = \frac{10.3^2}{4\pi^2} = \frac{90}{39.48} \simeq 2.3$ m

7. (i)

$$\frac{x}{y} = \frac{a+b}{a-b}$$

$$x(a-b) = y(a+b)$$

$$ax - bx = ay + by$$

$$ax - ay = bx + by$$

$$a(x-y) = b(x+y)$$

$$a = \frac{b(x+y)}{(x-y)}$$

(ii)

$$bc - ac = ac$$

$$-ac - ac = -bc$$

$$-2ac = -bc$$

$$a = \frac{-bc}{-2c} = \frac{b}{2}$$

8. (i) $y = \dfrac{3(u - v)}{4}$

$4y = 3u - 3v$

$3v = 3u - 4y$

$v = \dfrac{3u - 4y}{3}$

(ii) $s = \dfrac{t}{2}(u + v)$

$2s = tu + tv$

$-tv = tu - 2s$

$tv = 2s - tu$

$v = \dfrac{2s - tu}{t}$

9. $A = P\left(1 + \dfrac{i}{100}\right)^3$

$P\left(1 + \dfrac{i}{100}\right)^3 = A$

$\left(1 + \dfrac{i}{100}\right) = \dfrac{A}{P}$

$1 + \dfrac{i}{100} = \sqrt[3]{\dfrac{A}{P}}$

$100 + i = 100\sqrt[3]{\dfrac{A}{P}}$

$i = 100\sqrt[3]{\dfrac{A}{P}} - 100$

$P = 2500, \qquad A = 2650$

$\therefore i = 100\sqrt[3]{\dfrac{2650}{2500}} - 100$

$= 100(1.0196) - 100$

$= 1.961$

$\therefore \quad i = 2\%$

10. (i) $d = \sqrt{\dfrac{a - b}{ac}}$

$d^2 = \dfrac{a - b}{ac}$

$acd^2 = a - b$

$c = \dfrac{a - b}{ad^2}$

(ii) $b = \dfrac{2c - 1}{c - 1}$

$\Rightarrow b(c - 1) = 2c - 1$

$bc - b = 2c - 1$

$bc - 2c = b - 1$

$c(b - 2) = b - 1$

$c = \dfrac{b - 1}{b - 2}$

11. (i) From Pythagoras: $h^2 + r^2 = 15^2$

$h^2 = 15^2 - r^2$

$h = \sqrt{15^2 - r^2}$

(ii) at $r = 5$ cm: $h = \sqrt{15^2 - 5^2}$

$= \sqrt{225 - 25} = \sqrt{200}$

$= 10\sqrt{2}$ cm

(iii) $r = \dfrac{15}{2}$: $h = \sqrt{15^2 - \left(\dfrac{15}{2}\right)^2}$

$= \sqrt{168.75} = 12.99$ cm

$h = 13$ cm

12. (i) $2W + L = 300$

$L = 300 - 2W$

(ii) $A = L \times W$

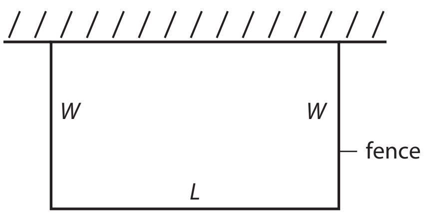
$= (300 - 2W) \cdot W = 300W - 2W^2$

(iii) $10\,000 = 300W - 2W^2$

$2W^2 - 300W + 10\,000 = 0$

$W^2 - 150W + 5000 = 0$

$(W - 50)(W - 100) = 0$



$\Rightarrow \quad W = 50 \quad \text{or} \quad W = 100$

hence, $L = 300 - 2(50) \quad \text{or} \quad L = 300 - 2(100)$

$= 200 \qquad\qquad = 100$

answer: (50, 200) or (100 100)

Exercise 1.8

1. (a) 4, 7, 10, 13, 16 … constant 1st difference ⇒ linear

(b) −2, 2, 6, 10, 14 … constant 1st difference ⇒ linear

(c) −4, −3, 0, 5, 12

+1, +3, +5, +7

+2, +2, +2 …. constant 2nd difference ⇒ quadratic

(d) 2, 1, −2, −7, −14, −23…

−1, −3, −5, −7,

−2, −2, −2 … constant 2nd difference ⇒ quadratic

(e) 2, 7, 22, 47

+5, +15, +25

+10, +10 …. constant 2nd difference ⇒ quadratic

(f) 3, 1, −5, −15, −29, ….

−2, −6, −10, −14

−4, −4, −4, … constant 2nd difference ⇒ quadratic

(g) 1, −4, −19, −44, −79 ….

−5, −15, −25, −35

−10, −10, −10 … constant 2nd difference ⇒ quadratic

(h) 3, −2, −7, −12, −17…

−5, −5, −5, −5… constant 1st difference ⇒ linear

(i) 0, 3, 12, 27, 48

+3, +9, +15, +21

+6, +6, +6 …. constant 2nd difference ⇒ quadratic

(j) 5, 17, 37, 65, 101

+12, +20, +28, +36

+8, +8, +8… constant 2nd difference ⇒ quadratic.

2. (a) −1, 3, 15, 35, 63

first differences = +4, +12, +20, +28

second differences = +8, +8, +8.

⇒ quadratic pattern of the form $ax^2 + bx + c$

also, $2a = 8 \Rightarrow a = 4$.

$\therefore 4x^2 + bx + c$

let $x = 1 \Rightarrow 4(1)^2 + b(1) + c = -1$

$\Rightarrow \quad b + c = -1 - 4 = -5$

let $x = 2 \Rightarrow 4(2)^2 + b(2) + c = 3$

$\Rightarrow 2b + c = 3 - 16 = -13$

using simultaneous equations: $2b + c = -13$

$$\underline{b + c = -5}$$

$$b \quad = -8$$

$$\Rightarrow -8 + c = -5$$

$$\Rightarrow \quad c = -5 + 8 = 3$$

$\therefore ax^2 + bx + c = 4x^2 - 8x + 3$ for $x = 1, 2, 3, ...$

Note also we could let $x = 0 \Rightarrow 4(0)^2 + b(0) + c = -1$

$\Rightarrow \quad c = -1$

$$\text{let } x = 1 \Rightarrow 4(1)^2 + b(1) + c = 3$$
$$4 + b + c = 3$$
$$4 + b - 1 = 3$$
$$b = 0.$$

$\therefore ax^2 + bx + c = 4x^2 - 1 \quad \text{for } x = 0, 1, 2, ...$

(b) $4, 3, 0, -5, -12, -21, -32$

first difference $= -1, -3, -5, -7, -9, -11$

second differnce $= -2, -2, -2, -2$

$\Rightarrow$ quadratic pattern of the form $ax^2 + bx + c$

also, $2a = -2 \Rightarrow a = -1.$

$\therefore -x^2 + bx + c$

let $x = 0 \quad \Rightarrow \quad -(0)^2 + b(0) + c = 4$

$c = 4$

let $x = 1 \quad \Rightarrow \quad -(1)^2 + b(1) + c = 3$

$-1 + b + 4 = 3$

$b = 0$

$\therefore ax^2 + bx + c = -x^2 + 4$ is the pattern for $x = 0, 1, 2, ...$

3. (i) $2, 7, 12, 17, 22, \ldots..$

first difference $= 5$, a constant $\Rightarrow$ a linear pattern.

$\therefore f(x) = ax + b = 5x + b.$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow f(0) = 5(0) + b = 2$

$\therefore \quad b = 2$

$\therefore f(x) = 5x + 2$ for $x = 0, 1, 2, \ldots$

(ii) $-6, -2, 2, 6, 10\ldots$

first difference $= 4$, a constant $\Rightarrow$ a linear pattern.

$\therefore f(x) = ax + b = 4x + b$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow f(0) = 4(0) + b = -6$

$\therefore \quad b = -6.$

$\therefore f(x) = 4x - 6$ for $x = 0, 1, 2, \ldots$

(iii) $3, 2, 1, 0, -1, -2, \ldots$

first difference $= -1$, a constant $\Rightarrow$ a linear relationship

$\therefore f(x) = ax + b = -x + b$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow \quad f(0) = -(0) + b = 3$

$\therefore \quad b = 3$

$\therefore f(x) = -x + 3$

$= 3 - x.$

(iv) $-2, 7, -12, -17, -22, \ldots$

first difference $= -5$, a constant $\Rightarrow$ a linear relationship

$\therefore f(x) = ax + b = -5x + b$

Let $x = 0$ be the first term of the pattern.

$\Rightarrow f(0) = -5(0) + b = -2$

$\therefore \quad b = -2$

$\therefore f(x) = -5x - 2$

(v) $3, 3.5, 4, 4.5, 5, \ldots$

first difference $= 0.5$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 0.5x + b$

Let $x = 0$ be the first term of the pattern.

$\therefore f(0) = 0.5(0) + b = 3$

$\Rightarrow \qquad b = 3$

$\therefore f(x) = 0.5(x) + 3$

$= \frac{x}{2} + 3$ for $x = 0, 1, 2, \ldots$

(vi) $-1, -0.8, -0.6, -0.4, -0.2, \ldots$

first difference $= 0.2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 0.2x + b$

Let $x = 0$ be the first term of the pattern.

$\therefore f(0) = 0.2(0) + b = -1$

$\therefore \qquad b = -1$

$\therefore f(x) = 0.2x - 1$ for $x = 0, 1, 2, \ldots$

4. 11, 13, 15, 17, 19, …

first difference $= 2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 2x + b$

Let $x = 3$ be the first term of the pattern.

$\Rightarrow f(3) = 2(3) + b = 11$

$\Rightarrow \qquad b = 5$

$\therefore f(x) = 2x + 5$ for $x = 3, 4, 5, \ldots$

5. 1, 3, 5, 7, 9, …

first difference $= 2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 2x + b$

Let $x = -2$ be the first term of the pattern.

$\therefore f(-2) = 2(-2) + b = 1$

$b = 1 + 4 = 5.$

$\therefore f(x) = 2x + 5$ for $x = -2, -1, 0, \ldots$

6. (a) 3, 6, 9, …

a first difference $= 3$ (a constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 3x + b$

Let $x = 1$ be the first element of the pattern.

$\therefore f(1) = 3(1) + b = 3$

$\Rightarrow \qquad b = 0.$

$\therefore f(x) = 3x$ for $x = 1, 2, 3, \ldots$

$\Rightarrow$ for the 15^{th} element, $x = 15$.

$\therefore f(15) = 3(15) = 45$ matchsticks are needed.

(b) 4, 8, 12, …

a first difference $= 4$ (a constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 4x + b$

Let $x = 1$ be the first element of the pattern.

$\therefore f(1) = 4(1) + b = 4$

$\Rightarrow \qquad b = 0.$

$\Rightarrow \therefore f(x) = 4x$ for $x = 1, 2, 3, \ldots$

$\Rightarrow$ for the 15^{th} element, $x = 15$.

$\therefore f(15) = 4(15) = 60$ matchsticks are needed.

(c) 3, 5, 7, …

a first difference $= 2$ (constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 2x + b$

Let $x = 1$ be the first element of the pattern.
$\therefore f(1) = 2(1) + b = 3$
$\Rightarrow \quad b = 1$
$\therefore f(x) = 2x + 1$ for $x = 1, 2, 3, \ldots$
$\Rightarrow$ For the 15th element, $x = 15$.
$\therefore f(15) = 2(15) + 1 = 31$ matchsticks are needed.

7. Plan $A = 35x + 70$
Plan $B = 24x + 125$
Both plans repay the same amount if
$35x + 70 = 24x + 125$
$\Rightarrow \quad 11x = 55$
$x = 5$ months.

8. 4, 7, 14, 25, 40
first difference: 3, 7, 11, 15
second difference : 4, 4, 4 $\Rightarrow$ a quadratic pattern, $f(t) = at^2 + bt + c$
$\therefore 2a = 4 \Rightarrow a = 2.$
$\therefore f(t) = 2t^2 + bt + c$
At $t = 0$, $f(0) = 2(0)^2 + b(0) + c = 4$ (i.e. at the start there were 4 bacteria)
$\Rightarrow \quad c = 4$
At $t = 1$, $f(1) = 2(1)^2 + b(1) + c = 7$
$\Rightarrow 2 + b + 4 = 7$
$\Rightarrow b = 1$
$\therefore f(t) = 2t^2 + t + 4$ for $t = 0, 1, 2, 3, \ldots.$
when is $f(t) = 529$, assuming $f(t) = 2t^2 + t + 4$?
if $t = 10 \Rightarrow 2(10)^2 + 10 + 4 = 214$ too small
$t = 15 \Rightarrow 2(15)^2 + 15 + 4 = 469$ too small
$t = 16 \Rightarrow 2(16)^2 + 16 + 4 = 532$ too large
$\therefore$ In the 16th hour, the number of bacteria was 529.

Exercise 1.9

1. (i) $y = 2x^2 + 2x - 1$ is not linear because the highest power is not 1.
(ii) $y = 2(x - 1)^{-1}$ is not linear because the highest power of x is not 1.
(iii) $y^2 = 3x + 4$
$\Rightarrow \quad y = \sqrt{3x + 4}$ is not linear because the highest power of x is not 1.
$= (3x + 4)^{\frac{1}{2}}$

2. (i) Solve $5x - 3 = 32$
$\Rightarrow 5x = 35$
$x = \frac{35}{5} = 7$
(ii) Solve $3x + 2 = x + 8$
$\Rightarrow \quad 3x - x = 8 - 2$
$2x = 6$
$x = 3$
(iii) Solve $2 - 5x = 8 - 3x$
$\Rightarrow \quad 3x - 5x = 8 - 2$
$-2x = 6$
$x = \frac{6}{-2} = -3$

3. (i) Solve $2(x - 3) + 5(x - 1) = 3$

$\Rightarrow 2x - 6 + 5x - 5 = 3$

$\Rightarrow 7x = 3 + 11 = 14$

$x = 2$

(ii) Solve $2(4x - 1) - 3(x - 2) = 14$

$\Rightarrow 8x - 2 - 3x + 6 = 14$

$5x = 14 - 4 = 10$

$x = 2$

(iii) Solve $3(x - 1) - 4(x - 2) = 6(2x + 3)$

$\Rightarrow 3x - 3 - 4x + 8 = 12x + 18$

$-x - 12x = 18 - 5$

$-13x = 13$

$x = -1$

(iv) Solve $3(x + 5) + 2(x + 1) - 3x = 22$

$\cancel{3x} + 15 + 2x + 2 - \cancel{3x} = 22$

$2x = 22 - 17$

$2x = 5$

$x = \frac{5}{2} = 2.5$

4. (i) $\frac{2x + 1}{5} = 1$

$\Rightarrow 2x + 1 = 5$

$2x = 4$

$x = 2$

(ii) $\frac{3x - 1}{4} = 8$

$\Rightarrow 3x - 1 = 32$

$3x = 33$

$x = 11$

(iii) $\frac{x - 3}{4} = \frac{x - 2}{5}$

$\Rightarrow 5(x - 3) = 4(x - 2)$

$5x - 15 = 4x - 8$

$5x - 4x = 15 - 8$

$x = 7$

5. (i) $\frac{2a}{3} - \frac{a}{4} = \frac{5}{6}$

$\Rightarrow$ multiplying each term by 12 : $4(2a) - 3(a) = 2(5)$

$\Rightarrow 8a - 3a = 10$

$5a = 10$

$a = 2$

(ii) $\frac{b + 2}{4} - \frac{b - 3}{3} = \frac{1}{2}$

$\Rightarrow$ multiplying each term by 12 : $3(b + 2) - 4(b - 3) = 6$

$\Rightarrow 3b + 6 - 4b + 12 = 6$

$-b = 6 - 18$

$-b = -12$

$b = 12$

(iii) $\frac{3c-1}{6}-\frac{c-3}{4}=\frac{4}{3}$

$\Rightarrow$ multiplying each term by 12 : $2(3c-1)-3(c-3)=4(4)$

$$\Rightarrow 6c-2-3c+9=16$$
$$3c=16-7$$
$$3c=9$$
$$c=3$$

6. (i) $\frac{x-2}{5}+\frac{2x-3}{10}=\frac{1}{2}$

multiplying each term by 10 we get : $2(x-2)+(2x-3)=5(1)$

$$\Rightarrow 2x-4+2x-3=5$$
$$4x=5+7$$
$$4x=12$$
$$x=3$$

(ii) $\frac{3y-12}{5}+3=\frac{3(y-5)}{2}$

multiplying each term by 10 we get : $2(3y-12)+10(3)=5.3(y-5)$

$$\Rightarrow 6y-24+30=15y-75$$
$$\therefore \quad 6y-15y=24-30-75$$
$$-9y=-81$$
$$y=\frac{-81}{-9}=+9$$

(iii) $\frac{3p-2}{6}-\frac{3p+1}{4}=\frac{2}{3}$

multiplying each term by 12 we get : $2(3p-2)-3(3p+1)=4(2)$

$$\Rightarrow 6p-4-9p-3=8$$
$$-3p=8+7$$
$$-3p=15$$
$$p=-5$$

(iv) $\frac{3r-2}{5}-\frac{2r-3}{4}=\frac{1}{2}$

multiplying each term by 20 we get : $4(3r-2)-5(2r-3)=10(1)$

$$\Rightarrow 12r-8-10r+15=10$$
$$2r=10-7$$
$$2r=3$$
$$r=\frac{3}{2}=1.5$$

7. (i) $\frac{3}{4}(2x-1)-\frac{2}{3}(4-x)=2$

multiplying each term by 12 we get : $3.3(2x-1)-4.2(4-x)=12.2$

$$\Rightarrow 18x-9-32+8x=24$$
$$26x=24+41$$
$$26x=65$$
$$x=\frac{65}{26}=2.5$$

(ii) $\frac{2}{3}(x-1)-\frac{1}{5}(x-3)=x+1$

multiplying each term by 15 we get : $5.2(x-1)-3.1(x-3)=15x+15$

$$\Rightarrow 10x-10-3x+9=15x+15$$
$$7x-15x=15+1$$
$$-8x=16$$
$$x=-2.$$

Exercise 1.10

1. (i) $3x - 2y = 8$, $x + y = 6$ $\Rightarrow$

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 2y &= 12 \\ \hline \text{(adding)} \quad 5x &= 20 \\ x &= 4 \end{aligned}$$

since $x + y = 6 \Rightarrow 4 + y = 6$

$$\therefore y = 2.$$

$\therefore$ solution $(x, y) = (4, 2)$

(ii) $3x - y = 1$, $x - 2y = -8$ $\Rightarrow$

$$\begin{aligned} 6x - 2y &= 2 \\ x - 2y &= -8 \\ \hline \text{(subtracting)} \quad 5x &= 10 \\ x &= 2 \end{aligned}$$

since $x - 2y = -8 \Rightarrow 2 - 2y = -8$

$$\begin{aligned} -2y &= -10 \\ y &= 5 \end{aligned}$$

$\therefore$ solution $(x, y) = (2, 5)$

(iii) $2x - 5y = 1$, $4x - 3y - 9 = 0$ $\Rightarrow$

$$\begin{aligned} 4x - 10y &= 2 \\ 4x - 3y &= 9 \\ \hline \text{(subtracting)} \quad -7y &= -7 \\ y &= 1 \end{aligned}$$

since $2x - 5y = 1 \Rightarrow 2x - 5(1) = 1$

$$\begin{aligned} \Rightarrow \quad 2x &= 6 \\ x &= 3 \end{aligned}$$

$\therefore$ solution $(x, y) = (3, 1)$

2. (i) $4x - 5y = 22$, $7x + 3y - 15 = 0$ $\Rightarrow$

$$\begin{aligned} 12x - 15y &= 66 \\ 35x + 15y &= 75 \\ \hline \text{(adding)} \quad 47x &= 141 \\ x &= \frac{141}{47} = 3 \end{aligned}$$

since $4x - 5y = 22 \Rightarrow 4(3) - 5y = 22$

$$\begin{aligned} -5y &= 22 - 12 \\ -5y &= 10 \\ y &= -2 \end{aligned}$$

$\therefore$ solution $(x, y) = (3, -2)$

(ii) $\frac{x}{2} - \frac{y}{6} = \frac{1}{6}$, $x - 2y = -8$ $\Rightarrow$ $3x - y = 1$, $x - 2y = -8$

$$\begin{aligned} \Rightarrow \quad 6x - 2y &= 2 \\ x - 2y &= -8 \\ \hline \text{(subtracting)} \quad 5x &= 10 \\ x &= 2 \end{aligned}$$

since $3x - y = 1 \Rightarrow 3(2) - y = 1$

$$\begin{aligned} 6 - y &= 1 \\ -y &= -5 \\ y &= 5 \end{aligned}$$

$\therefore$ solution $(x, y) = (2, 5)$

(iii) $\frac{4x-2}{5} = \frac{8y}{10}$ $\Rightarrow$ $8x - 4 = 8y$

$18x - 20y = 4$ $\qquad 18x - 20y = 4$

$$\Rightarrow \quad 8x - 8y = 4$$
$$18x - 20y = 4$$

$$\Rightarrow \quad 40x - 40y = 20$$
$$\underline{36x - 40y = 8}$$

(subtracting) $\qquad 4x = 12$

$$x = 3$$

since $\quad 18x - 20y = 4$

$$\Rightarrow 18(3) - 20y = 4$$

$$-20y = 4 - 54 = -50$$

$$y = 2\tfrac{1}{2}$$

$\therefore$ solution $\quad (x, y) = (3, 2\tfrac{1}{2})$

3. $\frac{2x-5}{3} + \frac{y}{5} = 6 \Rightarrow 5(2x-5) + 3y = 15.6$

$$\Rightarrow 10x - 25 + 3y = 90$$

$$10x + 3y = 115$$

$\frac{3x}{10} + 2 = \frac{3y-5}{2} \Rightarrow 3x + 10.2 = 5(3y - 5)$

$$\Rightarrow 3x + 20 = 15y - 25$$

$$\Rightarrow 3x - 15y = -45$$

$\therefore$ $10x + 3y = 115$ $\Rightarrow$ $50x + 15y = 575$

$3x - 15y = -45$ $\qquad \underline{3x - 15y = -45}$

(adding) $\qquad 53x = 530$

$$x = 10$$

since $\quad 10x + 3y = 115$

$$10(10) + 3y = 115$$

$$3y = 15$$

$$y = 5$$

$\therefore$ solution $\quad (x, y) = (10, 5)$

4. $y = 3x - 23 \quad \Rightarrow \quad y = 3x - 23$

$y = \frac{x}{2} + 2 \quad \Rightarrow \quad 2y = x + 4$

$$\Rightarrow \quad 2y = 6x - 46$$
$$\underline{2y = x + 4}$$

(subtracting) $\qquad 0 = 5x - 50$

$$\Rightarrow 5x = 50$$

$$x = 10$$

since $\quad y = 3x - 23$

$\Rightarrow \quad y = 3(10) - 23 = 7$

$\therefore$ solution $(x, y) = (10, 7)$

5. (i) $A: 2x + y + z = 8$ $\quad$ $3A: 6x + \cancel{3y} + 3z = 24$

$B: 5x - 3y + 2z = 3$ $\quad$ $B: \underline{5x - \cancel{3y} + 2z = 3}$

$C: 7x + y + 3z = 20$ $\quad$ D: (adding) $11x \quad + 5z = 27$

also $B: 5x - \cancel{3y} + 2z = 3$

$3C: \underline{21x + \cancel{3y} + 9z = 60}$

E: (adding) $26x \quad + 11z = 63$

$\Rightarrow 11D: 121x + \cancel{55z} = 297$

$5E: \underline{130x + \cancel{55z} = 315}$

(subtracting) $-9x = -18$

$x = 2$

since $11x + 5z = 27$

$11(2) + 5z = 27$

$5z = 27 - 22 = 5$

$z = 1$

since $2x + y + z = 8$

$2(2) + y + 1 = 8$

$y = 3$

$\therefore$ solution $(x, y, z) = (2, 3, 1)$

(ii) $A: 2x - y - z = 6$ $\quad$ $2A: 4x - \cancel{2y} - 2z = 12$

$B: 3x + 2y + 3z = 3$ $\Rightarrow$ $B: \underline{3x + \cancel{2y} + 3z = 3}$

$C: 4x + y - 2z = 3$ $\quad$ D: (adding) $7x \quad + z = 15$

also $B: 3x + \cancel{2y} + 3z = 3$

$2C: \underline{8x + \cancel{2y} - 4z = 6}$

E: (subtracting) $-5x \quad + 7z = -3$

since D: $7x + z = 15 \Rightarrow 7D = 49x + \cancel{7z} = 105$

also $E = \underline{-5x + \cancel{7z} = -3}$

subtracting: $54x = 108$

$x = 2.$

since $D: 7x + z = 15$

$\Rightarrow \quad 7(2) + z = 15$

$z = 1$

also, since $A: 2x - y - z = 6$

$\Rightarrow \quad 2(2) - y - 1 = 6$

$-y = 3$

$y = -3.$

$\therefore$ solution $(x, y, z) = (2, -3, 1)$

(iii) $A: 2x + y - z = 9$ $\quad$ $A: 2x + y - \cancel{z} = 9$

$B: x + 2y + z = 6$ $\Rightarrow$ $B: \underline{x + 2y + \cancel{z} = 6}$

$C: 3x - y + 2z = 17$ $\quad$ D (adding): $3x + 3y = 15$

also $2B: 2x + 4y + \cancel{2z} = 12$

$C: \underline{3x - y + \cancel{2z} = 17}$

E (subtracting): $-x + 5y = -5$

$$\begin{aligned}
\text{since } D: \quad 3x + 3y &= 15 \\
3E: -3x + 15y &= -15 \\
\text{adding:} \quad 18y &= 0 \\
y &= 0. \\
\text{since } E: \quad -x + 5y &= -5 \\
-x + 5(0) &= -5 \\
x &= 5.
\end{aligned}$$

$$\begin{aligned}
\text{also, } A: 2x + y - z &= 9 \\
2(5) + 0 - z &= 9 \\
-z &= 9 - 10 = -1 \\
+z &= 1
\end{aligned}$$

$\therefore$ solution $(x, y, z) = (5, 0, 1)$

6. (i) $A: 2a + b + c = 8 \quad \Rightarrow \quad 3A: 6a + \cancel{3b} + 3c = 24$

$B: 5a - 3b + 2c = 3 \qquad B: 5a - \cancel{3b} + 2c = -3$

$C: 7a - 3b + 3c = 1 \qquad D: \text{adding: } 11a \quad + 5c = 21$

also $\quad B: 5a - \cancel{3b} + 2c = -3$

$C: 7a - \cancel{3b} + 3c = 1$

$E: \text{subtracting: } -2a \quad - c = -4.$

$$\begin{aligned}
\text{since } D: \quad 11a + \cancel{5c} &= 21 \\
\text{and } 5E: -10a - \cancel{5c} &= -20 \\
\text{adding:} \quad a &= 1 \\
\text{since } \quad D: 11a + 5c &= 21 \\
\Rightarrow \quad 11(1) + 5c &= 21 \\
5c &= 10 \\
c &= 2.
\end{aligned}$$

$$\begin{aligned}
\text{also, } A: 2a + b + c &= 8 \\
2(1) + b + 2 &= 8 \\
b &= 4
\end{aligned}$$

$\therefore$ solution $(a, b, c) = (1, 4, 2)$

(ii) $A: x + y + 2z = 3 \quad \Rightarrow \quad 2A: 2x + \cancel{2y} + 4z = 6$

$B: 4x + 2y + z = 13 \qquad B: 4x + \cancel{2y} + z = 13$

$C: 2x + y - 2z = 9 \quad D: \text{(subtracting): } -2x \quad + 3z = -7$

also $\quad B: \cancel{4x} + \cancel{2y} + z = 13$

$2C: \cancel{4x} + \cancel{2y} - 4z = 18$

$E: \text{(subtracting):} \quad 5z = -5$

$\Rightarrow \quad z = -1$

$$\begin{aligned}
\text{since } \quad D: -2x + 3z &= -7 \\
\Rightarrow \quad -2x + 3(-1) &= -7 \\
-2x - 3 &= -7 \\
-2x &= -7 + 3 \\
-2x &= -4 \\
x &= 2.
\end{aligned}$$

$$\begin{aligned}
&\text{also, } A:\quad x + y + 2z = 3\\
&\qquad 2 + y + 2(-1) = 3\\
&\qquad\qquad y = 3
\end{aligned}$$

$\therefore$ solution $(x, y, z) = (2, 3, -1)$.

(iii) $A: x + y + z = 2$ $\qquad$ $A: x + y + z = 2$

$B: 2x + 3y + z = 7$

$C: \frac{x}{2} + \frac{y}{2} + \frac{z}{3} = \frac{2}{3} \quad \Rightarrow \quad C: \underline{3x - y + 2z = 4}$

D: (adding) $\quad 4x \qquad + 3z = 6$

also $B: 2x + 3y + z = 7$

$\underline{3C: 9x - 3y + 6z = 12}$

E: (adding) $11x \qquad + 7z = 19$

$\therefore$ $11D: 44x + 33z = 66$

$4E: \underline{44x + 28z = 76}$

subtracting: $\quad 5z = -10$

$z = -2.$

since $D: 4x + 3z = 6$

$4x + 3(-2) = 6$

$4x = 12$

$x = 3$

also, $A: x + y + z = 2$

$\Rightarrow \quad 3 + y - 2 = 2$

$y = 2 - 1$

$y = 1$

$\therefore$ solution $(x, y, z) = (3, 1, -2)$

7. $A: 6x + 4y - 2z - 5 = 0 \quad \Rightarrow \quad A: 6x + \cancel{4y} - 2z = 5$

$B: 3x - 2y + 4z + 10 = 0 \qquad \underline{2B: 6x - \cancel{4y} + 8z = -20}$

$C: 5x - 2y + 6z + 13 = 0 \qquad D$: (adding): $12x \quad + 6z = -15$

also $B: 3x - \cancel{2y} + 4z = -10$

$C: \underline{5x - \cancel{2y} + 6z = -13}$

E (subtracting): $-2x \qquad -2z = 3$

also, $D: \quad 12x + \cancel{6z} = -15$

$3E: \quad \underline{-6x - \cancel{6z} = 9}$

adding: $\quad 6x \qquad = -6$

$x = -1$

since $D: \quad 12x + 6z = -15$

$12(-1) + 6z = -15$

$6z = -3$

$z = -\frac{1}{2}$

also, $A: \quad 6x + 4y - 2z = 5$

$\Rightarrow 6(-1) + 4y - 2\left(-\frac{1}{2}\right) = 5$

$-6 + 4y + 1 = 5$

$$4y = 10 \qquad\qquad y = 2\tfrac{1}{2}$$

$\therefore$ solution $(x, y, z) = (-1, 2\frac{1}{2}, -\frac{1}{2})$.

8. Curve $f(x) = ax^2 + bx + c$

$(1, 2)$ on curve $\Rightarrow$ when $x = 1, f(x) = 2$

$$\Rightarrow 2 = a(1)^2 + b(1) + c$$
$$\Rightarrow 2 = a + b + c \qquad : A$$

$(2, 4)$ on curve $\Rightarrow$ when $x = 2, f(x) = 4$

$$\Rightarrow 4 = a(2)^2 + b(2) + c$$
$$\Rightarrow 4 = 4a + 2b + c \qquad : B$$

$(3, 8)$ on curve $\Rightarrow$ when $x = 3, f(x) = 8$

$$\Rightarrow 8 = a(3)^2 + b(3) + c$$
$$\Rightarrow 8 = 9a + 3b + c \qquad : C$$

since A: $\quad a + b + \cancel{c} = 2$

and B: $\quad \underline{4a + 2b + \cancel{c} = 4}$

D(subtracting): $\quad -3a - b \quad = -2$

since B: $4a + 2b + \cancel{c} = 4$

and C: $\underline{\ a + 3b + \cancel{c} = 8}$

E(subtracting): $-5a - b \quad = -4$

also $\quad D$: $\quad -3a - \cancel{b} = -2$

$\quad E$: $\quad \underline{-5a - \cancel{b} = -4}$

subtracting: $\quad 2a \quad = 2$

$\Rightarrow \quad a = 1$

since $\quad D$: $\quad -3a - b = -2$

$\Rightarrow \quad -3(1) - b = -2$

$-b = +1$

$b = -1$

also $\quad A: a + b + c = 2$

$\Rightarrow \quad 1 - 1 + c = 2$

$c = 2$

$\therefore$ solution $(a, b, c) = (1, -1, 2)$

9. Point 1, $\quad (1, 1)$
point 2, $\quad (0, -6)$
point 3, $\quad (-2, -8)$
Curve $= f(x) = ax^2 + bx + c$
$(1,1)$ on curve $\Rightarrow$ when $x = 1, f(x) = 1$

$$\Rightarrow 1 = a(1)^2 + b(1) + c$$
$$\Rightarrow 1 = a + b + c \qquad : A$$

$(0, -6)$ on curve $\Rightarrow$ when $x = 0, f(x) = -6$

$$\Rightarrow -6 = a(0) + b(0) + c$$
$$\Rightarrow -6 = c \qquad : B$$

$(-2, -8)$ on curve $\Rightarrow$ when $x = -2, f(x) = -8$

$$\Rightarrow -8 = a(-2)^2 + b(-2) + c$$

$$-8 = 4a - 2b + c \qquad :C$$

also $A:$ $\quad a + b - 6 = 1$

$C:$ $\quad 4a - \cancel{2b} - 6 = -8$

$\Rightarrow\ 2A:$ $\quad 2a + \cancel{2b} - 12 = 2$

adding: $\quad 6a - 18 = -6$

$$6a = 12$$

$$a = 2$$

since $A:$ $\quad a + b - 6 = 1$

$\Rightarrow \quad 2 + b - 6 = 1$

$$b = 1 + 4 = 5$$

$(a, b, c) = (2, 5, -6)$

solution: $f(x) = 2x^2 + 5x - 6$

10. Let x = number of people paying €20

Let y = number of people paying €30

$\therefore \quad x + y = 44\,000 : A$

also, $20x + 30y = 1\,200\,000 : B$

$\therefore\ 20A : \cancel{20x} + 20y = \quad 880\,000$

$B : \cancel{20x} + 30y = 1\,200\,000$

subtracting: $\quad -10y = -320\,000$

$$y = 32\,000$$

$\therefore$ 32 000 paid the higher price.

11. Let x be Lydia's age now.

$\therefore$ five years ago, Lydia was $(x - 5)$ years old.

Let Callum be y years old now.

$\therefore$ three years from now, Callum will be $(y + 3)$ years old

$\Rightarrow (y + 3) = 2(x - 5) \qquad : A$

also, $\dfrac{x + y}{2} = 16$

$\Rightarrow x + y = 32 \qquad : B$

From A: $\quad y + 3 = 2x - 10$

$$13 = 2x - y$$

$\therefore\ A:$ $\quad 2x - y = 13$

$B:$ $\quad x + y = 32$

adding: $\quad 3x = 45$

$$x = 15$$

since $x + y = 32$

$\Rightarrow 15 + y = 32$

$$y = 17$$

Lydia is 15 years old, Callum is 17 years old.

12. Equation of line: $y = ax + b$.

(6,7) on line $\Rightarrow$ when $x = 6, y = 7$

$\Rightarrow 7 = a(6) + b$

$\Rightarrow 6a + b = 7 \qquad : A$

also, $(-2, 3)$ on line $\Rightarrow$ when $x = -2, y = 3$

$\Rightarrow\ 3 = a(-2) + b$

$$\Rightarrow \quad -2a + b = 3 \qquad : B$$

$$\begin{array}{lll} \text{since} & A: & 6a + \cancel{b} = 7 \\ \text{and} & B: & \underline{-2a + \cancel{b} = 3} \\ \text{subtracting:} & & 8a \quad = 4 \\ \Rightarrow & & a \quad = \frac{4}{8} = \frac{1}{2}. \end{array}$$

since A: $6a + b = 7$

$$\Rightarrow 6\left(\frac{1}{2}\right) + b = 7$$

$$b = 4$$

$\therefore$ line $y = \frac{1}{2}x + 4$.

Verify (4, 6) is on line $\Rightarrow 6 = \frac{1}{2}(4) + 4$

$$= 6, \text{ which is true.}$$

13. $\frac{N_1}{4} - N_2 = 0 \quad \Rightarrow \quad N_1 - 4N_2 = 0 \qquad : A$

$N_1 + \frac{1}{2}N_2 - 99 = 0 \quad \Rightarrow \quad 2N_1 + N_2 = 198 \qquad : B$

$$\begin{array}{lll} \text{since} & A: & N_1 - 4N_2 = 0 \\ \text{and} & 4B: & \underline{8N_1 + 4N_2 = 792} \\ \text{adding:} & & 9N_1 \quad = 792 \\ & & N_1 \quad = 88 \end{array}$$

$$\begin{array}{ll} \text{also, } A: & N_1 - 4N_2 = 0 \\ \Rightarrow & 88 - 4N_2 = 0 \\ & -4N_2 = -88 \\ & N_2 = 22 \end{array}$$

$(N_1, N_2) = (88, 22)$

14. $\frac{a}{x-2} + \frac{b}{x+2} = \frac{4}{(x-2)(x+2)}$

$\Rightarrow \frac{a(x+2) + b(x-2)}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)}$

$\Rightarrow ax + 2a + bx - 2b = 4$

$$(a+b)x + 2a - 2b = 4 + 0.x$$

$$\begin{array}{lll} \Rightarrow & a + b = 0 & : A \\ \text{and} & 2a - 2b = 4 & : B \\ \text{also} & \underline{2a + 2b = 0} & : 2A \\ \text{adding:} & 4a \quad = 4 & \\ & a \quad = 1 & \\ \text{since} & a + b = 0 & \\ \Rightarrow & 1 + b = 0 & \\ & b = -1 & \end{array}$$

$\therefore (a, b) = (1, -1)$

$\therefore \frac{1}{x-2} + \frac{-1}{x+2} = \frac{4}{(x-2)(x+2)}$

$\frac{x + 2 - (x-2)}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)}$

$= \frac{\cancel{x} + 2 - \cancel{x} + 2}{(x-2)(x+2)}$

$\frac{4}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)}$ qed

15. $\dfrac{c}{z-3}+\dfrac{d}{z+2}=\dfrac{4}{(z-3)\,(z+2)}$

$\Rightarrow \dfrac{c(z+2)+d(z-3)}{(z-3)(z+2)}=\dfrac{4}{(z-3)(z+2)}$

$\Rightarrow cz+2c+dz-3d=4$

$(c+d)z+2c-3d=0.z+4$

$\therefore \quad c+d=0 \quad : A$

and $\quad 2c-3d=4 \quad : B$

$\Rightarrow \quad \underline{2c+2d=0} \quad : 2A$

subtracting: $\quad -5d=4$

$d=\dfrac{-4}{5}$

since $\quad c+d=0$

$\Rightarrow \quad c-\dfrac{4}{5}=0$

$\Rightarrow \quad c=\dfrac{4}{5}$

$\therefore (c,d)=\left(\dfrac{4}{5},-\dfrac{4}{5}\right)$

$\therefore \dfrac{4}{5(z-3)}+\dfrac{-4}{5(z+2)}=\dfrac{4}{(z-3)(z+2)}$

$\Rightarrow \dfrac{4(z+2)-4(z-3)}{5(z-3)(z+2)}$

$=\dfrac{\cancel{4z}+8-\cancel{4z}+12}{5(z-3)(z+2)}$

$=\dfrac{20}{5(z-3)(z+2)}$

$=\dfrac{4}{(z-3)(z+2)}=\dfrac{4}{(z-3)(z+2)}$ qed

16. Let x = number of litres of 70% alcohol

$\Rightarrow x(70\%)+50(40\%)=(x+50)(50\%)$

$\Rightarrow \quad 0.7x+0.4(50)=0.5x+0.5(50)$

$0.7x+20=0.5x+25$

$0.7x-0.5x=25-20$

$0.2x=5$

$x=25$ litres.

17. x = bigger number

y = smaller number

A: $\quad x+\ \ y=26$

B: $\quad 4x-5y=5$

$\Rightarrow 5A$: $\quad 5x+5y=130$

B: $\quad \underline{4x-5y=\ \ \ 5}$

adding: $\quad 9x \quad =135$

$x \quad =15$

since $\quad x+y=26$

$\Rightarrow \quad 15+y=26$

$y=11$

$\therefore (x, y) = (15, 11)$

18. $v = u + at$ $\quad v = \text{speed}$

$t = \text{time.}$

at $t = 7, v = 2 \quad \Rightarrow \quad 2 = u + 7a \quad :A$

at $t = 13, v = 5 \quad \Rightarrow \quad 5 = u + 13a \quad :B$

since $\quad A: u + 7a = 2$

$B: \underline{u + 13a = 5}$

subtracting: $\quad -6a = -3$

$6a = 3$

$a = \frac{3}{6} = \frac{1}{2}$

since $u + 7a = 2$

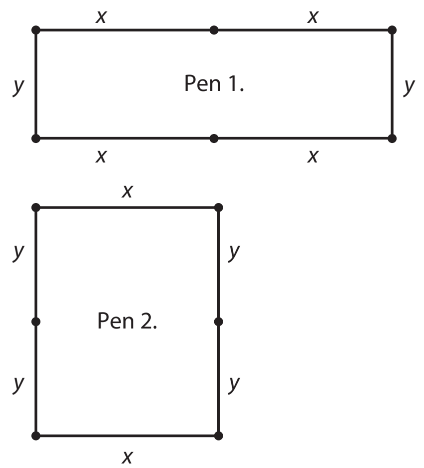
$\Rightarrow \quad u + 7\left(\frac{1}{2}\right) = 2$

$u = 2 - 3\frac{1}{2}$

$u = -1\frac{1}{2}$

19. $\quad \therefore \quad 4x + 2y = 60 \quad :A$

also, $2x + 4y = 42 \quad :B$



$\therefore \quad 4x + 2y = 60 \quad :A$

also $\quad \underline{4x + 8y = 84} \quad :2B$

subtracting: $\quad -6y = -24$

$y = 4$

since $4x + 2y = 60$

$\Rightarrow 4x + 2(4) = 60$

$4x = 52$

$x = 13$

$\therefore$ Original pen dimensions: 26×4

new pen dimensions: 13×8

Pen 1 requires $4x$ and $2y$ lengths

Pen 2 requires $2x$ and $4y$ lengths

$\Rightarrow$ they both have $2x$ and $2y$ length in common

$\therefore$ in pen 2, $2x$ lengths are swapped for $2y$ lengths

if $y < x$, less fencing is needed in pen 2.

Note: Area of pen 1 $= (2x) \times (y) = 2xy$

Area of pen 2 $= (x) \times (2y) = 2xy$

$\Rightarrow$ the areas are the same.

20. $y = ax^2 + bx + c$

the point $(0, 1) \in$ curve $\Rightarrow$ when $x = 0, y = 1$

$\Rightarrow 1 = a(0)^2 + b(0) + c$

$\Rightarrow 1 = c$

the point $(2, 9) \in$ curve $\Rightarrow$ when $x = 2, y = 9$

$\Rightarrow 9 = a(2)^2 + b(2) + c$

$\Rightarrow 9 = 4a + 2b + c$

$\Rightarrow 9 = 4a + 2b + 1 \quad$ since $c = 1$

$$\Rightarrow 8 = 4a + 2b$$
$$A: \quad \Rightarrow 4 = 2a + b \quad \text{dividing each term by 2.}$$

the point $(4, 41) \in$ curve $\Rightarrow$ when $x = 4, y = 41$

$$\Rightarrow 41 = a(4)^2 + b(4) + c$$
$$\Rightarrow 41 = 16a + 4b + c$$
$$\Rightarrow 41 = 16a + 4b + 1 \quad \text{since } c = 1$$
$$\Rightarrow 40 = 16a + 4b$$
$$B: \quad \Rightarrow 10 = 4a + b \quad \text{dividing each term by 4.}$$

since $\quad A: \quad 2a + \cancel{b} = 4$

and $\quad B: \quad \underline{4a + \cancel{b} = 10}$

subtracting $\quad -2a = -6$

$a = 3$

since $\quad A: \quad 2a + b = 4$

$\Rightarrow \underline{2(3) + b = 4}$

$b = -2$

$\therefore (a, b, c) = (3, -2, 1)$

21. (i) $A: \quad y - z = 3$

$B: \quad x - 2y + z = -4$

$C: \quad x + 2y = 11$

$\therefore \quad A: \quad y - \cancel{z} = 3$

$B: \quad \underline{x - 2y + \cancel{z} = -4}$

D(adding): $\quad \cancel{x} - y = -1$

$C: \quad \underline{\cancel{x} + 2y = 11}$

subtracting: $\quad -3y = -12$

$y = 4$

since $x - y = -1$

$\Rightarrow \quad x - 4 = -1$

$x = 3$

also $\quad y - z = 3$

$\Rightarrow \quad 4 - z = 3$

$-z = -1$

$z = 1$

solution $(x, y, z) = (3, 4, 1)$

(ii) $A: \quad \frac{x}{3} + \frac{y}{2} - z = 7 \Rightarrow 2x + 3y - 6z = 42$

$B: \quad \frac{x}{4} - \frac{3y}{2} + \frac{z}{2} = -6 \Rightarrow x - 6y + 2z = -24$

$C: \quad \frac{x}{6} - \frac{y}{4} - \frac{z}{3} = 1 \Rightarrow 2x - 3y - 4z = 12$

$\therefore \quad A: \quad \cancel{2x} + 3y - 6z = 42$

$2B: \quad \underline{\cancel{2x} - 12y + 4z = -48}$

D(subtracting): $\quad 15y - 10z = 90$

also $\quad 2B: \quad \cancel{2x} - 12y + 4z = -48$

$C: \quad \underline{\cancel{2x} - 3y - 4z = 12}$

E(subtracting): $\quad -9y + 8z = -60$

$\therefore \quad 4D: \quad 60y - \cancel{40z} = 360$

$5E: \quad \underline{-45y + \cancel{40z} = -300}$

$$\begin{aligned}
\text{adding:}\quad 15y &= 60\\
y &= 4
\end{aligned}$$

$$\begin{aligned}
\text{since}\quad D: 15 - 10z &= 90\\
15(4) - 10z &= 90\\
-10z &= 30\\
z &= -3
\end{aligned}$$

$$\begin{aligned}
\text{also } A:\quad 2x + 3y - 6z &= 42\\
2x + 3(4) - 6(-3) &= 42\\
2x + 12 + 18 &= 42\\
2x &= 12\\
x &= 6
\end{aligned}$$

$\therefore \quad (x, y, z) = (6, 4, -3)$

22. Curve: $x^2 + y^2 + ax + by + c = 0.$

$(1, 0) \in$ curve $\Rightarrow$ when $x = 1, y = 0$

$\Rightarrow 1^2 + 0^2 + a(1) + b(0) + c = 0$

$\Rightarrow 1 + a + c = 0$

$\Rightarrow a + c = -1 \qquad : A$

$(1, 2) \in$ curve $\Rightarrow$ when $x = 1, y = 2$

$\Rightarrow 1^2 + 2^2 + a(1) + b(2) + c = 0$

$\Rightarrow 1 + 4 + a + 2b + c = 0$

$\Rightarrow a + 2b + c = -5 \qquad : B$

$(2,1) \in$ curve $\Rightarrow$ when $x = 2, y = 1$

$\Rightarrow 2^2 + 1^2 + a(2) + b(1) + c = 0$

$\Rightarrow 4 + 1 + 2a + b + c = 0$

$\Rightarrow 2a + b + c = -5 \qquad : C$

$$\begin{aligned}
\text{since}\quad B:&\quad a + \cancel{2b} + c = -5\\
\text{and}\quad 2C:&\quad 4a + \cancel{2b} + 2c = -10\\
D(\text{subtracting}):&\quad -3a - \cancel{c} = 5\\
A:&\quad a + \cancel{c} = -1\\
\text{adding:}&\quad -2a = 4\\
&\quad a = -2
\end{aligned}$$

$$\begin{aligned}
\text{since } A:\quad a + c &= -1\\
\Rightarrow \quad -2 + c &= -1\\
c &= 1
\end{aligned}$$

$$\begin{aligned}
\text{also}\quad B:\quad a + 2b + c &= -5\\
-2 + 2b + 1 &= -5\\
2b &= -4\\
b &= -2
\end{aligned}$$

$\therefore (a, b, c) = (-2, -2, 1)$

23. Let x be the number of small bags, y be the number of medium bags and z be the number of large bags. Then

$x + y + z = 15 \qquad$ **... 1**

$3x + 5y + 7z = 77 \qquad$ **... 2**

$y = x + 2$

$-x + y = 2 \qquad$ **... 3**

Then

1: $\quad x + y + z = 15$

$$\begin{array}{llr} \mathbf{3}: & \underline{-x + y \quad\quad = 6} & \\ & 2y + z \quad = 17 & \mathbf{...\ 4} \end{array}$$

Also

$$\begin{array}{llr} \mathbf{2}: & 3x + 5y + 7z = 77 & \\ \mathbf{3} \times 3: & \underline{-3x + 3y \quad\quad = 6} & \\ & 8y + 7z = 83 & \mathbf{...\ 5} \end{array}$$

Then

$$\begin{array}{ll} \mathbf{4} \times -4: & -8y - 4z = -68 \\ \mathbf{5}: & \underline{8y + 7z = 83} \\ & 3z = 15 \\ & z = 5 \\ \mathbf{4}: & 2y + 5 = 17 \\ & 2y = 12 \\ & y = 6 \\ \mathbf{3}: & 6 = x + 2 \\ & x = 4 \end{array}$$

The shop sells 4 small, 6 medium and 5 large bags of popcorn.

24. Let €x be the amount in the mutual fund, €y be the amount in government bonds and €z be the amount in her local bank. Then

$$\begin{array}{lr} x + y + z = 10500 & \mathbf{...\ 1} \\ x = 2z & \\ x - 2z = 0 & \mathbf{...\ 2} \\ 0.11x + 0.07y + 0.05z = 825 & \\ 11x + 7y + 5z = 82500 & \mathbf{...\ 3} \end{array}$$

Then

$$\begin{array}{llr} \mathbf{1} \times 2: & 2x + 2y + 2z = 21\,000 & \\ \mathbf{2}: & \underline{x \quad\quad - 2z = 0} & \\ & 3x + 2y \quad = 21\,000 & \mathbf{...\ 4} \\ \mathbf{1} \times -5: & -5x - 5y - 5z = -52\,500 & \\ \mathbf{3}: & \underline{11x + 7y + 5z = 82\,500} & \\ & 6x + 2y \quad = 30\,000 & \mathbf{...\ 5} \end{array}$$

Then

$$\begin{array}{ll} \mathbf{4} \times -1: & -3x - 2y = -21\,000 \\ \mathbf{5}: & \underline{6x + 2y = 30\,000} \\ & 3x = 9000 \\ & x = 3000 \\ \mathbf{2}: & x = 2z \\ & 3000 = 2z \\ & z = 1500 \\ \mathbf{1}: & 3000 + y + 1500 = 10\,500 \\ & y = 6000 \end{array}$$

She invests €3000 in the mutual fund, €6000 in a government bond and €1500 in her local bank.

25. Let x be the number of white beads, y be the number of blue beads and z be the number of green beads. Then

$$y = (x + z) - 4$$
$$x - y + z = 4 \quad \ldots \mathbf{1}$$
$$z = x + y$$
$$x + y - z = 0 \quad \ldots \mathbf{2}$$
$$y = 2x$$
$$2x - y = 0 \quad \ldots \mathbf{3}$$

Then

1: $x - y + z = 4$

2: $\underline{x + y - z = 0}$

$2x = 4$

$x = 2$

3: $y = 2x$

$y = 4$

2: $z = x + y$

$z = 2 + 4$

$z = 6$

He uses 2 white, 4 blue and 6 green beads.

Revision Exercise 1 (Core)

1. (i) $\dfrac{12m^2n^3}{(6m^4n^5)^2} = \dfrac{\overset{1}{\cancel{12}}\, m^2n^3}{\underset{3}{\cancel{36}}\, m^8n^{10}} = \dfrac{1}{3m^6\, n^7}$

(ii) $\dfrac{3 + \frac{1}{x}}{\frac{5}{x} + 4} = \dfrac{\left(\frac{3x+1}{x}\right)}{\left(\frac{5+4x}{x}\right)} = \left(\dfrac{3x+1}{x}\right) \cdot \left(\dfrac{x}{5+4x}\right) = \dfrac{3x+1}{5+4x}$

(iii) $\dfrac{2 + \frac{x}{2}}{x^2 - 16} = \dfrac{\frac{4+x}{2}}{(x-4)(x+4)} = \dfrac{\cancel{(x+4)}}{2(x-4)\cancel{(x+4)}} = \dfrac{1}{2x-8}$

2. (i) $y = x + 4$
$5y + 2x = 6$ $\Rightarrow$

$$\begin{aligned} y - x &= 4 &&: A \\ 5y + \cancel{2x} &= 6 &&: B \\ \therefore \quad \underline{2y - \cancel{2x}} &\underline{= 8} &&: 2A \\ \text{adding} \quad 7y &= 14 \\ y &= 2 \end{aligned}$$

since $y = x + 4$

$\Rightarrow \quad 2 = x + 4$

$\Rightarrow \quad x = -2$

$\therefore (x, y) = (-2, 2)$

(ii) $3x + y = 7 \quad \Rightarrow \quad y = 7 - 3x$

$x^2 + y^2 = 13 \quad \Rightarrow \quad x^2 + (7 - 3x)^2 = 13$

$$\begin{aligned} \Rightarrow \quad x^2 + [49 - 42x + 9x^2] &= 13 \\ x^2 + 49 - 42x + 9x^2 &= 13 \\ 10x^2 - 42x + 36 &= 0 \\ 5x^2 - 21x + 18 &= 0 \\ (5x - 6)(x - 3) &= 0 \end{aligned}$$

$\therefore \quad x = 3 \quad$ or $\quad x = \dfrac{6}{5}$

$\Rightarrow \quad y = 7 - 3(3) \quad$ or $\quad y = 7 - 3\left(\dfrac{6}{5}\right)$

$y = 7 - 9 \quad$ or $\quad y = 7 - \dfrac{18}{5}$

$y = -2 \quad$ or $\quad y = \dfrac{17}{5}$

$\therefore (x, y) = (3, -2)$ or $\left(\frac{6}{5}, \frac{17}{5}\right)$

3.

$$\begin{array}{r|l} & x^2 + 2x - 1 \\ \hline x - 3 & x^3 - x^2 - 7x + 3 \\ & \underline{x^3 - 3x^2} \quad \text{(subtracting)} \\ & \quad 2x^2 - 7x + 3 \\ & \quad \underline{2x^2 - 6x} \quad \text{(subtracting)} \\ & \qquad -x + 3 \\ & \qquad \underline{-x + 3} \quad \text{(subtracting)} \\ & \qquad\quad 0 \end{array}$$

answer: $x^2 + 2x - 1$

4.

$$\begin{array}{r|l} & 3x^3 + 6x^2 + 3x + 33 \\ \hline x - 2 & 3x^4 \qquad - 9x^2 + 27x - 66 \\ & \underline{3x^4 - 6x^3} \quad \text{(subtracting)} \\ & \quad 6x^3 - 9x^2 + 27x - 66 \\ & \quad \underline{6x^3 - 12x^2} \quad \text{(subtracting)} \\ & \qquad 3x^2 + 27x - 66 \\ & \qquad \underline{3x^2 - 6x} \quad \text{(subtracting)} \\ & \qquad\quad 33x - 66 \\ & \qquad\quad \underline{33x - 66} \quad \text{(subtracting)} \\ & \qquad\qquad 0 \end{array}$$

answer: $3x^3 + 6x^2 + 3x + 33$

5. (i) $x^4 - 9x^2 = 0$

$\Rightarrow x^2 (x^2 - 9) = 0$

$\Rightarrow x^2 (x - 3)(x + 3) = 0$

$\therefore x = 0, 3, -3$

(ii) $(2x - 1)^3 (2 - x) = 0$

$\Rightarrow \quad (2x - 1)^3 = 0$

$\Rightarrow \quad (2x - 1) = 0$

$x = \frac{1}{2}$

or $\quad 2 - x = 0$

$x = 2$

$\therefore x = 2, \frac{1}{2}.$

6. $4x^2 + 20x + k$ is a perfect square

$\Rightarrow (2x + a)(2x + a) = 4x^2 + 20x + k$

$\Rightarrow \quad 4x^2 + 4ax + a^2 = 4x^2 + 20x + k$

$\Rightarrow \quad 4a = 20$

$a = 5$

$\therefore \quad a^2 = 5^2 = 25 = k$

$\therefore \quad k = 25(5^2)$

7. (i) $(2x + 3)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(3) + \binom{5}{2}(2x)^3(3)^2 + \binom{5}{3}(2x)^2(3)^3 + \binom{5}{4}(2x)(3)^4 + \binom{5}{5}(3)^5$

$= (1)(32x^5) + (5)(16x^4)(3) + (10)(8x^3)(9) + (10)(4x^2)(27) + (5)(2x)(81) + (1)(243)$

$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

(ii) $(x - 2)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + \binom{6}{3}x^3(-2)^3 + \binom{6}{4}x^2(-2)^4 + \binom{6}{5}x(-2)^5 + \binom{6}{6}(-2)^6$

$= (1)x^6 + (6)x^5(-2) + (15)x^4(4) + (20)x^3(-8) + (15)x^2(16) + (6)(x(-32) + (1)(64)$

$= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$

8. $x^3 - 27 = x^3 - 3^3 = (x - 3)(x^2 + 3x + 9)$

9. $p(x-q)^2+r=2x^2-12x+5$ for all values of x

$\Rightarrow p(x^2-2xq+q^2)+r=$

$\Rightarrow px^2-2pqx+pq^2+r=2x^2-12x+5.$

$\therefore p=2,\ -2pq=-12$ and $pq^2+r=5.$

$\Rightarrow \quad -2(2)q=-12$

$\Rightarrow \quad q=3$ and $2(3)^2+r=5$

$r=5-18$

$r=-13$

$(p,q,r)=(2,3,-13)$

10. $A: \quad 3x+5y-z=-3$

$B: \quad 2x+y-3z=-9$

$C: \quad x+3y+2z=7$

$$\begin{array}{rl} \Rightarrow 2A: & 6x+10y-\cancel{2z}=-6 \\ C: & \underline{x+\ \ 3y+\cancel{2z}=7} \\ D(\text{adding}): & 7x+13y \qquad =1 \end{array}$$

$$\begin{array}{lrl} \text{also} & 3A: & 9x+15y-\cancel{3z}=-9 \\ & B: & \underline{2x+\quad y-\cancel{3z}=-9} \\ E(\text{subtracting}): & & 7x+14y \qquad =0 \end{array}$$

$$\begin{array}{lrl} \therefore & D: & 7x+13y=1 \\ & E: & \underline{7x+14y=0} \\ (\text{subtracting}): & & \qquad -y=1 \\ & & \qquad\ \ y=-1 \end{array}$$

since $\quad 7x+14y=0$

$7x+14(-1)=0$

$7x=14$

$x=2.$

also, $C: \quad x+3y+2z=7$

$2+3(-1)+2z=7$

$2z=8$

$z=4.$

$\therefore \quad (x,y,z)=(2,-1,4)$

11. $(b+1)^3-(b-1)^3$

$=b^3+3b^2+3b+1-(b^3-3b^2+3b-1)$

$=\cancel{b^3}+3b^2+\cancel{3b}+1-\cancel{b^3}+3b^2-\cancel{3b}+1$

$=6b^2+2.$

12. (i) 3, 12, 27, 48, 75, …

first difference: 9, 15, 21, 27, …

second difference: 6, 6, 6… $\Rightarrow$ quadratic pattern of the form an^2+bn+c

$\Rightarrow \quad 2a=6$

$a=3$

$\therefore \quad 3n^2+bn+c$ represents the pattern

let $\quad n=1 \Rightarrow 3(1)^2+b(1)+c=3$

$b+c=0 \quad : \quad A$

let $\quad n=2 \Rightarrow 3(2)^2+b(2)+c=12$

$$
\begin{aligned}
& & 2b + c &= 0 \quad : \quad B \\
\Rightarrow \quad & A: & b + c &= 0 \\
& B: & \underline{2b + c} &\underline{= 0} \\
\text{(subtracting)}: & & -b \quad &= 0 \\
& \Rightarrow & b \quad &= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{since} \quad & b + c = 0 \\
& \Rightarrow 0 + c = 0 \\
& \Rightarrow \quad c = 0
\end{aligned}
$$

$\therefore \quad an^2 + bn + c = 3n^2$

(ii) 5, 20, 45, 80, 125, …

first difference: 15, 25, 35, 45, …

second difference: 10, 10, 10, …

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 10$

$a = 5 \qquad \therefore \quad 5n^2 + bn + c$ represents the pattern

We note that for $n = 1$: $5n^2 = 5$

$n = 2$: $5n^2 = 20$

$n = 3$: $5n^2 = 45$, etc.

$\Rightarrow b$ and c must equal zero

[alternatively, set up simultaneous equations in b and c and solve]

$\therefore$ the quadratic pattern is $5n^2$

(iii) 0.5, 2, 4.5, 8, 12.5, …

first difference = 1.5, 2.5, 3.5, 4.5, …

second difference = 1, 1, 1

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 1$

$a = \frac{1}{2} \qquad \therefore 0.5n^2 + bn + c$ represents this pattern

By comparison to part (ii), we can deduce that the quadratic pattern is $0.5n^2$.

13. 6, 12, 20, 30, 42, …

first difference: 6, 8, 10, 12, …

second difference: 2, 2, 2

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 2$

$a = 1 \quad \Rightarrow \quad n^2 + bn + c$ represents this pattern

$$
\begin{aligned}
\text{let} \quad n = 1 \quad \Rightarrow \quad & 1^2 + b(1) + c = 6 \\
& b + c = 5 \qquad : A \\
\text{let} \quad n = 2 \quad \Rightarrow \quad & 2^2 + b(2) + c = 12 \\
& 2b + c = 8 \qquad : B
\end{aligned}
$$

$$
\begin{aligned}
\text{since} \quad & A: & b + c &= 5 \\
\text{and} \quad & B: & \underline{2b + c} &\underline{= 8} \\
\text{subtracting}: & & -b \quad &= -3 \\
& & b \quad &= 3
\end{aligned}
$$

also $\quad A: \quad b + c = 5$

$$3 + c = 5$$
$$c = 2$$

$\therefore$ quadratic pattern is $n^2 + 3n + 2$

When $n = 100 \Rightarrow 100^2 + 3(100) + 2 = 10,302$

14. Let the width $= x$ cm.

Let the length $= y$ cm.

$\Rightarrow$ A: $3x = 2y + 3$

B: $4y = 2(x + y) + 12$

$\Rightarrow$ B: $4y = 2x + 2y + 12$

$\Rightarrow$ B: $2y = 2x + 12.$

since A: $3x - \cancel{2y} = 3$

and B: $2x - \cancel{2y} = -12$

(subtracting): $x = 15 \Rightarrow$ width $= 15$ cm.

also B: $2x - 2y = -12$

$$2(15) - 2y = -12$$
$$-2y = -42$$
$$y = 21 \Rightarrow \text{length} = 21 \text{ cm.}$$

15. A: $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{2}{r}$

B: $m = \dfrac{v - r}{r - u}$

since A: $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{2}{r}$

$$\Rightarrow \frac{v + u}{uv} = \frac{2}{r}$$

$$\Rightarrow \frac{2}{r} = \frac{v + u}{uv}$$

$$\Rightarrow \frac{r}{2} = \frac{u.v}{v + u}$$

$$\Rightarrow r = \frac{2uv}{v + u}$$

also, $m = \dfrac{v - r}{r - u} = \dfrac{v - \dfrac{2uv}{v + u}}{\dfrac{2uv}{v + u} - u}$

$$= \frac{\left(\dfrac{v(v + u) - 2uv}{v + u}\right)}{\left(\dfrac{2uv - u(v + u)}{v + u}\right)}$$

$$= \frac{v^2 + vu - 2uv}{\cancel{(v + u)}} \cdot \frac{\cancel{(v + u)}}{2uv - uv - u^2}$$

$$= \frac{v^2 - uv}{uv - u^2} = \frac{v\cancel{(v - u)}}{u\cancel{(v - u)}} = \frac{v}{u}$$

Revision Exercise 1 (Advanced)

1. 1, 3, 6, 10, …
first difference: 2, 3, 4, …
second difference: 1, 1, …
$\Rightarrow$ quadratic pattern of the form $an^2 + bn + c$
$\Rightarrow 2a = 1$
$a = \frac{1}{2} \Rightarrow \frac{1}{2}n^2 + bn + c$ represents this pattern

let $n = 1$: $\frac{1}{2}(1)^2 + b(1) + c = 1$
$\frac{1}{2} + b + c = 1$
$b + c = \frac{1}{2} \quad : A$

let $n = 2$: $\frac{1}{2}(2)^2 + b(2) + c = 3$
$2 + 2b + c = 3$
$2b + c = 1 \quad : B$

since A: $b + \cancel{c} = \frac{1}{2}$
and B: $\underline{2b + \cancel{c} = 1}$
(subtracting): $-b = -\frac{1}{2}$
$b = \frac{1}{2}$

also A: $b + c = \frac{1}{2}$
$\Rightarrow \quad \frac{1}{2} + c = \frac{1}{2}$
$\Rightarrow \quad c = 0$

$\therefore$ the quadratic pattern is $\frac{1}{2}n^2 + \frac{1}{2}n$

2. If x m^3 of soil is needed,
$\Rightarrow x(55\%) + 1(25\%) = (x + 1)(35\%)$
$\Rightarrow \quad 0.55x + 0.25 = 0.35x + 0.35$
$\Rightarrow \quad 0.55x - 0.35x = 0.35 - 0.25$
$0.2x = 0.1$
$x = 0.5$ m^3.

3. (i) Let x kg of alloy 1 be added to
y kg of alloy 2.
$\Rightarrow x + y = 8.4 \quad : A$
alloy 1 contains 60% gold
$\Rightarrow$ the amount of gold in the new alloy $= x(60\%) = 0.6x$
alloy 2 contains 40% gold
$\Rightarrow$ the amount of gold in the new alloy $= y(40\%) = 0.4y$
also, the total amount of gold in the new alloy $= (x + y)(50\%)$
$= 0.5(x + y)$

$\Rightarrow 0.6x + 0.4y = 0.5(x + y) \quad : B$
since A: $x + y = 8.4$
and B: $6x + 4y = 5(x + y)$
$\Rightarrow$ B: $x - \cancel{y} = 0$ (simplifying B)

and $A: \quad x + y = 8.4$
$\Rightarrow \quad 2x = 8.4$
$x = 4.2$ kg
since $B: \quad x - y = 0$
$\Rightarrow \quad 4.2 - y = 0$
$\Rightarrow \quad y = 4.2$ kg also.

4. $(3p - 2t)x + r - 4t^2 = 0$ for all x.
$\Rightarrow (3p - 2t)x + r - 4t^2 = 0.x + 0$
$\Rightarrow \quad 3p - 2t = 0 \qquad \Rightarrow t = \dfrac{3p}{2}$
and $\quad r - 4t^2 = 0.$
$\Rightarrow r - 4\left(\dfrac{3p}{2}\right)^2 = 0$
$\Rightarrow \quad r - \dfrac{\cancel{4}.9p^2}{\cancel{4}} = 0$
$\Rightarrow \quad r = 9p^2.$

5. $\dfrac{x + y^2}{x^2} + \dfrac{x - 1}{x} = -1$
$\Rightarrow x + y^2 + x(x - 1) = -x^2$ [multiplying each term by x^2]
$\Rightarrow x + y^2 + x^2 - x = -x^2$
$\Rightarrow \cancel{x} + y^2 + x^2 - \cancel{x} + x^2 = 0$
$2x^2 + y^2 = 0.$
also, $\dfrac{2x^2}{y^2} + 1 = 0.$
$\dfrac{2x^2}{y^2} = -1$
$\dfrac{x^2}{y^2} = -\dfrac{1}{2}$

6. Let the students take x litres of 10% solution and y litres of 30% solution.
$\Rightarrow \quad x + y = 10$ litres
also, $x(10\%) + y(30\%) = (x + y)15\%$
$\Rightarrow \quad 10x + 30y = 15(x + y)$ [multiplying each term by 100]
$\therefore \quad x + y = 10 \quad : A$
and $10x + 30y = 15x + 15y$
$\Rightarrow \quad -5x + 15y = 0 \quad : B$
since $A: \quad x + y = 10$
and $B: -\cancel{5x} + 15y = 0$
$\Rightarrow \quad 5A: \quad \cancel{5x} + 5y = 50$
adding : $\quad 20y = 50$
$y = \dfrac{50}{20} = 2\dfrac{1}{2}$ litres.
also, since $\quad x + y = 10$
$\Rightarrow \quad x + 2\dfrac{1}{2} = 10$
$x = 7\dfrac{1}{2}$ litres.

(i) $7\dfrac{1}{2}$ litres of 10% mixed with (ii) $2\dfrac{1}{2}$ litres of 30%.

7. $3x - y = 0 \qquad$ **... (1)**

$x + y + z = 23\,000$ ... **(2)**

$75x + 55y + 30z = 870\,000$

$15x + 11y + 6z = 174\,000$ ... **(3)**

(2) $\times$ 6: $6x + 6y + 6z = 138\,000$

$$
\begin{aligned}
15x + 11y + 6z &= 174\,000\\
6x + 6y + 6z &= 138\,000\\
\hline
9x + 5y &= 36\,000\\
\text{(1)} \times 3: \quad 9x - 3y &= 0\\
\hline
8y &= 36\,000\\
y &= 4500\\
\therefore \quad 3x &= 4500\\
x &= 1500
\end{aligned}
$$

$\therefore \quad x + y + z = 1500 + 4500 + z = 23\,000$

$\therefore \quad z = 17\,000$

$\therefore \quad x = 1500, y = 4500, z = 17\,000$

8. Brian's time for 50 m $= 50a$ seconds.

Luke's time for 50 m $= 50b$ seconds.

Luke faster than Brian $\Rightarrow 50a = 50b + 1$

Brian's time for 47 m $= 47a$ seconds.

Luke's time for 50 m $= 50b$ seconds.

Luke faster than Brian $\Rightarrow 47a = 50b + 0.1$

Subtracting : $3a = 0.9$

$a = 0.3$

$\Rightarrow 50(0.3) = 50b + 1$

$14 = 50b$

$0.28 = b$

Luke's speed $= \dfrac{1}{b} = \dfrac{1}{0.28} = 3.57$ m/sec.

9. (i) $\left(2 - \dfrac{x}{3}\right)^9$

General term: $T_{r+1} = \binom{9}{r}(2)^{9-r}\left(\dfrac{-x}{3}\right)^r$

$= \binom{9}{r}2^{9-r}\dfrac{(-1)^r x^r}{3^r}$

For the 6th term, $r = 5$. Thus the 6th term is

$\binom{9}{5}2^{9-5}\dfrac{(-1)^5x^5}{3^5}$

$= (126)(16)\dfrac{(-1)x^5}{243}$

$= -\dfrac{224}{27}x^5$

(ii) $(3x + 1)^{12}$

General term: $T_{r+1} = \binom{12}{r}(3x)^{12-r}(1)^r$

$= \binom{12}{r}3^{12-r}x^{12-r}$

For the 8th term, $r = 7$. Thus the 8th term is

$\binom{12}{7}3^5x^5$

$= (792)(243)x^5$

$= 192\,456x^5$

10. $(3x - 1)^{11}$

General term: $T_{r+1} = \binom{11}{r}(3x)^{11-r}(-1)^r$

$= (-1)^r\binom{11}{r}3x^{11-r}x^{11-r}$

For the 6th term, $r = 5$. Thus the 6th term is

$(-1)^5\binom{11}{5}3^6x^6$

$= (-1)(462)(729)x^6$

$= -336\,798x^6$

11. $(2x - 3)^{14}$

General term: $T_{r+1} = \binom{14}{r}(2x)^{14-r}(-3)^r$

$= \binom{14}{r}2^{14-r}x^{14-r}(-1)^r3^r$

For the term containing x^{10},

$14 - r = 10$

$r = 4$

The coefficient of x^{10} is then

$\binom{14}{4}2^{10}(-1)^4 3^4$

$= (1001)(1024)(1)(81)$

$= 83\,026\,944$

Revision Exercise 1 (Extended-Response Questions)

1. (a) Let x be the number of adults

y be the number of children.

$\therefore \quad x + y = 548 \quad : A$

also, $5x + 2.5y = 2460 : B$

since $A: \quad x + y = 548$

and $B: \quad \cancel{5x} + 2.5y = 2460$

$\Rightarrow \quad 5A: \quad \cancel{5x} + 5y = 2740$

(subtracting): $-2.5y = -280$

$2.5y = 280$

$y = \frac{280}{2.5} = 112.$

since $A: \quad x + y = 548$

$\Rightarrow \quad x + 112 = 548$

$\Rightarrow \quad x = 548 - 112$

$x = 436$

$\Rightarrow$ (i) Number of adult tickets $(x) = 436$

(ii) Number of children tickets $(y) = 112$.

(iii) Proportion of adult tickets sold $= \frac{436}{548} = 0.7956$.

(b) attendance (predicted) $= 13\,000$

$\Rightarrow$ adults $= 0.7956 \times 13\,000 = 10\,343$

$\Rightarrow$ children $= 13\,000 - 10\,343 = 2657$

Revenue $= 10\,343 \times (€5) + 2657 \times (€2.5) = €58\,357.50$

2. (i) x standard sofas

y deluxe sofas

Standard sofas require 2 hours of work $\Rightarrow 2x =$ time for x sofas

Deluxe sofas require 2.5 hours of work $\Rightarrow 2.5y =$ time for y sofas

$\Rightarrow$ with 48 hours of manufacturing time: $2x + 2.5y = 48 : A$

(ii) also, standard sofas need 1 hour finishing $\Rightarrow x =$ time for x sofas

deluxe sofas need 1.5 hours finishing $\Rightarrow 1.5y =$ time for y sofas

$\Rightarrow$ with 26 hours of finishing time: $x + 1.5y = 26 : B$

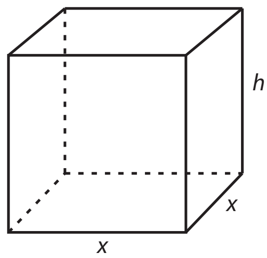
$$\begin{aligned}
\text{since } A:&\quad 2x + 2.5y = 48\\
\text{and } B:&\quad x + 1.5y = 26\\
\Rightarrow 2B:&\quad \cancel{2x} + 3y = 52\\
\text{and } A:&\quad \underline{\cancel{2x} + 2.5y = 48}\\
\text{(subtracting)}:&\quad 0.5y = 4\\
&\quad y = 8
\end{aligned}$$

also, since B: $x + 1.5y = 26$

$$\begin{aligned}
x + 1.5(8) &= 26\\
x + 12 &= 26\\
x &= 14
\end{aligned}$$

(iii) 14 standard sofas and 8 deluxe sofas.

3. Rectangular box with square base of length x cm.



Height of box $= h$ cm.

Volume of box $= l \times b \times h$

$= x \times x \times h = x^2h.$

(i) If $V = 40 \Rightarrow \quad 40 = x^2h$

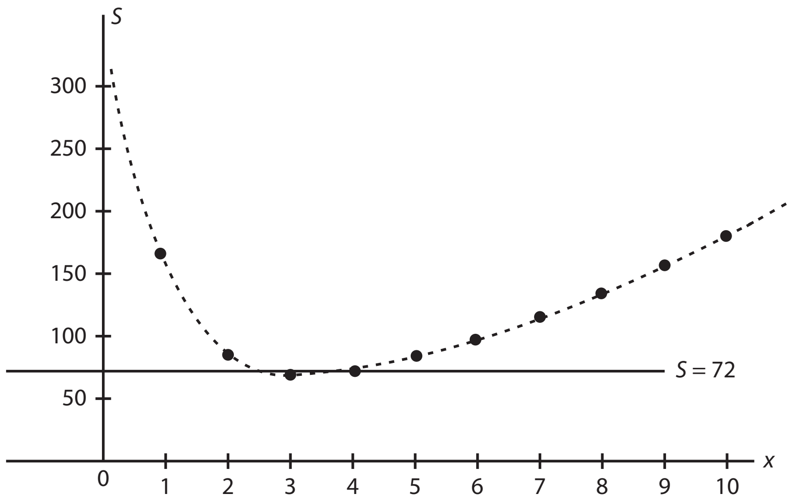
$$\Rightarrow \quad h = \frac{40}{x^2}.$$

(ii) Surface Area $= 4 \times (x \times h) + 2x^2$

$$= 4\cancel{x}\,.\,\frac{40}{x^{\cancel{2}}} + 2x^2$$

$$S = \frac{160}{x} + 2x^2.$$

(iii)



(iv) $72 = \dfrac{160}{x} + 2x^2$

$72x = 160 + 2x^3$

$\Rightarrow 2x^3 - 72x + 160 = 0.$

Using trail + error at $x = 2$: $2(2)^3 - 72(2) + 160 = 32 > 0$

$$\text{at } x = 2.8 \quad : \quad 2(2.8)^3 - 72(2.8) + 160 = 2.3 > 0$$
$$\text{at } x = 2.9 \quad : \quad 2(2.9)^3 - 72(2.9) + 160 = -0.02 < 0$$

$\therefore$ at $x = 2.9$ cm (approximately), $S = 72\text{ cm}^2$

also, at $x = 4$: $\quad 2(4)^3 - 72(4) + 160 = 0$

$\therefore$ at $x = 4$ cm, $\quad S = 72\text{ cm}^2$

when $x = 4$, $\quad h = \dfrac{40}{(4)^2} = 2.5$ cm.

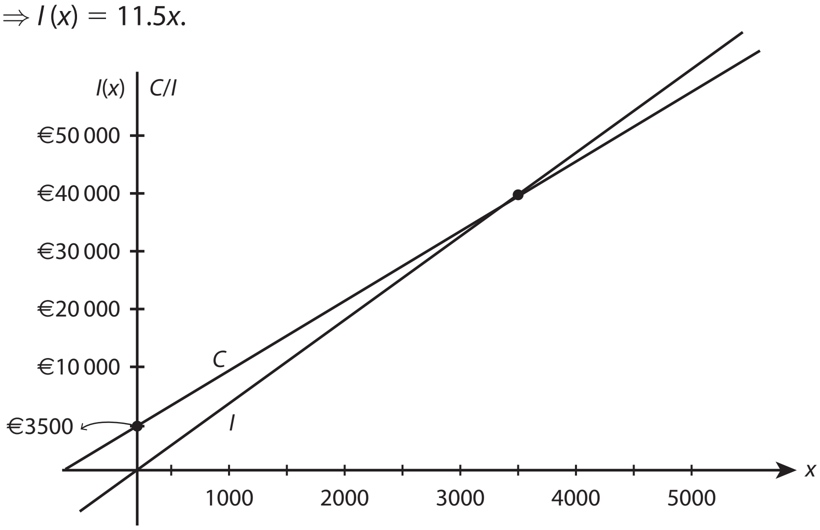
when $x = 2.9$, $h = \dfrac{40}{(2.9)^2} = 4.76$ cm.

4. Selling price of game $=$ €11.50.
Production cost for each game $=$ €10.50.
Initial production costs $=$ €3500.

(i) Cost of producing x games $= C(x)$
$\Rightarrow C(x) = 10.5x + 3500$

(ii) Income $= I(x)$
$\Rightarrow I(x) = 11.5x.$

(iii)



(iv) To recoup costs: $\quad 11.5x = 10.5x + 3500$
$\Rightarrow \quad x = 3500$ need to be sold.

(v) $P = I - C \quad \Rightarrow \quad P = \text{profit}.$

(vi) To make a profit of a 2000:
$$\Rightarrow \quad P = I - C$$
$$\therefore \quad 2000 = 11.5x - [10.5x + 3500]$$
$$2000 = 11.5x - 10.5x - 3500$$
$$5500 = x$$
$\therefore$ 5500 need to be sold.

5. 15 days to complete quilt.
x blue squares at a rate of 4 squares a day.
y white squares at a rate of 7 squares a day.
96 squares in quilt $\Rightarrow x + y = 96 \quad : A$

15 days to finish $\quad \Rightarrow \quad \dfrac{x}{4} + \dfrac{y}{7} = 15 : B$

$$\begin{array}{lrl}
\therefore & A: & x + y = 96 \\
 & 28B: & 7x + \cancel{4y} = 420 \\
\text{also} & 4A: & 4x + \cancel{4y} = 384 \\
\text{(subtracting):} & & 3x \quad = 36 \\
 & & x = 12
\end{array}$$

since A: $\quad x + y = 96$

$\Rightarrow \quad 12 + y = 96$

$y = 84$

(a) Cost $= x(0.8) + y(1.20)$

$= 12(0.8) + 84(1.20)$

$= €110.40$

(b) $l : w = 3x : 2x$

$\Rightarrow 3x.2x = 96$

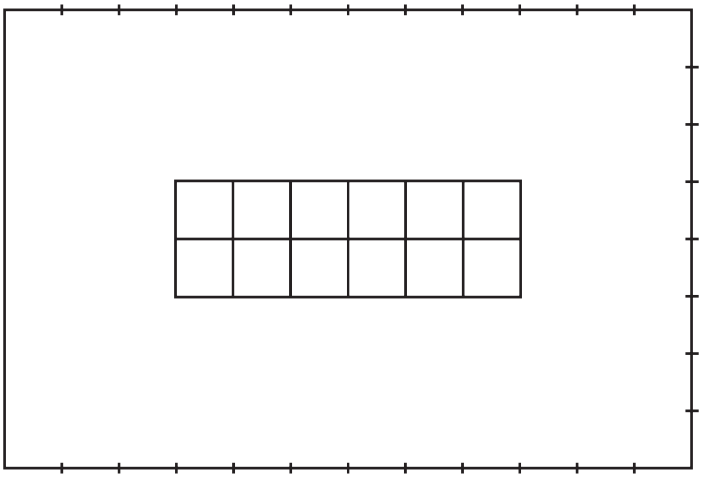
$6x^2 = 96$

$x^2 = 16$

$x = 4$

$\therefore \quad l = 3 \times 4 = 12$

$w = 2 \times 4 = 8$



The 12 blue squares could form 2 rows of 6 in the centre. (There are many different possibilities.)

6. Overheads $= €30\,000$ per year

Cost of manufacture $= €40$ per wheelbarrow

(i) $C(x) = 40x + 30\,000$

(ii) 6000 wheelbarrows per year $\Rightarrow \dfrac{30\,000}{6000} = €5$ overhead per wheelbarrow

$\Rightarrow$ Total cost per wheelbarrow $= €40 + €5 = €45$

(iii) To get a cost of €46 per wheelbarrow:

$$\frac{30\,000}{x} + 40 = 46.$$

$\Rightarrow \quad 30\,000 + 40x = 46x$

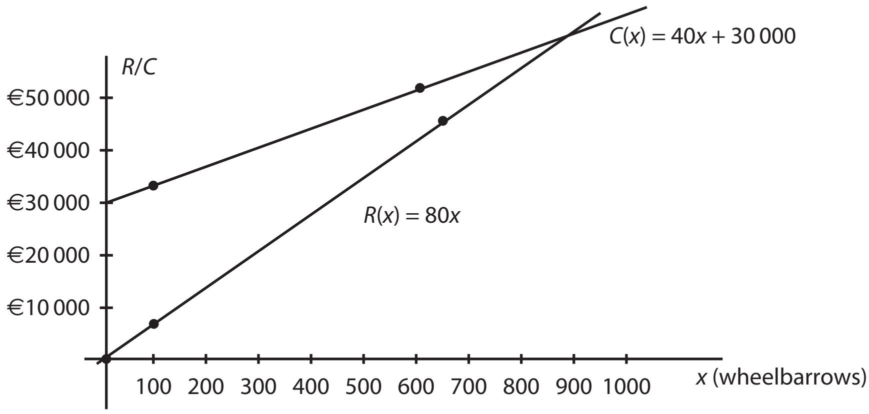
$\Rightarrow \quad 30\,000 = 6x$

$x = 5000$ wheelbarrows

(iv) Selling price $= €80$ per wheelbarrow

$\Rightarrow$ Revenue $= €80x$

(v)



(vi) To make a profit $80x > 40x + 30\,000$

$$\Rightarrow 40x > 30\,000$$

$$x > \frac{30\,00\cancel{0}}{4\cancel{0}} = 750$$

The minimum number of wheelbarrows = 751.

(vii) Profit €$P = 80x - [40x + 30\,000]$

€$P = 40x - 30\,000$

Chapter 2

Exercise 2.1

1. (a) (i) $x = 4, x = -5$

(ii) $(x - 3)(x - 4) = 0$

$\Rightarrow x = 3, x = 4$

(iii) $(x + 1)(x - 5) = 0$

$\Rightarrow x = -1, x = 5$

(b) (i) $(x + 3)(x - 5) = 0$

$\Rightarrow x = -3, x = 5$

(ii) $(2x - 3)(x + 5) = 0$

$\Rightarrow 2x = 3, x = -5$

$\Rightarrow x = \frac{3}{2}, x = -5$

(iii) $(3x + 2)(x - 5) = 0$

$\Rightarrow 3x = -2, x = 5$

$\Rightarrow x = \frac{-2}{3}, x = 5$

(c) (i) $(5x + 2)(x - 3) = 0$

$\Rightarrow 5x = -2, x = 3$

$\Rightarrow x = \frac{-2}{5}, x = 3$

(ii) $(3x + 5)(3x - 4) = 0$

$\Rightarrow 3x = -5, 3x = 4$

$\Rightarrow x = \frac{-5}{3}, x = \frac{4}{3}$

(iii) $(4x + 5)(2x - 3) = 0$

$\Rightarrow 4x = -5, 2x = 3$

$\Rightarrow x = \frac{-5}{4}, x = \frac{3}{2}$

(d) (i) $(x + 3)(x - 3) = 0$

$\Rightarrow x = -3, x = 3$

(ii) $x(3x - 10) = 0$

$\Rightarrow x = 0, 3x = 10$

$\Rightarrow x = 0, x = \frac{10}{3}$

(iii) $x(5x - 8) = 0$

$\Rightarrow x = 0, 5x = 8$

$\Rightarrow x = 0, x = \frac{8}{5}$

(e) (i) $(5 + x)(3 - 2x) = 0$

$\Rightarrow x = -5, 2x = 3$

$\Rightarrow x = -5, x = \frac{3}{2}$

(ii) $(5 + 3x)(2 - x) = 0$

$\Rightarrow 3x = -5, x = 2$

$\Rightarrow x = \frac{-5}{3}, x = 2$

(iii) $(2 + x)(1 - x) = 0$

$\Rightarrow x = -2, x = 1$

(f) (i) $(x + 5)(x + 4)(x - 4) = 0$

$\Rightarrow x = -5, x = -4, x = 4$

(ii) $(x - 3)(x + 1)(x - 1) = 0$

$\Rightarrow x = 3, x = -1, x = 1$

(g) (i) $(x + 2)(x - 2)(3x + 4) = 0$

$\Rightarrow x = -2, x = 2, 3x = -4$

$\Rightarrow x = -2, x = 2, x = \frac{-4}{3}$

(ii) $(x + 4)(x + 3)(x - 5) = 0$

$\Rightarrow x = -4, x = -3, x = 5$

2. (a) (i) $x = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)}$

$= \frac{2 \pm \sqrt{12}}{2}$

$= \frac{2 \pm \sqrt{4}\sqrt{3}}{2}$

$= 1 \pm \sqrt{3}$

$= 2.7, 20.7$

(ii) $x = \frac{-3 \pm \sqrt{9 - 4(+1)(-2)}}{2(1)}$

$= \frac{-3 \pm \sqrt{17}}{2}$

$= 0.56, -3.56$

$= 0.6, -3.6$

(iii) $x = \frac{6 \pm \sqrt{36 - 4(2)(3)}}{2(2)}$

$= \frac{6 \pm \sqrt{12}}{(4)}$

$= 2.36, 0.635$

$= 2.4, 0.6$

(b) (i) $x = \dfrac{6 \pm \sqrt{36 - 4(1)(3)}}{2(1)}$

$= \dfrac{6 \pm \sqrt{24}}{2}$

$= 5.44, 0.55$

$= 5.4, 0.6$

(ii) $x = \dfrac{8 \pm \sqrt{64 - 4(3)(1)}}{2(3)}$

$= \dfrac{8 \pm \sqrt{52}}{6}$

$= 2.535, 0.13$

$= 2.5, 0.1$

(iii) $x = \dfrac{-4 \pm \sqrt{16 - 4(2)(-5)}}{2(2)}$

$= \dfrac{-4 \pm \sqrt{56}}{4}$

$= 0.87, -2.87$

$= 0.9, -2.9$

3. (a) (i) $x = \dfrac{-4 \pm \sqrt{16 - 4(3)(-5)}}{2(3)}$

$= \dfrac{-4 \pm \sqrt{76}}{6}$

$= \dfrac{-2 \pm \sqrt{19}}{3}$

(ii) $x = \dfrac{12 \pm \sqrt{144 - 4(2)(-5)}}{2(2)}$

$= \dfrac{12 \pm \sqrt{184}}{4}$

$= \dfrac{6 \pm \sqrt{46}}{2}$

(iii) $4x^2 - 12x + 9 = 8$

$\Rightarrow 4x^2 - 12x + 1 = 0$

$\Rightarrow x = \dfrac{12 \pm \sqrt{144 - 4(4)(1)}}{2(4)}$

$= \dfrac{12 \pm \sqrt{128}}{8}$

$= \dfrac{3 \pm 2\sqrt{2}}{2}$

(b) (i) $x = \dfrac{-4 \pm \sqrt{16 - 4(1)(-8)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{48}}{2}$

$= \dfrac{-4 \pm 4\sqrt{3}}{2}$

$= -2 \pm 2\sqrt{3}$

(ii) $x = \dfrac{-4 \pm \sqrt{16 - 4(5)(-2)}}{2(5)}$

$= \dfrac{-4 \pm \sqrt{56}}{10}$

$= \dfrac{-2 \pm \sqrt{14}}{5}$

(iii) $x = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

4. (a) (i) $x(x + 7) + 2(3) = 4(3)(x)$

$\Rightarrow x^2 + 7x + 6 = 12x$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x = 2, x = 3$

(ii) $1(x) + 4(x - 1) = 3(x)(x - 1)$

$\Rightarrow x + 4x - 4 = 3x^2 - 3x$

$\Rightarrow 3x^2 - 8x + 4 = 0$

$\Rightarrow (3x - 2)(x - 2) = 0$

$\Rightarrow 3x = 2, x = 2$

$\Rightarrow x = \dfrac{2}{3}, x = 2$

(iii) $3(x + 1) - 2(x - 1) = 1(x - 1)(x + 1)$

$\Rightarrow 3x + 3 - 2x + 2 = x^2 - 1$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow x = -2, x = 3$

(b) (i) $1(x - 2) + 2(x) = 3(x)(x - 2)$

$\Rightarrow x - 2 + 2x = 3x^2 - 6x$

$\Rightarrow 3x^2 - 9x + 2 = 0$

$\Rightarrow x = \dfrac{9 \pm \sqrt{81 - 4(3)(2)}}{2(3)} = \dfrac{9 \pm \sqrt{57}}{6}$

(ii) $(x + 2)(x - 2) = (2x + 1)(x - 4)$

$\Rightarrow x^2 - 4 = 2x^2 - 7x - 4$

$\Rightarrow x^2 - 7x = 0$

$\Rightarrow x(x - 7) = 0$

$\Rightarrow x = 0, x = 7$

(iii) $2(x)(x - 4) + 3(x - 2)(x - 4)$

$= 5(x)(x - 2)$

$\Rightarrow 2x^2 - 8x + 3x^2 - 12x - 6x + 24$

$= 5x^2 - 10x$

$\Rightarrow 5x^2 - 26x + 24 = 5x^2 - 10x$

$\Rightarrow -16x = -24$

$\Rightarrow x = \dfrac{-24}{-16} = \dfrac{3}{2}$

5. (a) (i) $y = x^2 \Rightarrow y^2 - 7y + 10 = 0$

$\Rightarrow (y-2)(y-5) = 0$

$\Rightarrow y = 2, y = 5$

Hence: $x^2 = 2, x^2 = 5$

$\Rightarrow x = \pm\sqrt{2}, x = \pm\sqrt{5}$

(ii) $y = x + 1 \Rightarrow y^2 + 3y - 2 = 0$

$$\Rightarrow y = \frac{-3 \pm \sqrt{9 - 4(1)(-2)}}{2}$$

$$\Rightarrow y = \frac{-3 \pm \sqrt{17}}{2}$$

$$\Rightarrow x + 1 = \frac{-3 \pm \sqrt{17}}{2}$$

$$\Rightarrow x = \frac{-5 \pm \sqrt{17}}{2}$$

(iii) $y = x^2 \Rightarrow y^2 - 2y - 2 = 0$

$$\Rightarrow y = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2}$$

$$\Rightarrow y = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

Hence: $x^2 = 1 \pm \sqrt{3}$

$\Rightarrow x = \pm\sqrt{1 \pm \sqrt{3}}$

(iv) $y = k - 2 \Rightarrow 2y^2 - 3y - 4 = 0$

$$\Rightarrow y = \frac{3 \pm \sqrt{9 - 4(2)(-4)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

Hence: $k - 2 = \dfrac{3 \pm \sqrt{41}}{4}$

$$\Rightarrow k = \frac{11 \pm \sqrt{41}}{4}$$

(b) (i) $k = 2y - 1 \quad \Rightarrow k^2 - 3k - 28 = 0$

$\Rightarrow (k+4)(k-7) = 0$

$\Rightarrow k = -4, k = 7$

Hence: $2y - 1 = -4, 2y - 1 = 7$

$\Rightarrow 2y = -3, 2y = 8$

$\Rightarrow y = \dfrac{-3}{2}, y = 4$

(ii) $k = 2y - 3 \quad \Rightarrow k^2 - 1 = 0$

$\Rightarrow (k+1)(k-1) = 0$

$\Rightarrow k = -1, k = 1$

Hence: $2y - 3 = -1, 2y - 3 = 1$

$\Rightarrow 2y = 2, 2y = 4$

$\Rightarrow y = 1, y = 2$

(c) $k = y + \dfrac{4}{y} \Rightarrow k^2 - 9k + 20 = 0$

$\Rightarrow (k-4)(k-5) = 0$

$\Rightarrow k = 4, k = 5$

Hence: $y + \dfrac{4}{y} = 4, y + \dfrac{4}{y} = 5$

$\Rightarrow y^2 + 4 = 4y, y^2 + 4 = 5y$

$\Rightarrow y^2 - 4y + 4 = 0,$

$y^2 - 5y + 4 = 0$

$\Rightarrow (y-2)(y-2) = 0,$

$(y-1)(y-4) = 0$

$\Rightarrow y = 2, y = 1, y = 4$

(d) $x = 2t - \dfrac{5}{t} = 3, \Rightarrow x^2 - 12x + 27 = 0$

$\Rightarrow (x-3)(x-9) = 0$

$\Rightarrow x = 3, x = 9$

Hence: $2t - \dfrac{5}{t} = 3, 2t - \dfrac{5}{t} = 9$

$\Rightarrow 2t^2 - 5 = 3t, 2t^2 - 5 = 9t$

$\Rightarrow 2t^2 - 3t - 5 = 0, 2t^2 - 9t - 5 = 0$

$\Rightarrow (t+1)(2t-5) = 0, (2t+1)(t-5) = 0,$

$\Rightarrow t = -1, 2t = 5, 2t = -1, t = 5$

$\Rightarrow t = -1, t = \dfrac{5}{2}, t = -\dfrac{1}{2}, t = 5$

6. $$x = \frac{\sqrt{3} \pm \sqrt{3 - 4(2)(-3)}}{2(2)}$$

$$= \frac{\sqrt{3} \pm \sqrt{27}}{4}$$

$$= \frac{\sqrt{3} \pm 3\sqrt{3}}{4}$$

$$= \frac{4\sqrt{3}}{4}, -\frac{2\sqrt{3}}{4}$$

$$= \sqrt{3}, -\frac{\sqrt{3}}{2}$$

7. (a) $x = -4.2, 1.2$

(b) $x = -2.3, 1.3$

(c) $x = -9.5, -0.5$

(d) $-1.6 < x < 0.6$

(e) $x = -1.6, 0.6$

(f) $x = -3, 1$

(g) $x = -3.6, 0.6$

(h) $-4.1 < x < -0.9$

8. Graph does not intersect the x-axis

9. (a) $x_2 - x_1 = -22.5 + 2.5 = -20$
(b) $x_2 + x_1 = -22.5 - 2.5 = -25$

10. (a) $x = -0.5, 2$
(b) $x = -0.8$
(c) $x = -0.5, 2.4$

11. (i) $u = \sqrt{x}$ Equation is:

$$2u^2 + 3u - 5 = 0$$
$$(2u + 5)(u - 1) = 0$$
$$2u + 5 = 0 \quad \text{or} \quad u - 1 = 0$$
$$u = -\frac{5}{2} \quad \text{or} \quad u = 1$$

As $u = \sqrt{x}$ must be positive, $u = -\frac{5}{2}$ is not possible. Thus
$u = \sqrt{x} = 1$
$u^2 = x = 1$
is the only solution for x.

(ii) $u = \sqrt{x}$ Equation is:

$$u^2 - 3u - 4 = 0$$
$$(u + 1)(u - 4) = 0$$
$$u + 1 = 0 \quad \text{or} \quad u - 4 = 0$$
$$u = -1 \quad \text{or} \quad u = 4$$

As $u = \sqrt{x}$ must be positive, $u = -1$ is not possible. Thus
$u = \sqrt{x} = 4$
$u^2 = x = 16$
is the only solution for x.

12. (i) $x = \dfrac{\sqrt{7} \pm \sqrt{7 - 4(1)(-14)}}{2(1)}$

$$= \frac{\sqrt{7} \pm \sqrt{63}}{2}$$
$$= \frac{\sqrt{7} + 3\sqrt{7}}{2}, \frac{\sqrt{7} - 3\sqrt{7}}{2}$$
$$= 2\sqrt{7}, -\sqrt{7}$$

(ii) $x = \dfrac{-7\sqrt{5} \pm \sqrt{245 - 4(2)(15)}}{2(2)}$

$$= \frac{-7\sqrt{5} \pm \sqrt{125}}{4}$$
$$= \frac{-7\sqrt{5} + 5\sqrt{5}}{4}, \frac{-7\sqrt{5} - 5\sqrt{5}}{4}$$
$$= -\frac{\sqrt{5}}{2}, -3\sqrt{5}$$

Exercise 2.2

1. (i) f
(ii) h
(iii) g
(iv) Curve f has roots $= 1.6, 4.4$
Curve h has root $= 3$

2. $y = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow A = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right)$$
$$\Rightarrow B = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$$

3. (i) $b^2 - 4ac = (1)^2 - 4(2)(5) = -39 < 0 \Rightarrow$ Imaginary roots
(ii) $b^2 - 4ac = (3)^2 - 4(-2)(1) = 17 > 0$
$\Rightarrow$ Roots are real and different
(iii) $b^2 - 4ac = (2)^2 - 4(3)(-1) = 16 > 0$
$\Rightarrow$ Roots are real and different

(iv) $b^2 - 4ac = (2)^2 - 4(-1)(-3) = -8 < 0$

$\Rightarrow$ Imaginary roots

(v) $b^2 - 4ac = (8)^2 - 4(1)(16) = 0$

$\Rightarrow$ Roots are real and equal

(vi) $b^2 - 4ac = (-10)^2 - 4(1)(25) = 0$

$\Rightarrow$ Roots are real and equal

4. $y = x^2 - 2x + 5$ is positive for all values of x

$3x^2 - kx + 12$ is positive

$\Rightarrow$ Roots are imaginary

$\Rightarrow b^2 - 4ac < 0$

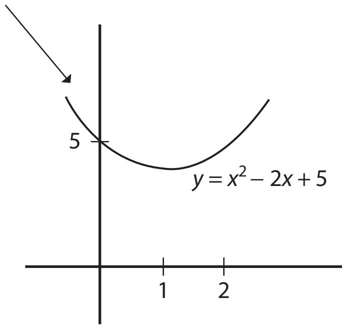
$\Rightarrow (-k^2) - 4(3)(12) < 0$

$\Rightarrow k^2 - 144 < 0$

$\Rightarrow$ Factors: $(k + 12)(k - 12)$

$\Rightarrow$ Roots: $k = -12, 12$

$\Rightarrow$ Solution for $k^2 - 144 < 0 \Rightarrow -12 < k < 12$



5. Equal Roots $\Rightarrow b^2 - 4ac = 0$

(i) $(-10)^2 - 4(1)(k) = 0$

$\Rightarrow 100 - 4k = 0$

$\Rightarrow -4k = -100$

$\Rightarrow k = 25$

(ii) $(k)^2 - 4(4)(9) = 0$

$\Rightarrow k^2 - 144 = 0$

$\Rightarrow (k + 12)(k - 12) = 0$

$\Rightarrow k = -12, 12$

(iii) $(-2k - 2)^2 - 4(1)(5k + 1) = 0$

$\Rightarrow 4k + 8k + 4 - 20k - 4 = 0$

$\Rightarrow 4k^2 - 12k = 0$

$\Rightarrow k^2 - 3k = 0$

$\Rightarrow k(k - 3) = 0$

$\Rightarrow k = 0, k = 3$

6. Equal Roots $\Rightarrow (2k + 2)^2 - 4(k^2)(4) = 0$

$\Rightarrow 4k^2 + 8k + 4 - 16k^2 = 0$

$\Rightarrow 12k^2 - 8k - 4 = 0$

$\Rightarrow 3k^2 - 2k - 1 = 0$

$\Rightarrow (3k + 1)(k - 1) = 0$

$\Rightarrow k = -\frac{1}{3}, k = 1$

7. (i) Real Roots $\Rightarrow (-3k)^2 - 4(1)(-k^2) \geq 0$

$\Rightarrow 9k^2 + 4k^2 \geq 0$

$\Rightarrow 13k^2 \geq 0$ True

(ii) Real Roots $\Rightarrow (2)^2 - 4(k)(2 - k) \geq 0$

$\Rightarrow 4 - 8k + 4k^2 \geq 0$

$\Rightarrow k^2 - 2k + 1 \geq 0$

$\Rightarrow (k - 1)^2 \geq 0$ True

8. Real Roots $\Rightarrow (-3)^2 - 4(1)(2 - c^2) \geq 0$

$\Rightarrow 9 - 8 + 4c^2 \geq 0$

$\Rightarrow 1 + 4c^2 \geq 0$ True

9. Real Roots $\Rightarrow (2)^2 - 4(k - 2)(-k) \geq 0$

$\Rightarrow 4 + 4k^2 - 8k \geq 0$

$\Rightarrow k^2 - 2k + 1 \geq 0$

$\Rightarrow (k - 1)^2 \geq 0$ True

10. Equal Roots $\Rightarrow (2k+1)^2 - 4(k-2)(k) = 0$
$\Rightarrow 4k^2 + 4k + 1 - 4k^2 + 8k = 0$
$\Rightarrow 12k = -1$
$\Rightarrow k = -1$

11. Equal Roots $\Rightarrow (6-2m)^2 - 4(m+3)(m-1) = 0$
$\Rightarrow 36 - 24m + 4m^2 - 4m^2 - 12m + 4m + 12 = 0$
$\Rightarrow -32m + 48 = 0$
$\Rightarrow -2m + 3 = 0$
$\Rightarrow -2m = -3$
$\Rightarrow m = 3$

12. Equal Roots $\Rightarrow b^2 - 4(a)(1) = 0$
$\Rightarrow b^2 - 4a = 0$
$\Rightarrow -4a = -b^2$
$\Rightarrow a = \dfrac{b^2}{4}$

Hence, $x = \dfrac{-b + \sqrt{0}}{2\left(\dfrac{b^2}{4}\right)}$
$= -b \cdot \dfrac{2}{b^2} = -\dfrac{2}{b}$

13. No Real Roots $\Rightarrow (-2p)^2 - 4(1)(3p^2 + q^2) < 0$
$\Rightarrow 4p^2 - 12p^2 - 4q^2 < 0$
$\Rightarrow -8p^2 - 4q^2 < 0$
$\Rightarrow -2p^2 - q^2 < 0$ True

Exercises 2.3

1. $y = -2x + 3 \cap y = x^2$
$\Rightarrow x^2 = -2x + 3$
$\Rightarrow x^2 + 2x - 3 = 0$
$\Rightarrow (x+3)(x-1) = 0$
$\Rightarrow x = -3, \quad x = 1$
$\Rightarrow y = (-3)^2, y = (1)^2$
$\Rightarrow y = 9, \quad y = 1$
ANS $= (-3, 9), (1, 1)$

2. $x = y - 1 \cap x^2 + y^2 = 5$
$\Rightarrow (y-1)^2 + y^2 - 5 = 0$
$\Rightarrow y^2 - 2y + 1 + y^2 - 5 = 0$
$\Rightarrow 2y^2 - 2y - 4 = 0$
$\Rightarrow y^2 - y - 2 = 0$
$\Rightarrow (y+1)(y-2) = 0$
$\Rightarrow y = -1, \quad y = 2$
$\Rightarrow x = -1 - 1, x = 2 - 1$
$\Rightarrow x = -2, x = 1$
ANS $= (-2, -1), (1, 2)$

3. $y = -2x + 2 \cap 4x^2 = y$
$\Rightarrow 4x^2 = -2x + 2$
$\Rightarrow 4x^2 + 2x - 2 = 0$
$\Rightarrow 2x^2 + x - 1 = 0$
$\Rightarrow (x+1)(2x-1) = 0$
$\Rightarrow x = -1, x = \dfrac{1}{2}$
$\Rightarrow y = -2(-1) + 2, y = -2\left(\dfrac{1}{2}\right) + 2$
$\Rightarrow y = 4, \quad y = 1$
ANS $= (-1, 4), \left(\dfrac{1}{2}, 1\right)$

4. $y = -x + 1 \cap y = x^2 - 6x + 5$
$\Rightarrow x^2 - 6x + 5 = -x + 1$
$\Rightarrow x^2 - 5x + 4 = 0$
$\Rightarrow (x-1)(x-4) = 0$
$\Rightarrow x = 1, \quad x = 4$
$\Rightarrow y = -1 + 1, \quad y = -4 + 1$
$\Rightarrow y = 0, \quad y = -3$
ANS $= (1, 0), (4, -3)$

5. $y = -x + 7 \cap x^2 + y^2 = 25$
$\Rightarrow x^2 + (x + 7)^2 - 25 = 0$
$\Rightarrow x^2 + x^2 - 14x + 49 - 25 = 0$
$\Rightarrow 2x^2 - 14x + 24 = 0$
$\Rightarrow x^2 - 7x + 12 = 0$
$\Rightarrow (x - 3)(x - 4) = 0$
$\Rightarrow x = 3, x = 4$
$\Rightarrow y = -3 + 7, y = -4 + 7$
$\Rightarrow y = 4, y = 3$
ANS = (3, 4), (4, 3)

6. $2x + 1 = y \cap 3x^2 - y^2 = 3$
$\Rightarrow 3x^2 - (2x + 1)^2 - 3 = 0$
$\Rightarrow 3x^2 - 4x^2 - 4x - 1 - 3 = 0$
$\Rightarrow -x^2 - 4x - 4 = 0$
$\Rightarrow x^2 + 4x + 4 = 0$
$\Rightarrow (x + 2)(x + 2) = 0$
$\Rightarrow x = -2$
$\Rightarrow y = 2(-2) + 1 = -3$
ANS = (−2, −3)

7. $y = 3x - 4 \cap y = x^2 - 4x + 6$
$\Rightarrow x^2 - 4x + 6 = 3x - 4$
$\Rightarrow x^2 - 7x + 10 = 0$
$\Rightarrow (x - 2)(x - 5) = 0$
$\Rightarrow x = 2, x = 5$
$\Rightarrow y = 3(2) - 4, y = 3(5) - 4$
$\Rightarrow y = 2, y = 11$
ANS = (2, 2), (5, 11)

8. $y = -x + 4 \cap x^2 + y^2 - 4x + 2 = 0$
$\Rightarrow x^2 + (-x + 4)^2 - 4x + 2 = 0$
$\Rightarrow x^2 + x^2 - 8x + 16 - 4x + 2 = 0$
$\Rightarrow 2x^2 - 12x + 18 = 0$
$\Rightarrow x^2 - 6x + 9 = 0$
$\Rightarrow (x - 3)(x - 3) = 0$
$\Rightarrow x = 3$
$\Rightarrow y = -3 + 4 = 1$
ANS = (3, 1)

9. $x = -2y + 2 \cap x^2 + 4y^2 = 4$
$\Rightarrow (-2y + 2)^2 + 4y^2 - 4 = 0$
$\Rightarrow 4y^2 - 8y + 4 + 4y^2 - 4 = 0$
$\Rightarrow 8y^2 - 8y = 0$
$\Rightarrow y^2 - y = 0$
$\Rightarrow y(y - 1) = 0$
$\Rightarrow y = 0, y = 1$
$\Rightarrow x = -2(0) + 2, x = -2(1) + 2$
$\Rightarrow x = 2, x = 0$
ANS = (2, 0), (0, 1)

10. $2x + 2 = y \cap xy = 4$
$\Rightarrow x(2x + 2) = 4$
$\Rightarrow 2x^2 + 2x - 4 = 0$
$\Rightarrow x^2 + x - 2 = 0$
$\Rightarrow (x + 2)(x - 1) = 0$
$\Rightarrow x = -2, x = 1$
$\Rightarrow y = 2(-2) + 2, y = 2(1) + 2$
$\Rightarrow y = -2, y = 4$
ANS = (−2, −2), (1, 4)

11. $y = -2x + 3 \cap y^2 + xy = 2$
$\Rightarrow (-2x + 3)^2 + x(-2x + 3) - 2 = 0$
$\Rightarrow 4x^2 - 12x + 9 - 2x^2 + 3x - 2 = 0$
$\Rightarrow 2x^2 - 9x + 7 = 0$
$\Rightarrow (2x - 7)(x - 1) = 0$
$\Rightarrow x = \frac{7}{2}, x = 1$
$\Rightarrow y = -2\left(\frac{7}{2}\right) + 3, y = -2(1) + 3$
$\Rightarrow y = -4, y = 1$
ANS $\left(\frac{7}{2}, -4\right)$, (1, 1)

12. $x = y - 3 \cap x^2 + y^2 + 2x - 4y + 3 = 0$
$\Rightarrow (y - 3)^2 + y^2 + 2(y - 3) - 4y + 3 = 0$
$\Rightarrow y^2 - 6y + 9 + y^2 + 2y - 6 - 4y + 3 = 0$
$\Rightarrow 2y^2 - 8y + 6 = 0$
$\Rightarrow y^2 - 4y + 3 = 0$
$\Rightarrow (y - 1)(y - 3) = 0$
$\Rightarrow y = 1, y = 3$
$\Rightarrow x = 1 - 3, x = 3 - 3$
$\Rightarrow x = -2, x = 0$
ANS = (−2, 1), (0, 3)

13. $s = 2t - 1 \cap 3t^2 - 2ts + s^2 = 9$
$\Rightarrow 3t^2 - 2t(2t - 1) + (2t - 1)^2 = 9$
$\Rightarrow 3t^2 - 4t^2 + 2t + 4t^2 - 4t + 1 - 9 = 0$
$\Rightarrow 3t^2 - 2t - 8 = 0$
$\Rightarrow (3t + 4)(t - 2) = 0$
$\Rightarrow t = -\frac{4}{3}, t = 2$
$\Rightarrow s = 2\left(-\frac{4}{3}\right) - 1, s = 2(2) - 1$
$\Rightarrow s = -\frac{11}{3}, s = 3$
ANS = $\left(-\frac{11}{3}, -\frac{4}{3}\right)$, (3, 2)

14. $2s + 3 = t \cap 2s^2 = t^2 + 1$

$\Rightarrow 2s^2 = (2s + 3)^2 + 1$

$\Rightarrow 2s^2 = 4s^2 + 12s + 9 + 1$

$\Rightarrow 2s^2 + 12s + 10 = 0$

$\Rightarrow s^2 + 6s + 5 = 0$

$\Rightarrow (s + 1)(s + 5) = 0$

$\Rightarrow s = -1, s = -5$

$\Rightarrow t = 2(-1) + 3, t = 2(-5) + 3$

$\Rightarrow t = 1, t = -7$

ANS $= (-1, 1), (-5, -7)$

15. $2t = 3s + 1 \cap t^2 + ts - 4s^2 = 2$

$\Rightarrow t = \dfrac{3s + 1}{2} \Rightarrow \left(\dfrac{3s + 1}{2}\right)^2 + s\left(\dfrac{3s + 1}{2}\right) - 4s^2 - 2 = 0$

$\Rightarrow \dfrac{9s^2 + 6s + 1}{2} + \dfrac{3s^2 + s}{2} - 4s^2 - 2 = 0$

$\Rightarrow 9s^2 + 6s + 1 + 6s^2 + 2s - 16s^2 - 8 = 0$

$\Rightarrow -s^2 + 8s - 7 = 0$

$\Rightarrow s^2 - 8s + 7 = 0$

$\Rightarrow (s - 1)(s - 7) = 0$

$\Rightarrow s = 1, s = 7$

$\Rightarrow t = \dfrac{3(1) + 1}{2}, t = \dfrac{3(7) + 1}{2}$

$\Rightarrow t = 2, t = 11$

ANS $= (1, 2), (7, 11)$

Exercise 2.4

1. $x, x + 1 \Rightarrow (x)^2 + (x + 1)^2 = 61$

$\Rightarrow x^2 + x^2 + 2x + 1 - 61 = 0$

$\Rightarrow 2x^2 + 2x - 60 = 0$

$\Rightarrow x^2 + x - 30 = 0$

$\Rightarrow (x + 6)(x - 5) = 0$

$\Rightarrow x = -6, x = 5$

$\Rightarrow x + 1 = -5, x + 1 = 6$

Numbers are $-6, -5$ or $5, 6$

2. $x, x + 2 \Rightarrow (x)^2 + (x + 2)^2 = 52$

$\Rightarrow x^2 + x^2 + 4x + 4 - 52 = 0$

$\Rightarrow 2x^2 + 4x - 48 = 0$

$\Rightarrow x^2 + 2x - 24 = 0$

$\Rightarrow (x + 6)(x - 4) = 0$

$\Rightarrow x = -6, x = 4$

$\Rightarrow x + 2 = -4, x + 2 = 6$

Numbers are $-6, -4$ or $4, 6$

3. (i) Perimeter $= 2x + 2y = 62$

$\Rightarrow \quad x + y = 31$

$\Rightarrow \quad y = -x + 31$

Area $= (x)(y) = 198$

$\Rightarrow xy = 198$

(ii) $\Rightarrow x(-x + 31) - 198 = 0$

$\Rightarrow -x^2 + 31x - 198 = 0$

$\Rightarrow x^2 - 31x + 198 = 0$

$\Rightarrow (x - 9)(x - 22) = 0$

$\Rightarrow x = 9$ m, $x = 22$ m

$\Rightarrow y = -9 + 31, y = -22 + 31$

$= 22$ m, $\quad = 9$ m

Dimensions $= 9$ m, 22 m

4. Sides $= x, x+1, x+2 \Rightarrow$ Pythagoras $\Rightarrow (x+2)^2 = (x)^2 + (x+1)^2$

$$\Rightarrow x^2 + 4x + 4 = x^2 + x^2 + 2x + 1$$
$$\Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3, x = -1 \text{ (Not valid)}$$
$$\Rightarrow x + 1 = 4$$
$$\Rightarrow x + 2 = 5$$

Perimeter $= 3 + 4 + 5 = 12$ units

5. $s = 25 \Rightarrow 12t - t^2 = 25$

$$\Rightarrow t^2 - 12t + 25 = 0$$
$$\Rightarrow t = \frac{12 \pm \sqrt{(-12)^2 - 4(1)(25)}}{2(1)}$$
$$= \frac{12 \pm \sqrt{144 - 100}}{2}$$
$$= \frac{12 \pm \sqrt{44}}{2}$$
$$= \frac{12 + 6.633}{2}, \frac{12 - 6.633}{2}$$
$$= 9.316, 2.683$$
$$= 9.32, 2.68$$

6. $x^2 - 15 = 2x$

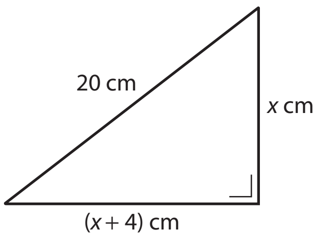
$$\Rightarrow x^2 - 2x - 15 = 0$$
$$\Rightarrow (x+3)(x-5) = 0$$
$$\Rightarrow x = -3, x = 5$$

7. $h = 6 \Rightarrow -16t^2 + 24t + 1 = 6$

$$\Rightarrow 16t^2 - 24t + 5 = 0$$
$$\Rightarrow (4t-1)(4t-5) = 0$$
$$\Rightarrow 4t = 1, 4t = 5$$
$$\Rightarrow t = \frac{1}{4} \text{ sec}, t = \frac{5}{4} \text{ sec}$$

8. Pythagoras $\Rightarrow (x)^2 + (x+4)^2 = (20)^2$

$$\Rightarrow x^2 + x^2 + 8x + 16 = 400$$
$$\Rightarrow 2x^2 + 8x - 384 = 0$$
$$\Rightarrow x^2 + 4x - 192 = 0$$
$$\Rightarrow (x-12)(x+16) = 0$$
$$\Rightarrow x = 12, x = -16 \text{ (Not valid)}$$



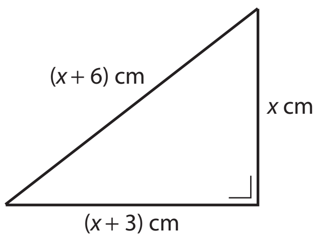
9. $x, x+2 \Rightarrow x(x+2) = 4(2x+2) - 1$

$$\Rightarrow x^2 + 2x = 8x + 8 - 1$$
$$\Rightarrow x^2 - 6x - 7 = 0$$
$$\Rightarrow (x+1)(x-7) = 0$$
$$\Rightarrow x = -1, x = 7$$
$$\Rightarrow x + 2 = 1, x + 2 = 9$$

ANS $= -1, 1$ or $7, 9$

10. Pythagoras $\Rightarrow (x)^2 + (x+3)^2 = (x+6)^2$

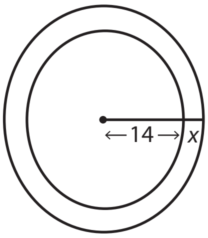
$$\Rightarrow x^2 + x^2 + 6x + 9 = x^2 + 12x + 36$$
$$\Rightarrow x^2 - 6x - 27 = 0$$
$$\Rightarrow (x-9)(x+3) = 0$$
$$\Rightarrow x = 9, x = -3 \text{ (Not valid)}$$



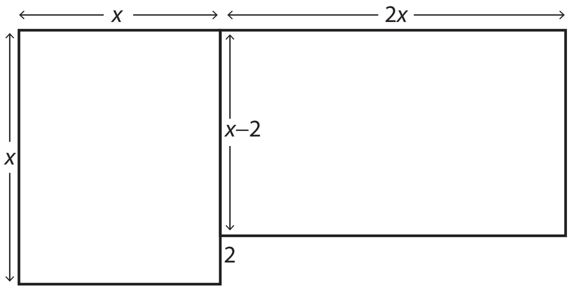
11. $x, x+4 \Rightarrow \text{Area} = x(x+4) = 60$
$\Rightarrow x^2 + 4x - 60 = 0$
$\Rightarrow (x-6)(x+10) = 0$
$\Rightarrow x = 6, x = -10$ (Not valid)
Dimensions are 6 m, 10 m

12. $x, x+1, x+2 \Rightarrow 3(x + x + 1 + x + 2) = (x+1)(x+2)$
$\Rightarrow 9x + 9 = x^2 + 3x + 2$
$\Rightarrow x^2 - 6x - 7 = 0$
$\Rightarrow (x+1)(x-7) = 0$
$\Rightarrow x = -1, x = 7$
ANS $= -1, 0, 1$ or $7, 8, 9$

13. Area $= \pi(x+14)^2 - \pi(14)^2 = 60\pi$
$\Rightarrow x^2 + 28x + 196 - 196 = 60$
$\Rightarrow x^2 + 28x - 60 = 0$
$\Rightarrow (x-2)(x+30) = 0$
$\Rightarrow x = 2, x = -30$ (Not valid)
width of desk = 2 m



14. Area $= x^2 + 96 = (2x)(x-2)$
$\Rightarrow x^2 + 96 = 2x^2 - 4x$
$\Rightarrow x^2 - 4x - 96 = 0$
$\Rightarrow (x-12)(x+8) = 0$
$\Rightarrow x = 12, x = -8$ (Not valid)
Dimensions = 10 m, 24 m



15. Point G $\Rightarrow 0.1x^2 - x + 2.5 = 1.5$
$\Rightarrow 0.1x^2 - x + 1 = 0$
$\Rightarrow x^2 - 10x + 10 = 0$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 4(1)(10)}}{2(1)}$$
$$= \frac{10 \pm \sqrt{60}}{2}$$
$$= \frac{10 - \sqrt{60}}{2}, \frac{10 + \sqrt{60}}{2}$$
$= 1.127, 8.873$ (Not valid)

Point C $\Rightarrow 0.1x^2 - x + 2.5 = 3$
$\Rightarrow 0.1x^2 - x - 0.5 = 0$
$\Rightarrow x^2 - 10x - 5 = 0$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 4(1)(-5)}}{2(1)}$$
$$= \frac{10 \pm \sqrt{120}}{2}$$
$$= \frac{10 + \sqrt{120}}{2}, \frac{10 - \sqrt{120}}{2}$$
$= 10.477, -0.477$ (Not valid)

Distance $= 10.477 - 1.127 = 9.35$ m

16. $3t - 4 = s \cap 2t^2 + s^2 = 43$

$\Rightarrow 2t^2 + (3t - 4)^2 - 43 = 0$

$\Rightarrow 2t^2 + 9t^2 - 24t + 16 - 43 = 0$

$\Rightarrow 11t^2 - 24t - 27 = 0$

$\Rightarrow (t - 3)(11t + 9) = 0$

$\Rightarrow t = 3, t = -\dfrac{9}{11}$ (Not valid)

$\Rightarrow s = 3(3) - 4$

$= 5$

Point = (3, 5)

17. $x + 3y = 5 \cap x^2 + 6y^2 = 40$

$\Rightarrow x = -3y + 5 \Rightarrow (-3y + 5)^2 + 6y^2 - 40 = 0$

$\Rightarrow 9y^2 - 30y + 25 + 6y^2 - 40 = 0$

$\Rightarrow 15y^2 - 30y - 15 = 0$

$\Rightarrow y^2 - 2y - 1 = 0$

$\Rightarrow y = \dfrac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)}$

$\Rightarrow y = \dfrac{2 \pm \sqrt{8}}{2}$

$\Rightarrow y = 1 + \sqrt{2}, 1 - \sqrt{2}$

$\Rightarrow y = 2.414, -0.414$

$\Rightarrow x = -3(2.414) + 5, x = -3(-0.414) + 5$

$= -2.242, \quad = 6.242$

Plane crosses front at 2 points; $(-2.2, 2.4)$ and $(6.2, -0.4)$

$k = 10 \Rightarrow x + 3y = 10 \cap x^2 + 6y^2 = 40$

$\Rightarrow x = -3y + 10 \Rightarrow (-3y + 10)^2 + 6y^2 - 40 = 0$

$\Rightarrow 15y^2 - 60y + 60 = 0$

$\Rightarrow y^2 - 4y + 4 = 0$

$\Rightarrow (y - 2)(y - 2) = 0$

$\Rightarrow y = 2$ and $x = -3(2) + 10 = 4$

Plane crosses front at one point; (4, 2)

$\Rightarrow$ Plane will avoid weather front when $k > 10$

$\Rightarrow$ Minimum value of $k = 11$

Exercise 2.5

1. (a) (i) sum $= -9$ (ii) product $= 4$

(b) (i) sum $= 2$ (ii) product $= -5$

(c) (i) sum $= 7$ (ii) product $= 2$

(d) (i) sum $= 9$ (ii) product $= -3$

(e) (i) sum $= \dfrac{7}{2}$ (ii) product $= -\dfrac{1}{2}$

(f) (i) sum $= -\dfrac{1}{7}$ (iii) product $= -\dfrac{1}{7}$

(g) (i) sum $= -\dfrac{10}{3}$ (ii) product $= -\dfrac{2}{3}$

(h) (i) sum $= -\dfrac{10}{5} = -2$ (ii) product $= \dfrac{1}{5}$

(i) (i) sum $= -2$ (ii) product $= -3$

(j) (i) sum $= \dfrac{3}{4}$ (ii) product $= \dfrac{5}{4}$

2. (a) $x^2 + 3x - 1 = 0$
(b) $x^2 - 6x - 4 = 0$
(c) $x^2 - 7x - 5 = 0$
(d) $x^2 + \frac{2}{3}x - \frac{7}{3} = 0$
$\Rightarrow 3x^2 + 2x - 7 = 0$
(e) $x^2 + \frac{5}{2}x - 2 = 0$
$\Rightarrow 2x^2 + 5x - 4 = 0$
(f) $x^2 + \frac{3}{2}x - 5 = 0$
$\Rightarrow 2x^2 + 3x - 10 = 0$
(g) $x^2 + \frac{1}{4}x - \frac{1}{3} = 0$
$\Rightarrow 12x^2 + 3x - 4 = 0$
(h) $x^2 + 1\frac{2}{3}x + \frac{1}{2} = 0$
$\Rightarrow 6x^2 + 10x + 3 = 0$

3. (i) $x^2 - (4 + 6)x + (4)(6) = 0$
$\Rightarrow x^2 - 10x + 24 = 0$
(ii) $x^2 - (2 - 3)x + (2)(-3) = 0$
$\Rightarrow x^2 + x - 6 = 0$
(iii) $x^2 - (-5 - 1)x + (-5)(-1) = 0$
$\Rightarrow x^2 + 6x + 5 = 0$
(iv) $x^2 - (4 + \sqrt{5})x + (4)(\sqrt{5}) = 0$
$\Rightarrow x^2 - (4 + \sqrt{5})x + 4\sqrt{5} = 0$
(v) $x^2 - (a + 3a)x + (a)(3a) = 0$
$\Rightarrow x^2 - 4ax + 3a^2 = 0$
(vi) $x^2 - \left(\frac{2}{5} + \frac{3}{5}\right)x + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = 0$
$\Rightarrow x^2 - x + \frac{6}{25} = 0$
$\Rightarrow 25x^2 - 25x + 6 = 0$
(vii) $x^2 - \left(\frac{2}{b} + \frac{3}{b}\right)x + \left(\frac{2}{b}\right)\left(\frac{3}{b}\right) = 0$
$\Rightarrow x^2 - \frac{5}{b}x + \frac{6}{b^2} = 0$
$\Rightarrow b^2x^2 - 5bx + 6 = 0$
(viii) $x^2 - \left(\frac{5}{2} + \frac{3}{5}\right)x + \left(\frac{5}{2}\right)\cdot\left(\frac{3}{5}\right) = 0$
$\Rightarrow x^2 - \frac{31}{10}x + \frac{3}{2} = 0$
$\Rightarrow 10x^2 - 31x + 15 = 0$

Exercise 2.6

1. (i) $c = \left(\frac{28}{2}\right)^2 = (14)^2 = 196$ (ii) $c = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ (iii) $c = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$

2. (i) $x^2 - 8x - 3 = 0$
$\Rightarrow x^2 - 8x + 16 - 16 - 3 = 0$
$\Rightarrow (x - 4)^2 - 19 = 0$
(ii) $x^2 - 2x - 5 = 0$
$\Rightarrow x^2 - 2x + 1 - 1 - 5 = 0$
$\Rightarrow (x - 1)^2 - 6 = 0$
(iii) $x^2 - 2x + 1 = 0$
$\Rightarrow (x - 1)^2 = 0$

3. (i) $x^2 + 4x - 6 = 0$
$\Rightarrow x^2 + 4x + 4 - 4 - 6 = 0$
$\Rightarrow (x + 2)^2 - 10 = 0$
(ii) $x^2 + 9x + 4 = 0$
$\Rightarrow x^2 + 9x + 20\frac{1}{4} - 20\frac{1}{4} + 4 = 0$
$\Rightarrow \left(x + 4\frac{1}{2}\right)^2 - 16\frac{1}{4} = 0$
(iii) $x^2 - 7x - 3 = 0$
$\Rightarrow x^2 - 7x + 12\frac{1}{4} - 12\frac{1}{4} - 3 = 0$
$\Rightarrow \left(x - 3\frac{1}{2}\right)^2 - 15\frac{1}{4} = 0$

4. (i) $2x^2 + 4x - 5 = 0$

$= 2\left(x^2 + 2x - \frac{5}{2}\right) = 0$

$= 2\left(x^2 + 2x + 1 - 1 - \frac{5}{2}\right) = 0$

$= 2\left[(x + 1)^2 - \frac{7}{2}\right] = 0$

$= 2(x + 1)^2 - 7 = 0$

$\Rightarrow$ Minimum Point $= (-1, -7)$

(ii) $3x^2 - 6x - 1 = 0$

$\Rightarrow 3\left(x^2 - 2x - \frac{1}{3}\right) = 0$

$\Rightarrow 3\left(x^2 - 2x + 1 - 1 - \frac{1}{3}\right) = 0$

$\Rightarrow 3\left[(x - 1)^2 - 1\frac{1}{3}\right] = 0$

$\Rightarrow 3(x - 1)^2 - 4 = 0$

$\Rightarrow$ Minimum Point $= (1, -4)$

(iii) $4x^2 + x + 3 = 0$

$\Rightarrow 4\left(x^2 + \frac{1}{4}x + \frac{3}{4}\right) = 0$

$\Rightarrow 4\left(x^2 + \frac{1}{4}x + \frac{1}{64} - \frac{1}{64} + \frac{3}{4}\right) = 0$

$\Rightarrow 4\left[\left(x + \frac{1}{8}\right)^2 + \frac{47}{64}\right] = 0$

$\Rightarrow 4\left(x + \frac{1}{8}\right)^2 + \frac{47}{16} = 0$

$\Rightarrow$ Minimum Point $= \left(-\frac{1}{8}, \frac{47}{16}\right)$

5. $x^2 - 6x + k = 0$

$= x^2 - 6x + 9 - 9 + k$

$= (x - 3)^2 + k - 9$

hence, $k - 9 > 0$

$\Rightarrow k > 9$

Minimum value of $k = 10$

6. $2x^2 - 12x + 7$

$= 2\left(x^2 - 6x + \frac{7}{2}\right)$

$= 2\left(x^2 - 6x + 9 - 9 + \frac{7}{2}\right)$

$= 2\left[(x - 3)^2 - \frac{11}{2}\right]$

$= 2(x - 3)^2 - 11$

7. $g(x) = x^2 + 8x + 20$

$= x^2 + 8x + 16 - 16 + 20$

$= (x + 4)^2 + 4$

Since $(x + 4)^2 \geqslant 0$

$\Rightarrow g(x) = (x + 4)^2 + 4 \geqslant 4$

8. (i) Minimum for $f(x) = (-1, -5)$

$\Rightarrow f(x) = (x + 1)^2 - 5$

$= x^2 + 2x + 1 - 5$

$= x^2 + 2x - 4$

Choose $(-2, -4) \Rightarrow f(-2) = (-2)^2 + 2(-2) - 4$

$= 4 - 4 - 4$

$= -4$ True

Minimum for $g(x) = (2, -1)$

$\Rightarrow g(x) = (x - 2)^2 - 1$

$= x^2 - 4x + 4 - 1$

$= x^2 - 4x + 3$

Choose (3, 0) $\Rightarrow g(3) = (3)^2 - 4(3) + 3$
$= 9 - 12 + 3$
$= 0$ True

Minimum for $h(x) = (4, 1)$
$\Rightarrow h(x) = (x - 4)^2 + 1$
$= x^2 - 8x + 16 + 1$
$= x^2 - 8x + 17$

Choose (5, 2) $\Rightarrow h(5) = (5)^2 - 8(5) + 17$
$= 25 - 40 + 17$
$= 42 - 40 = 2$ True

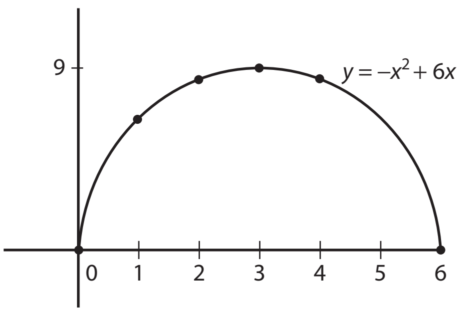
9. $f(x) = x^2 + 4x + 7$
$= x^2 + 4x + 4 + 3$
$= (x + 2)^2 + 3$

(i) smallest value of $f(x) = 3$

(ii) $x = -2$

(iii) $\dfrac{1}{(x^2 + 4x + 7)} = \dfrac{1}{3}$ (greatest value)

10. $y = -x^2 + 6x$
$= -(x^2 - 6x)$
$= -(x^2 - 6x + 9 - 9)$
$= -[(x - 3)^2 - 9]$
$= -(x - 3)^2 + 9$
maximum point $= (3, 9)$
greatest height $= 9$



11. (i) C : $x^2 - 6x + 8$
$= x^2 - 6x + 9 - 9 + 8$
$= (x - 3)^2 - 1$

(ii) B: $x^2 - 6x + 9$
$= (x - 3)^2$

(iii) A: $x^2 - 6x + 10$
$= x^2 - 6x + 9 - 9 + 10$
$= (x - 3)^2 + 1$

12. Curve C: Maximum point $= (2, 4) = (p, q)$
$\Rightarrow y = 4 - a(x - 2)^2$
point (0, 3) $\Rightarrow 3 = 4 - a(0 - 2)^2$
$\Rightarrow 3 = 4 - a(4)$
$\Rightarrow 4a = 1$
$\Rightarrow a = \dfrac{1}{4}$

Curve D: Maximum point $= (2, 4) = (p, q)$
$\Rightarrow y = 4 - a(x - 2)^2$
point (1, 3) $\Rightarrow 3 = 4 - a(1 - 2)^2$
$\Rightarrow 3 = 4 - a(1)$
$\Rightarrow a = 1$

13. Roots are $x = 6, x = -3$
Hence, $y = a(x - 6)(x + 3)$
$\Rightarrow y = a(x^2 - 3x - 18)$
point (1,10) $\Rightarrow 10 = a[(1)^2 - 3(1) - 18]$
$\Rightarrow 10 = a(1 - 3 - 18)$
$\Rightarrow 10 = -20a$
$\Rightarrow a = -\dfrac{1}{2}$

$\Rightarrow y = -\frac{1}{2}(x^2 - 3x - 18) = -\frac{1}{2}x^2 + \frac{3}{2}x + 9$

14. Minimum point $= (-1, 3)$

$\Rightarrow y = a(x + 1)^2 + 3$

point $(0, 4) \Rightarrow 4 = a(0 + 1)^2 + 3$

$\Rightarrow 4 = a + 3$

$\Rightarrow a = 1$

$\Rightarrow y = 1(x + 1)^2 + 3 \qquad = x^2 + 2x + 4$

15. (i) Maximum point $= (6, 4)$

$\Rightarrow f(x) = 4 - (0.1)(x - 6)^2$

(ii) $f(x) = 4 - (0.1)(x^2 - 12x + 36) = 0$

$\Rightarrow 40 - (x^2 - 12x + 36) = 0$

$\Rightarrow -x^2 + 12x + 4 = 0$

$$\Rightarrow x = \frac{-12 \pm \sqrt{144 - 4(-1)(4)}}{2(-1)}$$

$$= \frac{-12 \pm \sqrt{160}}{-2}$$

$$= \frac{-12 \pm 4\sqrt{10}}{-2}$$

$= 6 - 2\sqrt{10}, 6 + 2\sqrt{10}$

point where ball started is $(6 - 2\sqrt{10}, 0)$

and point where ball finished is $(6 + 2\sqrt{10}, 0)$

(iii) Horizontal Distance $= 6 + 2\sqrt{10} - (6 - 2\sqrt{10})$

$= 6 + 2\sqrt{10} - 6 + 2\sqrt{10}$

$= 4\sqrt{10}$

16. (i) Area of trapezium $= \frac{1}{2}(a + b)h$

$= \frac{1}{2}(x + 20)(2x)$

$= x(x + 20)$

$= x^2 + 20x$

(ii) Given: $x^2 + 20x = 400$

$x^2 + 20x + 100 = 400 + 100$

$(x + 10)^2 = 500$

$x + 10 = \pm\sqrt{500}$

$x + 10 = \pm 10\sqrt{5}$

$x = -10 + 10\sqrt{5} \qquad$ (as $x = -10 - 10\sqrt{5}$ is not possible)

$x = 10(\sqrt{5} - 1)$

17. $T = 36 + 4t - t^2$

$t^2 - 4t = -T + 36$

$t^2 - 4t + 4 = -T + 36 + 4$

$(t - 2)^2 = -T + 40$

$T = -(t - 2)^2 + 40$

The maximum temperature is then 40°.

Then

$T = 36$

$-(t - 2)^2 + 40 = 36$

$-(t - 2)^2 = -4$

$(t - 2)^2 = 4$

$(t - 2) = \pm\sqrt{4}$

$t = 2 \pm 2$

$t = 0 \quad$ or $\quad t = 4$

Thus it will take 4 hours for temperature returns to 36°

Exercise 2.7

1. (i) $\sqrt{8} = \sqrt{4}.\sqrt{2} = 2\sqrt{2}$

(ii) $\sqrt{27} = \sqrt{9}.\sqrt{3} = 3\sqrt{3}$

(iii) $\sqrt{45} = \sqrt{9}.\sqrt{5} = 3\sqrt{5}$

(iv) $\sqrt{200} = \sqrt{100}.\sqrt{2} = 10\sqrt{2}$

(v) $3\sqrt{18} = 3.\sqrt{9}.\sqrt{2} = 3.3\sqrt{2} = 9\sqrt{2}$

2. (i) $2\sqrt{2} + 6\sqrt{2} - 3\sqrt{2} = 5\sqrt{2}$

(ii) $2\sqrt{2} + \sqrt{18}$
$= 2\sqrt{2} + \sqrt{9}.\sqrt{2}$
$= 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$

(iii) $\sqrt{32} + \sqrt{18}$
$= \sqrt{16}.\sqrt{2} + \sqrt{9}.\sqrt{2}$
$= 4\sqrt{2} + 3\sqrt{2}$
$= 7\sqrt{2}$

(iv) $\sqrt{27} + \sqrt{48} - 2\sqrt{3}$
$= \sqrt{9}\sqrt{3} + \sqrt{16}\sqrt{3} - 2\sqrt{3}$
$= 3\sqrt{3} + 4\sqrt{3} - 2\sqrt{3} = 5\sqrt{3}$

(v) $\sqrt{8} + \sqrt{200} - \sqrt{18}$
$= \sqrt{4}\sqrt{2} + \sqrt{100}\sqrt{2} - \sqrt{9}\sqrt{2}$
$= 2\sqrt{2} + 10\sqrt{2} - 3\sqrt{2}$
$= 9\sqrt{2}$

(vi) $7\sqrt{5} + 2\sqrt{20} - \sqrt{80}$
$= 7\sqrt{5} + 2\sqrt{4}.\sqrt{5} - \sqrt{16}.\sqrt{5}$
$= 7\sqrt{5} + 4\sqrt{5} - 4\sqrt{5}$
$= 7\sqrt{5}$

3. (i) $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(ii) $\dfrac{2}{\sqrt{8}} = \dfrac{2}{\sqrt{4}\sqrt{2}} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

(iii) $\dfrac{2}{5\sqrt{2}} = \dfrac{2}{5\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{10} = \dfrac{\sqrt{2}}{5}$

(iv) $\dfrac{20}{\sqrt{50}} = \dfrac{20}{\sqrt{25}\sqrt{2}} = \dfrac{20}{5\sqrt{2}} = \dfrac{4}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$

(v) $\dfrac{8}{\sqrt{128}} = \dfrac{8}{\sqrt{64}\sqrt{2}} = \dfrac{8}{8\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

4. (i) $\sqrt{8} \times \sqrt{12} = \sqrt{96} = \sqrt{16}\sqrt{6} = 4\sqrt{6}$

(ii) $3\sqrt{2} \times 5\sqrt{2} = 15.\sqrt{4} = 15.2 = 30$

(iii) $\sqrt{2}(\sqrt{6} + 3\sqrt{2}) = \sqrt{12} + 3\sqrt{4} = \sqrt{4}\sqrt{3} + 3.2 = 2\sqrt{3} + 6$

(iv) $(5 - \sqrt{3})(5 + \sqrt{3})$
$= 5(5 + \sqrt{3}) - \sqrt{3}(5 + \sqrt{3})$
$= 25 + 5\sqrt{3} - 5\sqrt{3} - \sqrt{9}$
$= 25 - 3 = 22$

(v) $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})$
$= \sqrt{7}(\sqrt{7} - \sqrt{5}) + \sqrt{5}(\sqrt{7} - \sqrt{5})$
$= \sqrt{49} - \sqrt{35} + \sqrt{35} - \sqrt{25} = 7 - 5 = 2$

(vi) $(a + 2\sqrt{b})(a - 2\sqrt{b})$
$= a(a - 2\sqrt{b}) + 2\sqrt{b}(a - 2\sqrt{b})$
$= a^2 - 2a\sqrt{b} + 2a\sqrt{b} - 4\sqrt{b^2} = a^2 - 4b$

5. (i) $\dfrac{4}{\sqrt{5} + 1} . \dfrac{\sqrt{5} - 1}{\sqrt{5} - 1} = \dfrac{4(\sqrt{5} - 1)}{\sqrt{25} - 1} = \dfrac{4(\sqrt{5} - 1)}{5 - 1} = \dfrac{4(\sqrt{5} - 1)}{4} = \sqrt{5} - 1$

(ii) $\dfrac{12}{3 - \sqrt{2}} . \dfrac{3 + \sqrt{2}}{3 + \sqrt{2}} = \dfrac{36 + 12\sqrt{2}}{9 - \sqrt{4}} = \dfrac{36 + 12\sqrt{2}}{7}$

(iii) $\dfrac{2 - \sqrt{5}}{2 + \sqrt{5}} . \dfrac{2 - \sqrt{5}}{2 - \sqrt{5}} = \dfrac{4 - 2\sqrt{5} - 2\sqrt{5} + \sqrt{25}}{4 - \sqrt{25}}$
$= \dfrac{9 - 4\sqrt{5}}{-1} = -9 + 4\sqrt{5}$

(iv) $\dfrac{1}{\sqrt{8} - \sqrt{2}} = \dfrac{1}{\sqrt{4}\sqrt{2} - \sqrt{2}} = \dfrac{1}{2\sqrt{2} - \sqrt{2}} = \dfrac{1}{\sqrt{2}} . \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{4}} = \dfrac{\sqrt{2}}{2}$

6. (i) $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} = \frac{1(\sqrt{2}+1) - 1(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$= \frac{\sqrt{2}+1-\sqrt{2}+1}{\sqrt{4}+\sqrt{2}-\sqrt{2}-1} = \frac{2}{2-1} = 2$

(ii) $\frac{1}{2+\sqrt{3}} + \frac{1}{2-\sqrt{3}} = \frac{1(2-\sqrt{3}) + 1(2+\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$

$= \frac{2-\sqrt{3}+2+\sqrt{3}}{4-2\sqrt{3}+2\sqrt{3}-\sqrt{9}}$

$= \frac{4}{4-3} = 4$

7. (i) $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+\sqrt{5})$

$= 2\sqrt{3}(2\sqrt{3}+\sqrt{5}) - \sqrt{5}(2\sqrt{3}+\sqrt{5})$

$= 4.3 + 2\sqrt{15} - 2\sqrt{15} - \sqrt{25} = 12 - 5 = 7$

(ii) $\frac{4}{2-\sqrt{5}} + \frac{2}{2+\sqrt{5}} = \frac{4(2+\sqrt{5}) + 2(2-\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})}$

$= \frac{8+4\sqrt{5}+4-2\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-\sqrt{25}}$

$= \frac{12+2\sqrt{5}}{-1}$

$= -12 - 2\sqrt{5}$

8. (i) $X + Y = \frac{4+\sqrt{3}}{\sqrt{2}} + \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

(ii) $X - Y = 4\frac{+\sqrt{3}}{\sqrt{2}} - \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{4+\sqrt{3}-4+\sqrt{3}}{\sqrt{2}}$

$= \frac{2\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{6}}{2} = \sqrt{6}$

(iii) $XY = \frac{4+\sqrt{3}}{\sqrt{2}} \cdot \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{16-4\sqrt{3}+4\sqrt{3}-\sqrt{9}}{2} = \frac{16-3}{2} = \frac{13}{2}$

(iv) $\frac{X}{Y} = \frac{\frac{4+\sqrt{3}}{\sqrt{2}}}{\frac{4-\sqrt{3}}{\sqrt{2}}} = \frac{4+\sqrt{3}}{4-\sqrt{3}} \cdot \frac{4+\sqrt{3}}{4+\sqrt{3}} = \frac{16+4\sqrt{3}+4\sqrt{3}+\sqrt{9}}{16+4\sqrt{3}-4\sqrt{3}-\sqrt{9}}$

$= \frac{19+8\sqrt{3}}{13}$

9. $(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})$

$= 2\sqrt{5}(2\sqrt{5}+3\sqrt{2}) - 3\sqrt{2}(2\sqrt{5}+3\sqrt{2})$

$= 4\sqrt{25} + 6\sqrt{10} - 6\sqrt{10} - 9\sqrt{4}$

$= 20 - 18 = 2$

10. $\frac{5}{2+\sqrt{3}} = \frac{5}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \frac{5(2-\sqrt{3})}{4-3} = 5(2-\sqrt{3})$

11. $\frac{2+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{2-2\sqrt{2}+\sqrt{2}-2}{1+\sqrt{2}-\sqrt{2}-2} = \frac{-\sqrt{2}}{-1} = \sqrt{2}$

$\frac{\sqrt{5}-2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{5+\sqrt{5}-2\sqrt{5}-2}{5+\sqrt{5}-\sqrt{5}-1} = \frac{3-\sqrt{5}}{4}$

Hence, $\sqrt{2} \cdot \frac{(3-\sqrt{5})}{4} \cdot (3+\sqrt{5}) = \frac{\sqrt{2}}{4}[3(3+\sqrt{5}) - \sqrt{5}(3+\sqrt{5})]$

$= \frac{\sqrt{2}}{4}[9 + 3\sqrt{5} - 3\sqrt{5} - 5]$

$= \frac{\sqrt{2}}{4}[4] = \sqrt{2}$

12. $\dfrac{-1+\sqrt{3}}{1+\sqrt{3}} = \dfrac{-1+\sqrt{3}}{1+\sqrt{3}} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$= \dfrac{-1+\sqrt{3}+\sqrt{3}-3}{(1)^2-(\sqrt{3})^2}$

$= \dfrac{-4+2\sqrt{3}}{1-3}$

$= \dfrac{-4+2\sqrt{3}}{-2}$

$= 2-\sqrt{3}$

13. $\dfrac{\sqrt{3}}{1-\sqrt{3}} - \dfrac{1}{\sqrt{3}} = \dfrac{(\sqrt{3})^2-(1-\sqrt{3})}{\sqrt{3}(1-\sqrt{3})}$

$= \dfrac{3-1+\sqrt{3}}{\sqrt{3}-3}$

$= \dfrac{2+\sqrt{3}}{\sqrt{3}-3} \times \dfrac{\sqrt{3}+3}{\sqrt{3}+3}$

$= \dfrac{2\sqrt{3}+6+3+3\sqrt{3}}{(\sqrt{3})^2-9}$

$= \dfrac{9+5\sqrt{3}}{3-9}$

$= \dfrac{9+5\sqrt{3}}{-6}$

$= \dfrac{-9-5\sqrt{3}}{6}$

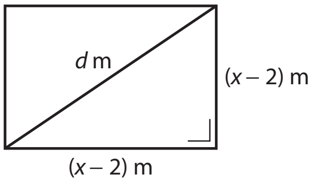
Exercise 2.8

1. $d^2 = (x+2)^2 + (x-2)^2$

$= x^2 + 4x + 4 + x^2 - 4x + 4$

$= 2x^2 + 8$

$\Rightarrow d = \sqrt{2x^2+8}$



2. (a) $|AC|^2 = (2+\sqrt{3})^2 + (2-\sqrt{3})^2$

$= 4 + 4\sqrt{3} + 3 + 4 - 4\sqrt{3} - 3$

$= 14$

$\Rightarrow |AC| = \sqrt{14}$

(b) (i) Distance, 1st Runner $= 2-\sqrt{3}+2+\sqrt{3}+2-\sqrt{3}+2+\sqrt{3} = 8$

Distance, 2nd Runner $= 2\sqrt{14}$

$\Rightarrow$ Difference $= 8 - 2\sqrt{14} = 2(4-\sqrt{14})$km

(ii) Time, 1st Runner $= \dfrac{8000}{1.5} = 5333.33$

Time, 2nd Runner $= \dfrac{2\sqrt{14}\times 1000}{1.4} = 5345.22$

$\Rightarrow$ Difference $= 5345.22 - 5333.33 = 11.89 = 12$ secs

3. $|EG|^2 = (4)^2 + (2)^2 = 16 + 4 = 20$

$\Rightarrow |EG| = \sqrt{20}$

$|KG|^2 = (\sqrt{20})^2 + (2)^2 = 20 + 4 = 24$

$\Rightarrow |KG| = \sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Hence: Distance $= 4 + 2 + 2 + 2\sqrt{6} = 8 + 2\sqrt{6}$

4. (i) $x + y = \sqrt{a} + \frac{1}{\sqrt{a}} + \sqrt{a} - \frac{1}{\sqrt{a}} = 2\sqrt{a}$

(ii) $x - y = \sqrt{a} + \frac{1}{\sqrt{a}} - \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)$

$= \sqrt{a} + \frac{1}{\sqrt{a}} - \sqrt{a} + \frac{1}{\sqrt{a}} = \frac{2}{\sqrt{a}}$

Hence, $\sqrt{x^2 - y^2} = \sqrt{(x + y)(x - y)}$

$= \sqrt{2\sqrt{a} \cdot \frac{2}{\sqrt{a}}} = \sqrt{4} = 2$

5. (i) $\sqrt{2x + 1} = 3$

$\Rightarrow 2x + 1 = 9$

$\Rightarrow 2x = 8$

$\Rightarrow x = 4$

Check $x = 4 \Rightarrow \sqrt{2(4) + 1} = \sqrt{9} = 3$ True

(ii) $\sqrt{3x + 10} = x$

$\Rightarrow 3x + 10 = x^2$

$\Rightarrow x^2 - 3x - 10 = 0$

$\Rightarrow (x + 2)(x - 5) = 0$

$\Rightarrow x = -2, x = 5$

Check $x = -2 \Rightarrow \sqrt{3(-2) + 10} = \sqrt{4} = 2 \neq -2$ False

Check $x = 5 \Rightarrow \sqrt{3(5) + 10} = \sqrt{25} = 5$ True

(iii) $\sqrt{2x - 1} = \sqrt{x + 8}$

$\Rightarrow 2x - 1 = x + 8$

$\Rightarrow \quad x = 9$

Check $x = 9 \Rightarrow \sqrt{2(9) - 1} = \sqrt{9 + 8}$

$\Rightarrow \quad \sqrt{17} = \sqrt{17}$ True

(iv) $\sqrt{3x - 5} = x - 1$

$\Rightarrow (\sqrt{3x - 5})^2 = (x - 1)^2$

$\Rightarrow 3x - 5 = x^2 - 2x + 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x = 2, x = 3$

Check $x = 2 \Rightarrow \sqrt{3(2) - 5} = 2 - 1$

$\Rightarrow \quad 1 = 1$ True

Check $x = 3 \Rightarrow \sqrt{3(3) - 5} = 3 - 1$

$\Rightarrow \quad 2 = 2$ True

(v) $\sqrt{2x + 5} = x + 1$

$\Rightarrow (\sqrt{2x - 5})^2 = (x + 1)^2$

$\Rightarrow 2x + 5 = x^2 + 2x + 1$

$\Rightarrow x^2 = 4$

$\Rightarrow x = -2, x = 2$

Check $x = -2 \Rightarrow \sqrt{2(-2) + 5} = -2 + 1$

$\Rightarrow \quad 1 = -1$ False

Check $x = 2 \Rightarrow \sqrt{2(2) + 5} = 2 + 1$

$\Rightarrow \quad 3 = 3$ True

(vi) $\sqrt{2x^2 - 7} = x + 3$

$\Rightarrow (\sqrt{2x^2 - 7})^2 = (x + 3)^2$

$\Rightarrow 2x^2 - 7 = x^2 + 6x + 9$

$\Rightarrow x^2 - 6x - 16 = 0$

$\Rightarrow (x + 2)(x - 8) = 0$

$\Rightarrow x = -2, x = 8$

Check $x = -2 \Rightarrow \sqrt{2(-2)^2 - 7} = -2 + 3$

$\Rightarrow \quad 1 = 1$ True

Check $x = 8 \Rightarrow \sqrt{2(8)^2 - 7} = 8 + 3$

$\Rightarrow \quad 11 = 11$ True

6. (i) $\sqrt{x + 5} = 5 - \sqrt{x}$

$\Rightarrow (\sqrt{x + 5})^2 = (5 - \sqrt{x})^2$

$\Rightarrow x + 5 = 25 - 10\sqrt{x} + x$

$\Rightarrow 10\sqrt{x} = 20$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4$

Check $x = 4 \Rightarrow \sqrt{4 + 5} = 5 - \sqrt{4}$

$\Rightarrow \quad 3 = 3$ True

(ii) $\sqrt{5x+6} = \sqrt{2x} + 2$

$\Rightarrow (\sqrt{5x+6})^2 = (\sqrt{2x}+2)^2$

$\Rightarrow 5x + 6 = 2x + 4\sqrt{2x} + 4$

$\Rightarrow 3x + 2 = 4\sqrt{2x}$

$\Rightarrow (3x+2)^2 = (4\sqrt{2x})^2$

$\Rightarrow 9x^2 + 12x + 4 = 16(2x)$

$\Rightarrow 9x^2 - 20x + 4 = 0$

$\Rightarrow (9x-2)(x-2) = 0$

$\Rightarrow x = \frac{2}{9}, x = 2$

Check $x = \frac{2}{9} \Rightarrow \sqrt{5\left(\frac{2}{9}\right) + 6} = \sqrt{\frac{4}{9}} + 2$

$\Rightarrow \sqrt{\frac{64}{9}} = \frac{2}{3} + 2$

$\Rightarrow \frac{8}{3} = 2\frac{2}{3}$ True

Check $x = 2 \Rightarrow \sqrt{5(2)+6} = \sqrt{2(2)} + 2$

$\Rightarrow \sqrt{16} = 2 + 2$

$\Rightarrow 4 = 4$ True

(iii) $\sqrt{x+7} = 7 - \sqrt{x}$

$\Rightarrow (\sqrt{x+7})^2 = (7-\sqrt{x})^2$

$\Rightarrow x + 7 = 49 - 14\sqrt{x} + x$

$\Rightarrow 14\sqrt{x} = 42$

$\Rightarrow \sqrt{x} = 3$

$\Rightarrow x = 9$

Check $x = 9 \Rightarrow \sqrt{9+7} + \sqrt{9} = 7$

$\Rightarrow 4 + 3 = 7$ True

(iv) $\sqrt{3x-2} = \sqrt{x-2} + 2$

$\Rightarrow (\sqrt{3x-2})^2 = (\sqrt{x-2}+2)^2$

$\Rightarrow 3x - 2 = x - 2 + 4\sqrt{x-2} + 4$

$\Rightarrow 2x - 4 = 4\sqrt{x-2}$

$\Rightarrow x - 2 = 2\sqrt{x-2}$

$\Rightarrow (x-2)^2 = (2\sqrt{x-2})^2$

$\Rightarrow x^2 - 4x + 4 = 4(x-2) = 4x - 8$

$\Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x-2)(x-6) = 0$

$\Rightarrow x = 2, x = 6$

Check $x = 2 \Rightarrow \sqrt{3(2)-2} = \sqrt{2-2} + 2$

$\Rightarrow \sqrt{4} = 0 + 2$

$\Rightarrow 2 = 2$ True

Check $x = 6 \Rightarrow \sqrt{3(6)-2} = \sqrt{6-2} + 2$

$\Rightarrow \sqrt{16} = \sqrt{4} + 2$

$\Rightarrow 4 = 2 + 2$ True

7. $\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4x+8}} = \frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4(x+2)}}$

$= \frac{1}{\sqrt{x+2}} - \frac{1}{2\sqrt{x+2}}$

$= \frac{2-1}{2\sqrt{x+2}}$

$= \frac{1}{2\sqrt{x+2}}$

Thus

$\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4x+8}} = 2$

$\frac{1}{2\sqrt{x+2}} = 2$

$1 = 4\sqrt{x+2}$

$1 = 16(x+2)$

$1 = 16x + 32$

$-31 = 16x$

$x = \frac{-31}{16}$

8. $x = \sqrt{4x + 5}$

$x^2 = 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5 \quad \text{or} \quad x = -1$

check: $x = 5$: $5 = \sqrt{25}$

$5 = 5$ True.

$x = -1$: $-1 = \sqrt{1}$

$-1 = 1$ False.

Thus the only solution is $x = 5$.

9. $x = \sqrt{a} + \frac{1}{\sqrt{a}} + 1$

$\Rightarrow x - 2 = \sqrt{a} + \frac{1}{\sqrt{a}} + 1 - 2 = \sqrt{a} + \frac{1}{\sqrt{a}} - 1$

Hence, $x^2 - 2x = x(x - 2)$

$$= \left(\sqrt{a} + \frac{1}{\sqrt{a}} + 1\right)\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right)$$

$$= \sqrt{a}\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right) + \frac{1}{\sqrt{a}}\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right) + 1\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right)$$

$$= a + 1 - \sqrt{a} + 1 + \frac{1}{a} - \frac{1}{\sqrt{a}} + \sqrt{a} + \frac{1}{\sqrt{a}} - 1$$

$$= a + \frac{1}{a} + 1$$

10. $(a + \sqrt{3})(b - \sqrt{3}) = 7 + 3\sqrt{3}$

$\Rightarrow ab - \sqrt{3}a + \sqrt{3}b - 3 = 7 + 3\sqrt{3}$

$\Rightarrow (ab - 3) + (-a + b)\sqrt{3} = 7 + 3\sqrt{3}$

$\Rightarrow ab - 3 = 7 \text{ and } -a + b = 3$

$\Rightarrow \quad b = a + 3$

$\Rightarrow a(a + 3) - 3 - 7 = 0$

$\Rightarrow a^2 + 3a - 10 = 0$

$\Rightarrow (a + 5)(a - 2) = 0$

$\Rightarrow a = -5$ (Not valid), $a = 2$ (Valid)

$\Rightarrow b = 2 + 3 = 5$

11. (i) $|IC|^2 = (x + 2)^2 + (x - 2)^2$

$= x^2 + 4x + 4 + x^2 - 4x + 4$

$= 2x^2 + 8$

$\Rightarrow |IC| = \sqrt{2x^2 + 8}$

(ii) $|ID|^2 = (\sqrt{2x^2 + 8})^2 + x^2$

$= 2x^2 + 8 + x^2$

$= 3x^2 + 8$

$\Rightarrow |ID| = \sqrt{3x^2 + 8}$

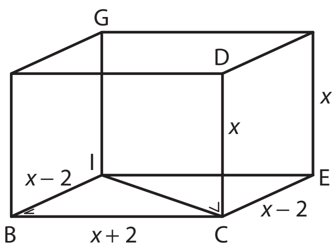
Hence, $\sqrt{3x^2 + 8} = \sqrt{56}$

$\Rightarrow \quad 3x^2 + 8 = 56$

$\Rightarrow \quad 3x^2 = 48$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow x = -4$ (Not valid), $x = 4$ (Valid)



Exercises 2.9

1. $f(x) = x^2 - 8x + 15$ If $(x - 3)$ is a factor, then $x = 3$ is a root.

$\Rightarrow f(3) = (3)^2 - 8(3) + 15$

$= 9 - 24 + 15$

$= 24 - 24 = 0 \quad \Rightarrow (x - 3)$ is a factor

2. If $(x - 1)$ is a factor, then $x = 1$ is a root.

$f(x) = x^3 - x^2 - 9x + 9$

$\Rightarrow f(1) = (1)^3 - (1)^2 - 9(1) + 9$

$= 1 - 1 - 9 + 9 = 10 - 10 = 0$

$\Rightarrow (x - 1)$ is a factor

3. If $(x + 2)$ is a factor, then $x = -2$ is a root.

$f(x) = x^3 + 6x^2 + 11x + 6$

$\Rightarrow f(-2) = (-2)^3 + 6(-2)^2 + 11(-2) + 6$

$= -8 + 24 - 22 + 6$

$= 30 - 30 = 0$

$\Rightarrow (x + 2)$ is a factor

4. If $(x - 2)$ is a factor, then $x = 2$ is a root.

$f(x) = 2x^3 - 3x^2 - 12x + 20$

$\Rightarrow f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 20$

$= 16 - 12 - 24 + 20$

$= 36 - 36 = 0 \quad \Rightarrow x - 2$ is a factor

5. If $(x - 2)$ is a factor, then $x = 2$ is a root.

$f(x) = x^3 - 5x^2 + 8x - 4$

$\Rightarrow f(2) = (2)^3 - 5(2)^2 + 8(2) - 4$

$= 8 - 20 + 16 - 4$

$= 24 - 24 = 0 \quad \Rightarrow (x - 2)$ is a factor

6. If $(2x - 1)$ is a factor $\Rightarrow x = \frac{1}{2}$ is a root.

$f(x) = 2x^3 + 7x^2 + 2x - 3$

$\Rightarrow f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 3$

$= \frac{1}{4} + \frac{7}{4} + 1 - 3$

$= 3 - 3 = 0 \quad \Rightarrow (2x - 1)$ is a factor

7. If $(2x + 1)$ is a factor $\Rightarrow x = -\frac{1}{2}$ is a root.

$f(x) = 2x^3 - x^2 - 5x - 2$

$\Rightarrow f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 2$

$= -\frac{1}{4} - \frac{1}{4} + \frac{5}{2} - 2$

$= \frac{5}{2} - \frac{5}{2} = 0 \quad \Rightarrow (2x + 1)$ is a factor

8. If $(x - 1)$ is a factor $\Rightarrow x = 1$ is a root.

$f(x) = x^3 + kx^2 - x - 8$

$\Rightarrow f(1) = (1)^3 + k(1)^2 - (1) - 8 = 0$

$\Rightarrow 1 + k - 1 - 8 = 0$

$\Rightarrow k = 8$

9. If $(x + 2)$ is a factor $\Rightarrow x = -2$ is a root.

$f(x) = x^3 + 6x^2 + px + 6$

$\Rightarrow f(-2) = (-2)^3 + 6(-2)^2 + p(-2) + 6 = 0$

$\Rightarrow -8 + 24 - 2p + 6 = 0$

$-2p = -22$

$\Rightarrow p = 11$

10. If $(x - 3)$ is a factor $\Rightarrow x = 3$ is a root.

$f(x) = x^3 - 2x^2 - 5x + 6$

$\Rightarrow f(3) = (3)^3 - 2(3)^2 - 5(3) + 6$

$= 27 - 18 - 15 + 6 = 33 - 33 = 0 \Rightarrow x = 3$ is a root.

$$\begin{array}{r|l} & x^2 + x - 2 \\ \hline x - 3 & x^3 - 2x^2 - 5x + 6 \\ & \underline{x^3 - 3x^2} \\ & \quad x^2 - 5x \\ & \quad \underline{x^2 - 3x} \\ & \qquad -2x + 6 \\ & \qquad \underline{-2x + 6} \\ & \qquad\qquad 0 \end{array}$$

$\Rightarrow x^2 + x - 2$

$= (x + 2)(x - 1)$

$\therefore$ the other two factors are $(x + 2)$ and $(x - 1)$

11. If $(x + 3)$ is a factor $\Rightarrow x = -3$ is a root.

$f(x) = x^3 - 2x^2 - 9x + 18$

$\Rightarrow f(-3) = (-3)^3 - 2(-3)^2 - 9(-3) + 18$

$= -27 - 18 + 27 + 18 = 45 - 45 = 0 \Rightarrow x = -3$ is a root.

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x + 3 & x^3 - 2x^2 - 9x + 18 \\ & \underline{x^3 + 3x^2} \\ & \quad -5x^2 - 9x \\ & \quad \underline{-5x^2 - 15x} \\ & \qquad\quad 6x + 18 \\ & \qquad\quad \underline{6x + 18} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 - 5x + 6$

$= (x - 2)(x - 3)$

$\therefore$ the other two factors are $(x - 2)$ and $(x - 3)$

12. (i) $f(x) = x^3 - 4x^2 - x + 4$

$\Rightarrow f(1) = (1)^3 - 4(1)^2 - (1) + 4$

$= 1 - 4 - 1 + 4 = 5 - 5 = 0$

$\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r|l} & x^2 - 3x - 4 \\ \hline x - 1 & x^3 - 4x^2 - x + 4 \\ & \underline{x^3 - x^2} \\ & \quad -3x^2 - x \\ & \quad \underline{-3x^2 + 3x} \\ & \qquad\quad -4x + 4 \\ & \qquad\quad \underline{-4x + 4} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 - 3x - 4 = (x + 1)(x - 4)$

(ii) $f(x) = x^3 - 8x^2 + 19x - 12$

$\Rightarrow f(1) = (1)^3 - 8(1)^2 + 19(1) - 12$

$= 1 - 8 + 19 - 12 = 20 - 20 = 0$ $\quad\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ \hline x - 1 & x^3 - 8x^2 + 19x - 12 \\ & \underline{x^3 - x^2} \\ & \quad -7x^2 + 19x \\ & \quad \underline{-7x^2 + 7x} \\ & \qquad\quad 12x - 12 \\ & \qquad\quad \underline{12x - 12} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 - 7x + 12$

$= (x - 3)(x - 4)$

(iii) $f(x) = x^3 + 6x^2 - x - 30$

$\Rightarrow f(2) = (2)^3 + 6(2)^2 - (2) - 30$

$= 8 + 24 - 2 - 32 = 32 - 32 = 0$ $\quad\Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$\begin{array}{r|l} & x^2 + 8x + 15 \\ \hline x - 2 & x^3 + 6x^2 - x - 30 \\ & \underline{x^3 - 2x^2} \\ & \quad 8x^2 - x \\ & \quad \underline{8x^2 - 16x} \\ & \qquad\quad 15x - 30 \\ & \qquad\quad \underline{15x - 30} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 + 8x + 15$

$= (x + 3)(x + 5)$

(iv) $f(x) = 3x^3 - 4x^2 - 3x + 4$

$\Rightarrow f(1) = 3(1)^3 - 4(1)^2 - 3(1) + 4$

$= 3 - 4 - 3 + 4 = 7 - 7 = 0$ $\quad\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r|l} & 3x^2 - x - 4 \\ \hline x - 1 & 3x^3 - 4x^2 - 3x + 4 \\ & \underline{3x^3 - 3x^2} \\ & \quad -x^2 - 3x \\ & \quad \underline{-x^2 + x} \\ & \qquad\quad -4x + 4 \\ & \qquad\quad \underline{-4x + 4} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow 3x^2 - x - 4$

$= (3x - 4)(x + 1)$

(v) $f(x) = 2x^3 - 3x^2 - 8x - 3$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 8(-1) - 3$

$= -2 - 3 + 8 - 3 = 8 - 8 = 0$ $\quad\Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$\begin{array}{r|l} & 2x^2 - 5x - 3 \\ \hline x + 1 & 2x^3 - 3x^2 - 8x - 3 \\ & \underline{2x^3 + 2x^2} \\ & \quad -5x^2 - 8x \\ & \quad \underline{-5x^2 - 5x} \\ & \qquad\quad -3x - 3 \\ & \qquad\quad \underline{-3x - 3} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow 2x^2 - 5x - 3$

$= (2x + 1)(x - 3)$

(vi) $f(x) = 2x^3 - 3x^2 - 12x + 20$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 20$

$= 16 - 12 - 24 + 20 = 36 - 36 = 0$ $\qquad \Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$
\begin{array}{r l}
 & 2x^2 + x - 10 \\
x - 2 & \overline{)\,2x^3 - 3x^2 - 12x + 20} \\
 & \underline{2x^3 - 4x^2} \\
 & \quad x^2 - 12x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad -10x + 20 \\
 & \qquad \underline{-10x + 20} \\
 & \qquad \qquad 0
\end{array}
$$

$\Rightarrow 2x^2 + x - 10$

$\Rightarrow (2x + 5)(x - 2)$

13. $f(x) = 2x^3 + 13x^2 + 13x - 10$

$f(-2) = 2(-2)^3 + 13(-2)^2 + 13(-2) - 10$

$= -16 + 52 - 26 - 10 = 52 - 52 = 0$ $\qquad \Rightarrow x = -2$ is a root

$\Rightarrow (x + 2)$ is a factor.

$$
\begin{array}{r l}
 & 2x^2 + 9x - 5 \\
x + 2 & \overline{)\,2x^3 + 13x^2 + 13x - 10} \\
 & \underline{2x^3 + 4x^2} \\
 & \quad 9x^2 + 13x \\
 & \quad \underline{9x^2 + 18x} \\
 & \qquad -5x - 10 \\
 & \qquad \underline{-5x - 10} \\
 & \qquad \qquad 0
\end{array}
$$

$\Rightarrow 2x^2 + 9x - 5$

$\Rightarrow (2x - 1)(x + 5)$

14. $(x + 2)$ is a factor $\quad \Rightarrow x = -2$ is a root

$f(x) = x^3 + ax^2 - x - 2$

$\Rightarrow f(-2) = (-2)^3 + a(-2)^2 - (-2) - 2 = 0$

$\Rightarrow -8 + 4a + 2 - 2 = 0$

$\Rightarrow 4a = 8$

$\Rightarrow a = 2$

$$
\begin{array}{r l}
 & x^2 - 1 \\
x + 2 & \overline{)\,x^3 + 2x^2 - x - 2} \\
 & \underline{x^3 + 2x^2} \\
 & \qquad 0 - x - 2 \\
 & \qquad \underline{-x - 2} \\
 & \qquad \qquad 0
\end{array}
$$

$\Rightarrow x^2 - 1 = (x + 1)(x - 1)$

15. $f(x) = x^3 - x^2 - 14x + 24$

$f(2) = (2)^3 - (2)^2 - 14\,(2) + 24$

$= 8 - 4 - 28 + 24 = 32 - 32 = 0$ $\qquad \Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$
\begin{array}{r l}
 & x^2 + x - 12 \\
x - 2 & \overline{)\,x^3 - x^2 - 14x + 24} \\
 & \underline{x^3 - 2x^2} \\
 & \quad x^2 - 14x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad -12x + 24 \\
 & \qquad \underline{-12x + 24} \\
 & \qquad \qquad 0
\end{array}
$$

$\Rightarrow x^2 + x - 12$

$= (x - 3)(x + 4)$

Hence, roots $x = 2, 3, -4$

16. $f(x) = x^3 + 5x^2 + 2x - 8$

$\Rightarrow f(1) = (1)^3 + 5(1)^2 + 2(1) - 8$

$= 1 + 5 + 2 - 8 = 8 - 8 = 0$ $\quad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$
\begin{array}{r|l}
 & x^2+6x+8 \\ \hline
x-1 & x^3+5x^2+2x-8 \\
 & \underline{x^3-x^2} \\
 & \quad 6x^2+2x \\
 & \quad \underline{6x^2-6x} \\
 & \qquad 8x-8 \\
 & \qquad \underline{8x-8} \\
 & \qquad\quad 0
\end{array}
$$

$\Rightarrow x^2 + 6x + 8 = 0$

$\Rightarrow (x + 2)(x + 4) = 0$

$\Rightarrow$ roots $x = -2, -4$

17. (i) $f(x) = x^3 - 4x^2 - x + 4$

$f(1) = (1)^3 - 4(1)^2 - (1) + 4$

$= 1 - 4 - 1 + 4 = 5 - 5 = 0$ $\quad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$
\begin{array}{r|l}
 & x^2-3x-4 \\ \hline
x-1 & x^3-4x^2-x+4 \\
 & \underline{x^3-x^2} \\
 & \quad -3x^2-x \\
 & \quad \underline{-3x^2+3x} \\
 & \qquad -4x+4 \\
 & \qquad \underline{-4x+4} \\
 & \qquad\quad 0
\end{array}
$$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x + 1)(x - 4) = 0$

$\Rightarrow$ roots $x = -1, 4$

(ii) $f(x) = x^3 + 2x^2 - 11x - 12$

$f(-1) = (-1)^3 + 2(-1)^2 - 11(-1) - 12$

$= -1 + 2 + 11 - 12 = 13 - 13 = 0$ $\quad \Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$
\begin{array}{r|l}
 & x^2+x-12 \\ \hline
x+1 & x^3+2x^2-11x-12 \\
 & \underline{x^3+x^2} \\
 & \quad x^2-11x \\
 & \quad \underline{x^2+x} \\
 & \qquad -12x-12 \\
 & \qquad \underline{-12x-12} \\
 & \qquad\quad 0
\end{array}
$$

$\Rightarrow x^2 + x - 12 = 0$

$\Rightarrow (x + 4)(x - 3) = 0$

$\Rightarrow$ roots $x = -4, 3$

(iii) $f(x) = 3x^3 - 4x^2 - 3x + 4$

$f(1) = 3(1)^3 - 4(1)^2 - 3(1) + 4$

$= 3 - 4 - 3 + 4 = 7 - 7 = 0$ $\quad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$
\begin{array}{r|l}
 & 3x^2-x-4 \\ \hline
x-1 & 3x^3-4x^2-3x+4 \\
 & \underline{3x^3-3x^2} \\
 & \quad -x^2-3x \\
 & \quad \underline{-x^2+x} \\
 & \qquad -4x+4 \\
 & \qquad \underline{-4x+4} \\
 & \qquad\quad 0
\end{array}
$$

$\Rightarrow 3x^2 - x - 4 = 0$

$\Rightarrow (3x - 4)(x + 1) = 0$

$3x = 4$ or $x = -1$

$\Rightarrow$ roots $x = \frac{4}{3}, -1$

(iv) $f(x) = x^3 - 7x - 6$

$f(-1) = (-1)^3 - 7(-1) - 6$

$= -1 + 7 - 6 = 7 - 7 = 0$ $\Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x + 1 & x^3 - 7x - 6 \\ & \underline{x^3 + x^2} \\ & \quad -x^2 - 7x \\ & \quad \underline{-x^2 - x} \\ & \qquad -6x - 6 \\ & \qquad \underline{-6x - 6} \\ & \qquad\qquad 0 \end{array}$$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow$ roots $x = -2, 3$

18. If $(x + 1)$ is a factor $\Rightarrow x = -1$ is a root

$f(x) = 2x^3 + ax^2 + bx - 3$

$f(-1) = 2(-1)^3 + a(-1)^2 + b(-1) - 3 = 0$

$\Rightarrow -2 + a - b - 3 = 0 \Rightarrow a - b = 5$

If $(x + 3)$ is a factor $\Rightarrow x = -3$ is a root

$\Rightarrow f(-3) = 2(-3)^3 + a(-3)^2 + b(-3) - 3 = 0$

$\Rightarrow -54 + 9a - 3b - 3 = 0$

$\Rightarrow$ $9a - 3b = 57$

$\Rightarrow$ $3a - b = 19$

and $\underline{a - b = 5}$

$\Rightarrow$ $2a = 14$

$\Rightarrow$ $a = 7$

$\Rightarrow$ $7 - b = 5$

$\Rightarrow$ $b = 2$

Hence, $(x + 1)(x + 3) = x^2 + 4x + 3$

$$\begin{array}{r|l} & 2x - 1 \\ \hline \Rightarrow x^2 + 4x + 3 & 2x^3 + 7x^2 + 2x - 3 \\ & \underline{2x^3 + 8x^2 + 6x} \\ & \quad -x^2 - 4x - 3 \\ & \quad \underline{-x^2 - 4x - 3} \\ & \qquad\qquad 0 \end{array}$$

$2x - 1 = 0$

$\Rightarrow 3^{rd}$ root $x = \frac{1}{2}$

19. If $(x + 1)$ is a factor $\Rightarrow x = -1$ is a root

$f(x) = x^3 + 5x^2 + kx - 12$

$f(-1) = (-1)^3 + 5(-1)^2 + k(-1) - 12 = 0$

$\Rightarrow -1 + 5 - k - 12 = 0$

$-k - 8 = 0$

$\Rightarrow k = -8$

$$\begin{array}{r|l} & x^2 + 4x - 12 \\ \hline \Rightarrow x + 1 & x^3 + 5x^2 - 8x - 12 \\ & \underline{x^3 + x^2} \\ & \quad 4x^2 - 8x \\ & \quad \underline{4x^2 + 4x} \\ & \qquad -12x - 12 \\ & \qquad \underline{-12x - 12} \\ & \qquad\qquad 0 \end{array}$$

$\Rightarrow x^2 + 4x - 12$

$\Rightarrow (x + 6)(x - 2)$ other 2 factors

20. If $(x+2)$ is a factor $\Rightarrow x=-2$ is a root

$f(x) = 2x^3 + ax^2 - 17x + b$

$f(-2) = 2(-2)^3 + a(-2)^2 - 17(-2) + b = 0$

$\Rightarrow -16 + 4a + 34 + b = 0 \Rightarrow 4a + b = -18$

If $(x-3)$ is a factor $\Rightarrow x = 3$ is a root

$\Rightarrow f(3) = 2(3)^3 + a(3)^2 - 17(3) + b = 0$

$\Rightarrow 54 + 9a - 51 + b = 0$

$\Rightarrow \quad 9a + b = -3$

and $\underline{4a + b = -18}$

$\Rightarrow \quad 5a = 15$

$\Rightarrow \quad a = 3$

$\Rightarrow 4(3) + b = -18$

$b = -30$

Hence, $(x+2)(x-3) = x^2 - x - 6$

$$\begin{array}{r|l} & 2x+5 \\ \hline x^2-x-6 & 2x^3+3x^2-17x-30 \\ & \underline{2x^3-2x^2-12x} \\ & \quad 5x^2-5x-30 \\ & \quad \underline{5x^2-5x-30} \\ & \qquad 0 \end{array}$$

$\Rightarrow$ 3rd factor $(2x+5)$

21. $x^2 - 2x - 3 = (x+1)(x-3) = 0$

$=$ roots $x = -1, x = 3$

$f(x) = ax^3 + 8x^2 + bx + 6$

$\Rightarrow f(-1) = a(-1)^3 + 8(-1)^2 + b(-1) + 6 = 0$

$\Rightarrow -a + 8 - b + 6 = 0$

$\Rightarrow a + b = 14$

$f(3) = a(3)^3 + 8(3)^2 + b(3) + 6 = 0$

$\Rightarrow 27a + 72 + 3b + 6 = 0$

$\Rightarrow \quad 27a + 3b = -78$

$\Rightarrow \quad 9a + b = -26$

and $\quad \underline{a + b = 14}$

$8a = -40$

$\Rightarrow \quad a = -5$

$\Rightarrow -5 + b = 14$

$\Rightarrow \quad b = 19$

Hence,
$$\begin{array}{r|l} & -5x-2 \\ \hline x^2-2x-3 & -5x^3+8x^2+19x+6 \\ & \underline{-5x^3+10x^2+15x} \\ & \quad -2x^2+4x+6 \\ & \quad \underline{-2x^2+4x+6} \\ & \qquad 0 \end{array}$$

$\Rightarrow$ 3rd factor $= -5x - 2$

$\Rightarrow$ 3rd root: $-5x = 2$

$\Rightarrow x = -\frac{2}{5}$

22. (i) $ax^3 - b = c$

$\Rightarrow ax^3 = b + c$

$\Rightarrow x^3 = \frac{b+c}{a}$

$\Rightarrow x = \sqrt[3]{\frac{b+c}{a}}$

(ii) $a(x+b)^3 = c$

$\Rightarrow (x+b)^3 = \frac{c}{a}$

$\Rightarrow x + b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$

$\Rightarrow x = \left(\frac{c}{a}\right)^{\frac{1}{3}} - b$

Exercise 2.10

1. (i) Graph crosses the x-axis at $x = -1, 1, 3$

$\Rightarrow f(x) = a(x+1)(x-1)(x-3)$

Point $(0, 3) \Rightarrow f(0) = a(0+1)(0-1)(0-3) = 3$

$\Rightarrow 3a = 3$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+1)(x-1)(x-3) = x^3 - 3x^2 - x + 3$

(ii) Graph crosses the x-axis at $x = -4, 1, 2$

$\Rightarrow f(x) = a(x+4)(x-1)(x-2)$

Point $(0, 8) \Rightarrow f(0) = a(0+4)(0-1)(0-2) = 8$

$\Rightarrow 8a = 8$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+4)(x-1)(x-2) = x^3 + x^2 - 10x + 8$

2. (i) Graph crosses the x-axis at $x = -3, 0, 2$

$\Rightarrow f(x) = a(x+3)(x)(x-2)$

Point $(1, -4) \Rightarrow f(1) = a(1+3)(1)(1-2) = -4$

$\Rightarrow -4a = -4$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+3)(x)(x-2) = x^3 + x^2 - 6x$ (Green Graph)

Point $(1, -12) \Rightarrow f(1) = a(1+3)(1)(1-2) = -12$

$\Rightarrow -4a = -12$

$\Rightarrow \quad a = 3$

$\therefore f(x) = 3(x+3)(x)(x-2) = 3x^3 + 3x^2 - 18x$ (Blue Graph)

(ii) Graph crosses the x-axis at $x = 1, 2, 3$

$\Rightarrow f(x) = a(x-1)(x-2)(x-3)$

Point $(0, 6) \Rightarrow f(0) = a(0-1)(0-2)(0-3) = 6$

$\Rightarrow -6a = 6 \Rightarrow a = -1$

$\therefore f(x) = -1(x-1)(x-2)(x-3) = -x^3 + 6x^2 - 11x + 6$ (Red Graph)

Point $(0, 12) \Rightarrow f(0) = a(0-1)(0-2)(0-3) = 12$

$\Rightarrow -6a = 12$

$\Rightarrow \quad a = -2$

$\therefore f(x) = -2(x-1)(x-2)(x-3) = -2x^3 + 12x^2 - 22x + 12$ (Blue Graph)

3. Graph crosses the x-axis at $x = 1, -2, \frac{1}{2}$

$\Rightarrow f(x) = a(x-1)(x+2)(2x-1)$

Point $(0, 6) \Rightarrow f(0) = a(0-1)(0+2)(0-1) = 6$

$\Rightarrow 2a = 6 \Rightarrow a = 3$

$\therefore f(x) = 3(x-1)(x+2)(2x-1)$

$= 6x^3 + 3x^2 - 15x + 6$

4. $f(x) = (x-3)(x+1)(x+2)$

$= (x-3)(x^2 + 3x + 2)$

$= x^3 - 7x - 6 \Rightarrow a = 0, b = -7, c = -6$

5. (i) $f(x) = x^3 + 2$ Point $(0, 2) \Rightarrow f(0) = 0^3 + 2 = 2$ True

(ii) $g(x) = x^3$ Point $(1,1) \Rightarrow f(1) = (1)^3 = 1$ True

(iii) $h(x) = 2x^3$ Point $(1, 2) \Rightarrow f(1) = 2(1)^3 = 2$ True

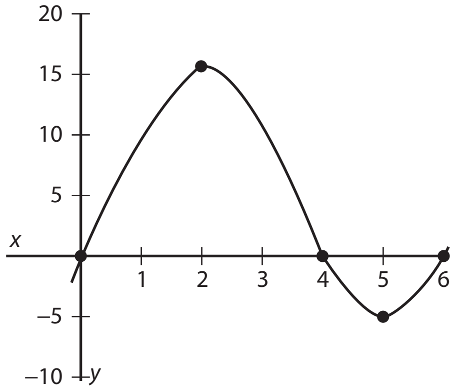
A$(2\frac{1}{3}, 4)$

6. $f(x) = x(x - 4)(x - 6)$

$f(2) = 2(2 - 4)(2 - 6)$
$= 2(-2)(-4) = 16$

$f(5) = 5(5 - 4)(5 - 6)$
$= 5(1)(-1) = -5$

Points: (0, 0) (4, 0) (6, 0) (2, 16) (5, −5)



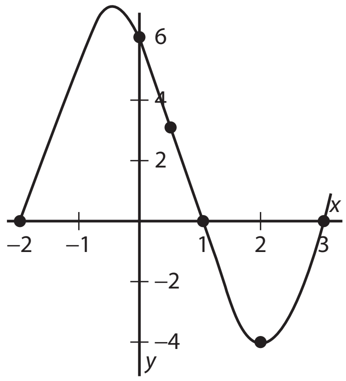
7. $f(x) = (x + 2)(x - 1)(x - 3)$

$f(0) = (0 + 2)(0 - 1)(0 - 3) = 6$

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2} + 2\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 3\right) = 3\frac{1}{8}$

$f(2) = (2 + 2)(2 - 1)(2 - 3) = -4$

Points: $(-2, 0)$ $(1, 0)$ $(3, 0)$ $(0, 6)$ $\left(\frac{1}{2}, 3\frac{1}{8}\right)$ $(2, -4)$



8. (i) Roots are $x = -1, 1, 2$

$\Rightarrow f(x) = a(x + 1)(x - 1)(x - 2)^2$

(ii) Point $(0, 4) \Rightarrow f(0) = a(0 + 1)(0 - 1)(0 - 2)^2 = 4$

$\Rightarrow -4a = 4$

$a = -1$

Hence, $f(x) = -1(x^2 - 1)(x^2 - 4x + 4)$

$= -1x^4 + 4x^3 - 3x^2 - 4x + 4 = ax^4 + bx^3 + cx^2 + dx + e$

$\Rightarrow a = -1, b = 4, c = -3, d = -4, e = 4$

9. (i) $(0, 4) \in f(x) \Rightarrow f(0) = 4$

$(0, -2) \in g(x) \Rightarrow g(0) = -2$

Hence, $4 = a(-2)$

$\Rightarrow a = -2$

(ii) $f(x)$ has roots $x = -2, 1$ (double roots)

$\Rightarrow f(x) = a(x + 2)^2(x - 1)^2$

Point $(0, 4) \Rightarrow f(0) = a(0 + 2)^2(0 - 1)^2 = 4$

$\Rightarrow 4a = 4$

$\Rightarrow a = 1$

Hence, $f(x) = (x + 2)^2(x - 1)^2$

$g(x)$ has roots $x = -2, 1$ (double roots)

$\Rightarrow g(x) = a(x + 2)^2(x - 1)^2$

Point $(0, -2) \Rightarrow g(0) = a(0 + 2)^2(0 - 1)^2 = -2$

$\Rightarrow 4a = -2$

$\Rightarrow a = -\frac{1}{2}$

Hence, $g(x) = -\frac{1}{2}(x + 2)^2 (x - 1)^2$

10. (i) Roots are $x = -1, 2, 5$

$\Rightarrow f(x) = (x + 1)(x - 2)(x - 5) = 0$

$\Rightarrow x^3 - 6x^2 + 3x + 10 = 0$

(ii) Roots are $x = -3, -1, 0$

$\Rightarrow f(x) = (x + 3)(x + 1)(x - 0) = 0$

$\Rightarrow x^3 - 4x^2 + 3x = 0$

(iii) Roots are $x = -2, \frac{1}{4}, 3$

$\Rightarrow f(x) = (x + 2)(x - \frac{1}{4})(x - 3) = 0$

$\Rightarrow (x + 2)(4x - 1)(x - 3) = 0$

$\Rightarrow 4x^3 - 5x^2 - 23x + 6 = 0$

(iv) Roots are $x = \frac{1}{2}, 2, 4$

$\Rightarrow f(x) = \left(x - \frac{1}{2}\right)(x - 2)(x - 4) = 0$

$\Rightarrow (2x - 1)(x - 2)(x - 4) = 0$

$\Rightarrow 2x^3 - 13x^2 + 22x - 8 = 0$

11. (i) Roots are $x = -\frac{1}{2}, 3, 6$

$\Rightarrow f(x) = a\left(x + \frac{1}{2}\right)(x - 3)(x - 6)$

$\Rightarrow f(x) = a(2x + 1)(x - 3)(x - 6)$

Point $(1, 30) \Rightarrow f(1) = a(2(1) + 1)(1 - 3)(1 - 6) = 30$

$\Rightarrow 30a = 30$

$\Rightarrow \quad a = 1$

$\Rightarrow f(x) = 1(2x + 1)(x - 3)(x - 6)$

$\Rightarrow f(x) = 2x^3 - 17x^2 + 27x + 18$

(ii) Roots are $x = -4, -\frac{1}{2}, 2\frac{1}{2}$

$\Rightarrow f(x) = a(x + 4)(x + \frac{1}{2})(x - 2\frac{1}{2})$

$\Rightarrow f(x) = a(x + 4)(2x + 1)(2x - 5)$

Point $(0, 20) \Rightarrow f(0) = a(0 + 4)(0 + 1)(0 - 5) = 20$

$\Rightarrow -20a = 20 \Rightarrow a = -1$

$\Rightarrow f(x) = -1(x + 4)(2x + 1)(2x - 5)$

$\Rightarrow f(x) = -4x^3 - 8x^2 + 37x + 20$

12. 2 Roots are $x = 2, 4$

$\Rightarrow (x - 2)(x - 4) = (x^2 - 6x + 8)$

$$
\begin{array}{r|l}
 & \;\; -3x - 1 \\
\text{Hence, } x^2 - 6x + 8 & \overline{-3x^3 + 17x^2 + bx - 8} \\
 & \underline{-3x^3 + 18x^2 - 24x} \\
 & \qquad\quad -x^2 + (b + 24)x - 8 \\
 & \qquad\quad \underline{-x^2 + 6x \qquad\quad - 8} \\
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Hence, $b + 24 - 6 = 0$ and $-3x - 1 = 0$

$\Rightarrow b = -18 \qquad \Rightarrow x = -\frac{1}{3}$

$\Rightarrow a = -\frac{1}{3}$

13. (i) Graph $f(x) = 0$ meets x-axis at $x = -1, 0, 2$

(ii) $f(x) = g(x) \Rightarrow$ Curve meets line at $x = -1.3, 0, 2.3$

(iii) $f(x) = g(x) \Rightarrow x^3 - x^2 - 2x = x$

$\Rightarrow x^3 - x^2 - 3x = 0$

$\Rightarrow x(x^2 - x - 3) = 0$

$\Rightarrow x = 0$ OR $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$

$x = \dfrac{1 \pm \sqrt{13}}{2}$

$x = \dfrac{1 + 3.60555}{2}, \dfrac{1 - 3.60555}{2}$

$= 2.303, -1.303$

$= 2.3, \quad -1.3$

$\Rightarrow x = 0, 2.3$ or -1.3

14. (i) Volume $= x(x - 1)(x + 1) = (x^3 - x)$ cm^3

(ii) Volume $= 24 = x^3 - x$

$\Rightarrow f(x) = x^3 - x - 24$

$f(3) = (3)^3 - (3) - 24 = 27 - 27 = 0$

$\Rightarrow x = 3$ is a root

15. $h = 2r \Rightarrow r = \dfrac{h}{2}$

$V = \pi\left(\dfrac{h}{2}\right)^2 \cdot h = \dfrac{\pi}{4}h^3$

(i) Hence, $ah^3 = \dfrac{\pi}{4}h^3 \Rightarrow a = \dfrac{3.14}{4} = 0.785 = 0.79$

(ii) Volume cylinder $= ah^3 = (0.79)(11)^3 = 1051.49$ cm^3

(iii) Volume $= 215.58 = ah^3$

$\Rightarrow (0.79)h^3 = 215.58$

$\Rightarrow h^3 = \dfrac{215.58}{0.79} = 272.886$

$\Rightarrow h = \sqrt[3]{272.886} = 6.486 = 6.5$ cm

16. Volume of cube $= (3)^3 = 27$

Volume of sphere $= (4.19)(27) = 113.13$

Volume of cube $= x^3 = 113.13$

$\Rightarrow x = \sqrt[3]{113.13} = 4.836 = 4.8$ cm

Hence, $4.19x^3 - 150 = x^3$

$\Rightarrow 3.19x^3 = 150$

$\Rightarrow x^3 = \dfrac{150}{3.19} = 47.0219$

$\Rightarrow x = \sqrt[3]{47.0219} = 3.609 = 3.6$ cm

Revision Exercise 2 (Core)

1. $x^2 - 6x + 5 = 0$

$\Rightarrow (x - 1)(x - 5) = 0$

$\Rightarrow x = 1, \quad x = 5$

Hence, $t - \frac{6}{t} = 1$ OR $t - \frac{6}{t} = 5$

$\Rightarrow t^2 - 6 = t$ OR $t^2 - 6 = 5t$

$\Rightarrow t^2 - t - 6 = 0$ OR $t^2 - 5t - 6 = 0$

$\Rightarrow (t + 2)(t - 3) = 0$ OR $(t + 1)(t - 6) = 0$

$\Rightarrow t = -2, 3, -1, 6$

2. $2(x + 1)(x - 4) - (x - 2)^2 = 0$

$\Rightarrow 2x^2 - 6x - 8 - (x^2 - 4x + 4) = 0$

$\Rightarrow x^2 - 2x - 12 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{(2)^2 - 4(1)(-12)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{52}}{2}$$

$$= \frac{2 \pm 2\sqrt{13}}{2} = 1 \pm \sqrt{13}$$

3. $px^2 + 2x + 1 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{(2)^2 - 4(p)(1)}}{2(p)}$$

$$= \frac{-2 \pm \sqrt{4 - 4p}}{2p}$$

No solution $\Rightarrow 4 - 4p < 0$

$\Rightarrow -4p < -4$

$\Rightarrow 4p > 4$

$\Rightarrow p > 1$

4. Real roots $\Rightarrow b^2 - 4ac \geq 0$

$\Rightarrow (-a - d)^2 - 4(1)(ad - b^2) \geq 0$

$\Rightarrow a^2 + 2ad + d^2 - 4ad + 4b^2 \geq 0$

$\Rightarrow a^2 - 2ad + d^2 + 4b^2 \geq 0$

$\Rightarrow (a - d)^2 + (2b)^2 \geq 0$ True

5. $(x + 1)$ and $(x - 2)$ are factors $\Rightarrow x = -1$ and $x = 2$ are roots

$f(x) = 6x^4 - x^3 + ax^2 - 6x + b$

$\Rightarrow f(-1) = 6(-1)^4 - (-1)^3 + a(-1)^2 - 6(-1) + b = 0$

$\Rightarrow 6 + 1 + a + 6 + b = 0$

$\Rightarrow a + b = -13$

$f(2) = 6(2)^4 - (2)^3 + a(2)^2 - 6(2) + b = 0$

$\Rightarrow 96 - 8 + 4a - 12 + b = 0$

$\Rightarrow 4a + b = -76$

and $\underline{a + b = -13}$

$\Rightarrow 3a = -63$

$\Rightarrow a = -21$

$\Rightarrow -21 + b = -13 \Rightarrow b = 8$

6. (i) $f(x) = x^3 - 4x^2 - 11x + 30$

$f(2) = (2)^3 - 4(2)^2 - 11(2) + 30$

$= 8 - 16 - 22 + 30 = 38 - 38 = 0$

$\Rightarrow x = 2$ is a root $\Rightarrow (x - 2)$ is a factor

(ii)
$$\begin{array}{r|l} & x^2 - 2x - 15 \\ x-2 & x^3 - 4x^2 - 11x + 30 \\ & \underline{x^3 - 2x^2} \\ & \quad -2x^2 - 11x \\ & \quad \underline{-2x^2 + 4x} \\ & \qquad -15x + 30 \\ & \qquad \underline{-15x + 30} \\ & \qquad\qquad 0 \end{array}$$

$\Rightarrow x^2 - 2x - 15$
$= (x + 3)(x - 5)$

(iii) $f(x) = (x - 2)(x + 3)(x - 5) = 0$
$\Rightarrow$ Roots : $x = 2, -3, 5$

7. (i) $b^2 - 4ac = (-2)^2 - (4)(1)(-5)$
$= 24 > 0 \quad \Rightarrow$ 2 distinct real roots

(ii) $b^2 - 4ac = (-4)^2 - 4(1)(6)$
$= 16 - 24 = -8 < 0 \quad \Rightarrow$ 2 imaginary roots

(iii) $b^2 - 4ac = (4)^2 - 4(-1)(-6)$
$= 16 - 24 = -8 < 0 \quad \Rightarrow$ 2 imaginary roots

8. $3^{2x} - 12(3^x) + 27 = 0$
$\Rightarrow (3^x)^2 - 12(3^x) + 27 = 0$
If $y = 3^x \Rightarrow y^2 - 12y + 27 = 0$
$\Rightarrow (y - 3)(y - 9) = 0$
$\Rightarrow y = 3 \quad$ OR $\quad y = 9$
$\Rightarrow 3^x = 3^1 \quad$ OR $\quad 3^x = 9 = 3^2$
$\Rightarrow x = 1 \quad$ OR $\quad x = 2$

Revision Exercise 2 (Advanced)

1. $2x^2 - 4x - 5$
$= 2(x^2 - 2x) - 5$
$= 2(x^2 - 2x + 1) - 2 - 5$
$= 2(x - 1)^2 - 7$

(i) $2x^2 - 4x - 5 = 0$
$\Rightarrow 2(x - 1)^2 = 7$
$\Rightarrow (x - 1)^2 = \frac{7}{2}$
$\Rightarrow x - 1 = \pm\sqrt{\frac{7}{2}}$
$\Rightarrow x = 1 \pm \sqrt{\frac{7}{2}}$

(ii) Minimum Point $= (1, -7)$

2. $(2\sqrt{2} - \sqrt{3})^2 = (2\sqrt{2})^2 - 2(2\sqrt{2})(\sqrt{3}) + (-\sqrt{3})^2$
$= 8 - 4\sqrt{6} + 3$
$= 11 - 4\sqrt{6}$

3. $\dfrac{\sqrt{7} + \sqrt{5}}{\sqrt{80} + \sqrt{5}} = \dfrac{\sqrt{7} + \sqrt{5}}{4\sqrt{5} + \sqrt{5}} = \dfrac{\sqrt{7} + \sqrt{5}}{5\sqrt{5}}$
$= \dfrac{\sqrt{7} + \sqrt{5}}{5\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{35} + 5}{25}$

4. $\sqrt{x + 2} = x - 4$
$\Rightarrow x + 2 = (x - 4)^2 = x^2 - 8x + 16$
$\Rightarrow x^2 - 9x + 14 = 0$
$\Rightarrow (x - 2)(x - 7) = 0$

$\Rightarrow x = 2, 7$

Test $x = 2 \quad \Rightarrow \sqrt{2+2} = 2 - 4$

$\Rightarrow \sqrt{4} = -2 \qquad$ False

Test $x = 7 \quad \Rightarrow \sqrt{7+2} = 7 - 4$

$\Rightarrow \sqrt{9} = 3 \qquad$ True

5. (i) Test $t = 1 \Rightarrow s = 8(1)^2 + 4\,(1) = 8 + 4 = 12$ m

$\Rightarrow t < 1 \Rightarrow$ Test $t = 0.9 \Rightarrow s = 8(0.9)^2 + 4\,(0.9) = 10.08$ m

(ii) Solve $8t^2 + 4t = 10$

$$\Rightarrow 4t^2 + 2t - 5 = 0$$

$$\Rightarrow t = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-5)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{84}}{8}$$

$$= \frac{-2 + 9.165}{8}, \frac{-2 - 9.165}{8}$$

$= 0.8956$ (Valid), -1.3956 (Not valid)

$= 0.90$

(iii) Answer $= 0.8956$

Corrected answer $= 0.90 \quad \Rightarrow$ error $= 0.0044$

$\Rightarrow$ Percentage error $= \dfrac{0.0044}{0.8956} \times 100$

$= 0.491\%$

$= 0.49\%$

6. $\sigma = \dfrac{\sqrt{p\,(1+p)}}{n}$

$$\Rightarrow \sqrt{p + p^2} = \sigma n$$

$$\Rightarrow p + p^2 = \sigma^2 n^2$$

$$\Rightarrow p^2 + p - \sigma^2 n^2 = 0$$

$$\Rightarrow p = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-\sigma^2 n^2)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{(1)^2 + 4\sigma^2 n^2}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{1 + 4\sigma^2 n^2}}{2}$$

7.

	$k < 0$	$0 < k < \frac{1}{4}$	$k > \frac{1}{4}$
k	Negative	Positive	Positive
$4k$	Negative	Positive	Positive
$4k - 1$	Negative	Negative	Positive
$k(4k - 1)$	Positive	Negative	Positive

Solve $x^2 + 4kx + k = 0$.

$$\Rightarrow x = \frac{-4k \pm \sqrt{(4k)^2 - 4(1)(k)}}{2(1)}$$

$$\Rightarrow x = \frac{-4k \pm \sqrt{16k^2 - 4k}}{2}$$

$$\Rightarrow x = \frac{-4k \pm \sqrt{4} \cdot \sqrt{4k^2 - k}}{2}$$

$\Rightarrow$ Roots: $x = -2k \pm \sqrt{k\,(4k - 1)}$

Hence, $x^2 + 4kx + k > 0$ for $0 < k < \dfrac{1}{4}$

8. $ax^2 + 2bx + c = 0$

$\Rightarrow$ Roots: $x = \dfrac{-2b \pm \sqrt{(2b)^2 - 4(a)(c)}}{2a}$

$\Rightarrow x = \dfrac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$

$\Rightarrow x = \dfrac{-2b \pm \sqrt{4} \cdot \sqrt{b^2 - ac}}{2a}$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - ac}}{a}$ (Real)

Similarly, $bx^2 + 2cx + a = 0 \Rightarrow$ Roots: $x = \dfrac{-c \pm \sqrt{c^2 - ab}}{b}$ (Real)

True if $c > a$ $\Rightarrow bc > a^2$

and $b > a$ $\Rightarrow a^2 - bc < 0$

Hence, $cx^2 + 2ax + b = 0 \Rightarrow$ Roots: $x = \dfrac{-a \pm \sqrt{a^2 - bc}}{c}$

Since $a^2 - bc < 0 \Rightarrow$ Roots are not real

9. Points of intersection at A and B occur when $g(x) = f(x)$.

Hence, $x^2 + 5x - 1 = -x^2 + 5x + 3$

$\Rightarrow 2x^2 - 4 = 0$

$\Rightarrow x^2 - 2 = 0$

$\Rightarrow x^2 = 2$

$\Rightarrow x = -\sqrt{2}, x = \sqrt{2}$

$\Rightarrow g(-\sqrt{2}) = (-\sqrt{2})^2 + 5(-\sqrt{2}) - 1$

$= 2 - 5\sqrt{2} - 1 = 1 - 5\sqrt{2}$

and $g(\sqrt{2}) = (\sqrt{2})^2 + 5(\sqrt{2}) - 1$

$= 2 + 5\sqrt{2} - 1 = 1 + 5\sqrt{2}$

Hence, A $= (\sqrt{2}, 1 - 5\sqrt{2})$, B $= (-\sqrt{2}, 1 + 5\sqrt{2})$

10. $kx^2 - 2kx - 3k - 12 = 0$

$[a = k, b = -2k, c = -3k - 12]$

For equal roots:

$b^2 - 4ac = 0$

$(-2k)^2 - 4(k)(-3k - 12) = 0$

$4k^2 + 12k^2 + 48k = 0$

$16k^2 + 48k = 0$

$k^2 + 3k = 0$

$k(k + 3) = 0$

$k = 0$ or $k = -3$

When $k = 0$, the equation is untrue.

Thus the only solution is $k = -3$.

11. $x^2 - \sqrt{3}x - 6 = 0$

$\Rightarrow x = \dfrac{\sqrt{3} \pm \sqrt{(\sqrt{3})^2 - 4(1)(-6)}}{2(1)}$

$= \dfrac{\sqrt{3} \pm \sqrt{27}}{2} = \dfrac{\sqrt{3} + 3\sqrt{3}}{2}, \dfrac{\sqrt{3} - 3\sqrt{3}}{2}$

$= 2\sqrt{3}, -\sqrt{3}$

Hence, $r_1 r_2 = (2\sqrt{3})(-\sqrt{3}) = -6$

12. $3x + y = -1 \cap x^2 + y^2 = 53$

$\Rightarrow y = -3x - 1 \Rightarrow x^2 + (-3x - 1)^2 = 53$

$\Rightarrow x^2 + 9x^2 + 6x + 1 - 53 = 0$

$\Rightarrow 10x^2 + 6x - 52 = 0$

$\Rightarrow 5x^2 + 3x - 26 = 0$

$\Rightarrow (5x + 13)(x - 2) = 0$

$\Rightarrow 5x = -13, x = 2$

$\Rightarrow x = -\dfrac{13}{5}$ $\qquad \Rightarrow y = -3(2) - 1$

$\qquad\qquad\qquad\qquad = -7$

$\Rightarrow y = -3\left(\dfrac{-13}{5}\right) - 1$

$= \dfrac{34}{5}$

$\therefore$ the solutions are: $\left(\dfrac{-13}{5}, \dfrac{34}{5}\right)$ and $(2, -7)$

13. length $= x$ and width $= y$.

$\Rightarrow 2x + 2y = 48$ and $x = \dfrac{1}{2}y^2$

$\Rightarrow x + y = 24$

$\Rightarrow x = -y + 24 \qquad \Rightarrow -y + 24 = \dfrac{1}{2}y^2$

$\Rightarrow -2y + 48 = y^2$

$\Rightarrow y^2 + 2y - 48 = 0$

$\Rightarrow (y + 8)(y - 6) = 0$

$\Rightarrow y = -8$(Not valid), $y = 6$ m (Valid)

$\Rightarrow x = -6 + 24$

$= 18$ m

$\Rightarrow$ Dimensions are 18 m and 6 m

14. $\dfrac{y}{1} = \dfrac{x^2}{x+1} \qquad \Rightarrow x^2 = xy + y$

$\Rightarrow x^2 - xy - y = 0$

$\Rightarrow x = \dfrac{y \pm \sqrt{(-y)^2 - 4(1)(-y)}}{2(1)}$

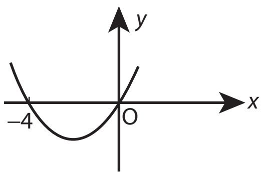
$= \dfrac{y \pm \sqrt{y^2 + 4y}}{2}$

Real roots $\Rightarrow b^2 - 4ac \geqslant 0 \qquad \Rightarrow y^2 + 4y \geqslant 0$

Factors: $y(y + 4)$

$\Rightarrow$ Roots: $y = 0, -4$

For $y^2 + 4y \geqslant 0 \quad \Rightarrow y \leqslant -4 \quad$ or $\quad y \geqslant 0$



15. $y = ax^2 + bx + c$

Point $(-2, -1) \Rightarrow -1 = a(-2)^2 + b(-2) + c$

$\Rightarrow -1 = 4a - 2b + c$

Point $(1, 2) \qquad \Rightarrow 2 = a(1)^2 + b(1) + c$

$\Rightarrow 2 = a + b + c$

Point $(3, -16) \Rightarrow -16 = a(3)^2 + b(3) + c$

$\Rightarrow -16 = 9a + 3b + c$

$\Rightarrow 4a - 2b + c = -1$

and $\underline{a + b + c = 2}$

$\Rightarrow 3a - 3b = -3$

$\Rightarrow a - b = -1$

$9a + 3b + c = -16$

$\underline{a + b + c = 2}$

$8a + 2b = -18$

$\Rightarrow 4a + b = -9$

and $\underline{a - b = -1}$

$5a = -10$

$\Rightarrow a = -2$

$\Rightarrow -2 - b = -1$

$\Rightarrow \qquad b = -1$

and $\qquad a + b + c = 2$

$\Rightarrow -2 - 1 + c = 2$

$\Rightarrow \qquad c = 5$

Hence, $\qquad y = -2x^2 - x + 5$

16. (ii) $f(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow f(0) = (0)^3 - 6(0)^2 + 11(0) - 6 = -6$

$\Rightarrow f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 12 - 12 = 0$

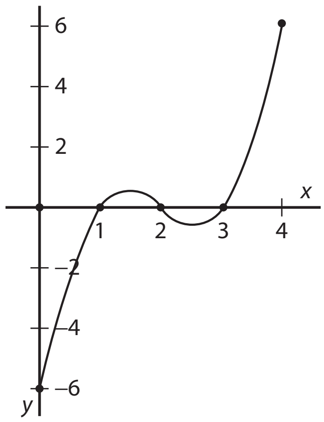
$\Rightarrow f(2) = (2)^3 - 6(2)^2 + 11(2) - 6 = 30 - 30 = 0$

$\Rightarrow f(3) = (3)^3 - 6(3)^2 + 11(3) - 6 = 60 - 60 = 0$

$\Rightarrow f(4) = (4)^3 - 6(4)^2 + 11(4) - 6 = 108 - 102 = 6$

$\Rightarrow$ Roots are $x = 1, 2, 3$

(iii) Points are $(0, -6)(1, 0)$
$(2, 0)(3, 0)$
$(4, 6)$



17. (i) Roots $x = 2, 5$ (double root)

(ii) $f(x) = a(x - 2)(x - 5)^2$

(iii) $f(x) = a(x^3 - 12x^2 + 45x - 50)$

Point $(3, 8) \Rightarrow f(3) = a[(3)^3 - 12(3)^2 + 45(3) - 50] = 8$

$\Rightarrow a[27 - 108 + 135 - 50] = 8$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Hence, $f(x) = 2(x^3 - 12x^2 + 45x - 50)$

$= 2x^3 - 24x^2 + 90x - 100 \qquad \Rightarrow a = 2, b = -24$

$c = 90, d = -100$

(iv) Point $(3, -8) \Rightarrow f(3) = a[(3)^3 - 12(3)^2 + 45(3) - 50] = -8$

$\Rightarrow 4a = -8 \Rightarrow a = -2$

$\Rightarrow f(x) = -2(x^3 - 12x^2 + 45x - 50)$

$= -2x^3 + 24x^2 - 90x + 100$

(v) Reflection in y-axis $\Rightarrow x$ is replaced by $(-x)$

$f(x) = 2x^3 - 24x^2 + 90x - 100$

$\Rightarrow f(-x) = 2(-x)^3 - 24(-x)^2 + 90(-x) - 100$

$= -2x^3 - 24x^2 - 90x - 100$

Revision Exercise 2 (Extended–Response)

1. (i) $C(t) = 0.02t - at^3$

$t = 5 \Rightarrow C(5) = (0.02)(5) - a(5)^3 = 0.075$

$\Rightarrow 0.1 - 125a = 0.075$

$\Rightarrow -125a = -0.025$

$\Rightarrow a = \dfrac{-0.025}{-125} = 0.0002$

(ii) $C(t) = 0 \Rightarrow 0.02t - 0.0002t^3 = 0$

$\Rightarrow 200t - 2t^3 = 0$

$\Rightarrow 100t - t^3 = 0$

$\Rightarrow t(100 - t^2) = 0$

$\Rightarrow t = 0$ (not valid), $t = 10$ (valid)

(iii) Beyond $t = 10 \Rightarrow C(t)$ is negative

2. (i) Area $= (2x + y)(3x + 2y)$

$= 6x^2 + 7xy + 2y^2$

(ii) Area dividing strips $=$ Area full poster $-$ Area 6 purple squares

$= 6x^2 + 7xy + 2y^2 - 6x^2$

$= 7xy + 2y^2 \quad \Rightarrow k = 7$

(iii) Area 6 purple squares $= 6x^2 = 1.5$

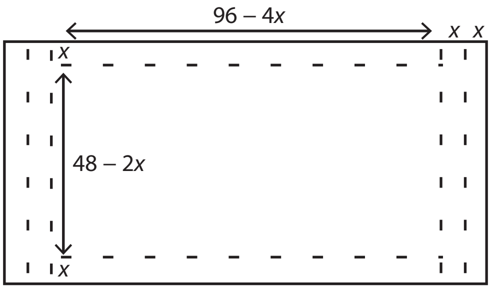
$$\Rightarrow x^2 = 0.25 \quad \Rightarrow x = 0.5 \text{ m} = \frac{1}{2}\text{ m}$$

Hence, $7xy + 4y^2 = 3.5y + 2y^2 = 1$

$$\Rightarrow 4y^2 + 7y - 2 = 0$$
$$\Rightarrow (4y - 1)(y + 2) = 0$$
$$\Rightarrow y = \frac{1}{4} \text{ (valid)}, y = -2 \text{ (Not valid)}$$

3. (a) Volume $= (96 - 4x)(48 - 2x)(x)$

$V = 4608x - 384x^2 + 8x^3$



(b) (i) $0 < x < 24$

(ii) At C, $x = 0$ $\Rightarrow$ Volume $= 0$

At A, $x = 8$ $\Rightarrow$ Maximum volume occurs ($= 16\,384$ cm^3)

At B, $x = 24$ $\Rightarrow$ Minimum volume occurs ($= 0$ cm^3)

(iii) Maximum volume $= 16\,300$ cm^3 (using graph)

(iv) $x = 10 \Rightarrow V = [96 - 4(10)][48 - 2(10)][10]$

$\Rightarrow (56)(28)(10) = 15\,680$ cm^3

(v) $x = 5 \Rightarrow$ Volume $= 14\,440$ cm^3

$x < 5 \Rightarrow$ Volume $< 14\,440$

$\Rightarrow$ Volume $= 14\,439.9$

(vi) If $5 \leq x \leq 15$, minimum volume occurs at $x = 15$ (using graph)

$\Rightarrow x = 15 \Rightarrow$ Volume $= [96 - 4(15)][48 - 2(15)][15]$

$\Rightarrow (36)(18)(15) = 9720$ cm^3

(c) Area $= 2(x)(96 - 4x) + 2(x)(48 - 2x) + 2(96 - 4x)(48 - 2x)$

$= 8x(24 - x) + 4x(24 - x) + 8(24 - x)(24 - x)$

$= 4(24 - x)(2x + x + 48 - 2x)$

$= 4(24 - x)(48 + x)$

$\Rightarrow a = 4, b = 24, c = 48$

4. (i) $4l + 3w = 120$

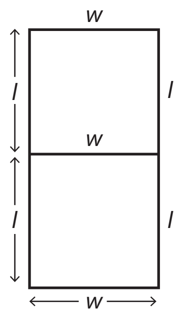
$\Rightarrow 4l = 120 - 3w$

$\Rightarrow l = 30 - \frac{3}{4}w$

Hence, area $= 2lw$

$= 2w\left(30 - \frac{3}{4}w\right)$

$= 60w - \frac{3}{4}w^2$



(ii) $60w - \frac{3}{2}w^2 = 0$

$\Rightarrow w\left(60 - \frac{3}{2}w\right) = 0$

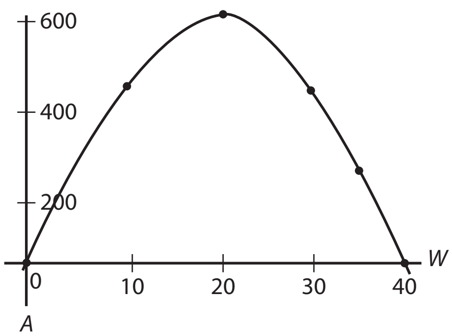
$\Rightarrow w = 0, 60 - \frac{3}{2}w = 0$

$120 - 3w = 0$

$\Rightarrow 3w = 120$

$\Rightarrow w = 40$

$w =$	0	10	20	30	40
$A =$	0	450	600	450	0



(iii) $A = -\frac{3}{2}w^2 + 60w$

$= -\frac{3}{2}(w^2 - 40w)$

$= -\frac{3}{2}(w^2 - 40w + 400) + 600$

$= 600 - \frac{3}{2}(w - 20)^2$

Hence, maximum area $= 600\ \text{m}^2$

(iv) $w = 20$ m

(v) Dimensions are: $w = 20$ m, $l = 30 - \frac{3}{4}(20) = 15$ m.

5. (a) (i) $h = 5 \Rightarrow t = 1, 3$

(ii) $t = 4.5$ secs

(b) $2 + 4t - t^2 = 0$

$\Rightarrow t = \dfrac{-4 \pm \sqrt{(4)^2 - 4(-1)(2)}}{2(-1)} = \dfrac{-4 \pm \sqrt{16 + 8}}{-2}$

$\Rightarrow t = \dfrac{-4 + \sqrt{24}}{-2} = 2 + \sqrt{6}$ (valid), $t = 2 - \sqrt{6} = -0.449$ (Not valid)

$\Rightarrow t = 4.449$

$\Rightarrow t = 4.45$

(c) $h = 2 + 4t - t^2$

$= 2 - (t^2 - 4t)$

$= 2 - (t^2 - 4t + 4) + 4$

$= 6 - (t - 2)^2 \Rightarrow (p, q) = (2, 6)$

6.

No. of Price hikes	Price per Rental	Number of Rentals	Total Income (I)
	€12	36	€432
1 Price hike	€12.50	34	€425
2 Price hikes	€13	32	€416
3 Price hikes	€13.50	30	€405
x price hikes	€$(12 + 0.5x)$	$36 - 2x$	€$(12 + 0.5x)(36 - 2x)$

(i) $\text{I} = (12 + 0.5x)(36 - 2x) = 432 - 6x - x^2$

(ii) $\text{I} = 432 - 6x - x^2$

$= 432 - (x^2 + 6x)$

$= 432 - (x^2 + 6x + 9) + 9$

$= 441 - (x + 3)^2$

(iii) Maximum Income = €441

(iv) Price per rental

7. (a) Area of garden = 2 Area BAC + Area ACED

$= 2\left(\frac{\pi}{4}x^2\right) + (x)(y)$

$= \frac{\pi}{4}x^2 + xy$

(b) (i) Length $= 2\left(\frac{2\pi x}{4}\right) + y = 100$

$\Rightarrow \pi x + y = 100$

$\Rightarrow y = 100 - \pi x$

(ii) Area $= \frac{\pi}{2}x^2 + x(100 - \pi x)$

$= \frac{\pi}{2}x^2 + 100x - \pi x^2$

$A = 100x - \frac{\pi}{2}x^2$

(iii) $100 - \pi x = 0$

$\Rightarrow \quad \pi x = 100$

$\Rightarrow \quad x = \dfrac{100}{\pi}$

Domain: $0 < x < \dfrac{100}{\pi}$

(c) Area $= 100x - \dfrac{\pi}{2}x^2 = 1000$

$\Rightarrow 200x - \pi x^2 = 2000$

$\Rightarrow \pi x^2 - 200x + 2000 = 0$

$$\Rightarrow x = \frac{200 \pm \sqrt{(-200)^2 - 4(\pi)(2000)}}{2(\pi)}$$

$$= \frac{200 \pm \sqrt{40\,000 - 8000\pi}}{2\pi}$$

$$= \frac{200 \pm \sqrt{14\,867.25877}}{2\pi}$$

$$= \frac{200 + 121.9314}{2\pi}, \frac{200 - 121.9314}{2\pi}$$

$= 51.236, 12.425$

$= 51.2, 12.4$

(d) (i) Volume $=$ (Area)(height)

$$= \left(100x - \frac{\pi}{2}x^2\right)\left(\frac{x}{50}\right)$$

$$= 2x^2 - \frac{\pi}{100}x^3$$

(ii) $x = 12.4 \Rightarrow$ Volume $= 2(12.4)^2 - \dfrac{\pi}{100}(12.4)^3$

$= 307.52 - 59.898$

$= 247.62 = 247.6\ \text{m}^3$

$x = 51.2 \Rightarrow$ Volume $= 2(51.2)^2 - \dfrac{\pi}{100}(51.2)^3$

$= 5242.88 - 4216.574$

$= 1026.306 = 1026.3\ \text{m}^3$

(iii) Volume $= 2x^2 - \dfrac{\pi}{100}x^3 = 500$

$\Rightarrow 200x^2 - \pi x^3 = 50\,000$

$\Rightarrow \quad f(x) = \pi x^3 - 200x^2 + 50\,000$

$\Rightarrow \quad f(18) = 18321.77 - 64\,800 + 50\,000 = 3521.77$

$\Rightarrow f(18.8) = \pi(18.8)^3 - 200(18.8)^2 + 50\,000$

$= 20\,874.85 - 70\,688 + 50\,000 = 186$

$\Rightarrow f(18.9) = \pi(18.9)^3 - 200(18.9)^2 + 50\,000$

$= 21\,209.74 - 71\,442 + 50\,000 = -232.26$

Hence, $x = 18.8$

8. (a) $y = 2x \cap y = 6 + 5x - x^2$

$\Rightarrow 2x = 6 + 5x - x^2$

$\Rightarrow x^2 - 3x - 6 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{(3)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{33}}{2}$$

$$\Rightarrow x_1 = \frac{3 + \sqrt{33}}{2} \Rightarrow y_1 = 2\left(\frac{3 + \sqrt{33}}{2}\right) = 3 + \sqrt{33}$$

Hence, A $= \left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right)$

$\Rightarrow x_2 = \frac{3-\sqrt{33}}{2} \Rightarrow y_2 = 2\left(\frac{3-\sqrt{33}}{2}\right) = 3-\sqrt{33}$

Hence, B $= \left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

(b) $(x, 6+5x-x^2)$ and $(x, 2x)$

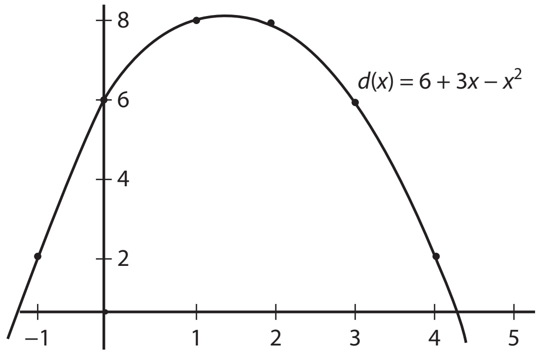
$d = \sqrt{(x-x^2)+(6+5x-x^2-2x)^2}$

$= \sqrt{(6+3x-x^2)^2}$

$\Rightarrow d(x) = 6+3x-x^2$

(c)

$\boldsymbol{x} =$	−1	0	1	2	3	4
$\boldsymbol{d(x)} =$	2	6	8	8	6	2



(d) $d(x) = 6+3x-x^2$

$= 6 - \left(x^2 - 3x - \left(1\frac{1}{2}\right)^2\right) + 2\frac{1}{4}$

$= 8\frac{1}{4} - \left(x - 1\frac{1}{2}\right)^2$

(e) Maximum point $= \left(1\frac{1}{2}, 8\frac{1}{4}\right)$

$\Rightarrow$ Maximum vertical distance $= 8\frac{1}{4}$ occurs at $x = 1\frac{1}{2}$

(f) $0 \leq d(x) \leq 8\frac{1}{4}$

9. (i) $y = x \cap y = \sqrt{x-b} + c$

$\Rightarrow x = \sqrt{x-b} + c$

$\Rightarrow x - c = \sqrt{x-b}$

$\Rightarrow (x-c)^2 = x-b$

$\Rightarrow x^2 - 2cx + c^2 - x + b = 0$

$\Rightarrow x^2 - x(2c+1) + c^2 + b = 0$

At $(a, a) \Rightarrow x = a \Rightarrow a^2 - a(2c+1) + c^2 + b = 0$

(ii) Tangent $\Rightarrow b^2 - 4ac = 0$

$\Rightarrow [-(2c+1)]^2 - 4(1)(c^2+b) = 0$

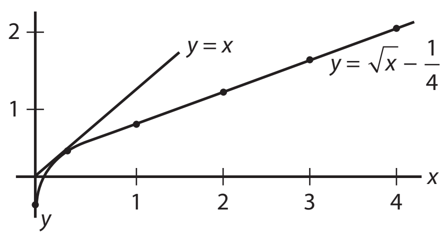
$\Rightarrow 4c^2 + 4c + 1 - 4c^2 - 4b = 0$

$\Rightarrow \quad 4c = 4b - 1$

$\Rightarrow \quad c = \frac{4b-1}{4}$

(iii)

$\boldsymbol{x} =$	0	1	2	3	4
$\boldsymbol{y} =$	−0.25	0.75	1.16	1.48	1.75



(iv) $y = x \cap y = \sqrt{x} - \frac{1}{4}$

$\Rightarrow \sqrt{x} - \frac{1}{4} = x$

$\Rightarrow \sqrt{x} = x + \frac{1}{4}$

$\Rightarrow x = \left(x + \frac{1}{4}\right)^2 = x^2 + \frac{1}{2}x + \frac{1}{16}$

$\Rightarrow x^2 - \frac{1}{2}x + \frac{1}{16} = 0$

$\Rightarrow \left(x + \frac{1}{4}\right)^2 = 0 \quad \Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4} \Rightarrow \left(\frac{1}{4}, \frac{1}{4}\right)$

(v) $y = x + k \cap y = \sqrt{x} - \frac{1}{4}$

$\Rightarrow x + k = \sqrt{x} - \frac{1}{4}$

$\Rightarrow x + \left(k + \frac{1}{4}\right) = \sqrt{x}$

$\Rightarrow x^2 + 2\left(k + \frac{1}{4}\right) + k^2 + \frac{1}{2}k + \frac{1}{16} = x$

$\Rightarrow x^2 + \left(2k - \frac{1}{2}\right) + x + k^2 + \frac{1}{2}k + \frac{1}{16} = 0$

(a) Twice $\Rightarrow b^2 - 4ac > 0$

$\Rightarrow \left(2k - \frac{1}{2}\right)^2 - 4(1)\left(k^2 + \frac{1}{2}k + \frac{1}{16}\right) > 0$

$\Rightarrow 4k^2 - 2k + \frac{1}{4} - 4k^2 - 2k - \frac{1}{4} > 0$

$\Rightarrow -4k^2 > 0$

$\Rightarrow k > 0$

(b) Once $\Rightarrow b^2 - 4ac = 0$

$\Rightarrow k = 0$

(c) At no point $\Rightarrow b^2 - 4ac < 0$

$\Rightarrow k > 0$

Chapter 3

Exercise 3.1

1. (i) $30° = \frac{\pi}{6}$

(ii) $45° = \frac{\pi}{4}$

(iii) $150° = 5(30°) = \frac{5\pi}{6}$

(iv) $135° = 3(45°) = \frac{3\pi}{4}$

(v) $36° = \frac{\pi}{180} \times 36° = \frac{\pi}{5}$

(vi) $240° = 4(60°) = \frac{4\pi}{3}$

(vii) $390° = 13(30°) = \frac{13\pi}{6}$

2. (i) $180°$

(ii) $90°$

(iii) $30°$

(iv) $\frac{5\pi}{6} = \frac{5 \times 180°}{6} = 150°$

(v) $\frac{4\pi}{9} = \frac{4 \times 180°}{9} = 80°$

(vi) $\frac{11\pi}{6} = \frac{11 \times 180°}{6} = 330°$

(vii) $\frac{5\pi}{12} = \frac{5 \times 180°}{12} = 75°$

3. (i) $l = r\theta \quad \Rightarrow l = (4)(2) = 8$ cm

(ii) $l = (4)(4) = 16$ cm

(iii) $l = (4)\left(2\frac{1}{2}\right) = 10$ cm

(iv) $l = 4\left(\frac{5}{4}\right) = 5$ cm

4. (i) $r\theta = l \quad \Rightarrow 6\theta = 6 \quad \Rightarrow \theta = 1$ radian

(ii) $6\theta = 12 \quad \Rightarrow \theta = 2$ radians

(iii) $6\theta = 3 \quad \Rightarrow \theta = \frac{1}{2}$ radian

(iv) $6\theta = 9 \quad \Rightarrow \theta = 1\frac{1}{2}$ radians

(v) $6\theta = 7\frac{1}{2} \quad \Rightarrow \theta = 1\frac{1}{4}$ radians

5. $r\theta = l \quad \Rightarrow 2r = 15 \quad \Rightarrow r = 7\frac{1}{2}$ cm

6. $5\theta = 6 \quad \Rightarrow \theta = 1\frac{1}{5}$ radians

Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(5)^2 1\frac{1}{5} = \frac{1}{2} \cdot \frac{25}{1} \cdot \frac{6}{5} = 15 \text{ cm}^2$

7. Area $= \frac{1}{2}(8)^2 \cdot \theta = 40$

$\Rightarrow \quad 32\theta = 40$

$\Rightarrow \quad \theta = \frac{40}{32} = \frac{5}{4} = 1\frac{1}{4}$ radians

8. $l = 2\pi r \Rightarrow 2\pi r = 12\pi$

$\Rightarrow \quad r = 6$ cm

Area $= \frac{1}{2}(6)^2\theta = 3\pi$

$\Rightarrow \quad 18\theta = 3\pi$

$\Rightarrow \quad \theta = \frac{3\pi}{18} = \frac{\pi}{6}$ radians

9. Area $= \frac{1}{2}(6)^2\theta = 27$

$\Rightarrow \quad 18\theta = 27$

$\theta = \frac{27}{18} = \frac{3}{2}$ radians

10. Shaded Area = Area large sector − Area small sector

$= \frac{1}{2}(8)^2 \cdot \frac{\pi}{4} - \frac{1}{2}(2)^2 \cdot \frac{\pi}{4}$

$= 8\pi - \frac{\pi}{2}$

$= \frac{15\pi}{2}$ cm^2

11. (i) $4\theta = 10$

$\Rightarrow \theta = \frac{10}{4} = \frac{5}{2}$ radians

(ii) $\theta = \frac{5}{2} \cdot \frac{180}{\pi} = 143.239° = 143°$

12. Shaded Area = Area square − Area sector

$= (2)(2) - \frac{1}{2}(2)^2 \cdot \frac{\pi}{2}$

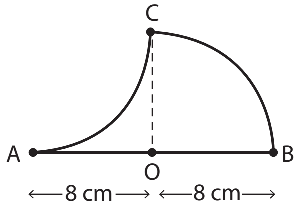
$= (4 - \pi)$ cm^2

13. Shaded region AOC = Area square − Area sector

$= (8)^2 - \frac{1}{2}(8)^2 \cdot \frac{\pi}{2}$

$= (64 - 16\pi)$ cm^2

Shaded region COB $= \frac{1}{2}(8)^2 \cdot \frac{\pi}{2} = 16\pi$ cm^2



Shaded figure = Shaded region AOC + Shaded region COB

$= 64 - 16\pi + 16\pi$

$= 64$ cm^2

14. (i) A is a centre $\Rightarrow |AB| = 6$ cm (radius)

C is a centre $\Rightarrow |CB| = 6$ cm (radius)

$\Rightarrow \quad \triangle ABC$ is an equilateral triangle

$\Rightarrow \quad |\angle ABC| = 60°$

(ii) Length arc AB $= 6 \cdot \frac{\pi}{3} = 2\pi$ cm

(iii) Shaded region AB = Area sector − Area $\triangle ABC$

$$= \frac{1}{2}(6)^2 \cdot \frac{\pi}{3} - \frac{1}{2}(6)(6)\sin 60^\circ$$

$$= 6\pi - 18 \cdot \frac{\sqrt{3}}{2}$$

$$= (6\pi - 9\sqrt{3})\ \text{cm}^2$$

Total shaded region $= 2(6\pi - 9\sqrt{3}) = (12\pi - 18\sqrt{3})\ \text{cm}^2$

15. (i) Length $= r\theta + 2r = 40$

$$r\theta = 40 - 2r$$

$$\theta = \frac{40 - 2r}{r}$$

(ii) Area $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{40 - 2r}{r}\right) = 100$

$$\Rightarrow\quad 40r - 2r^2 = 200$$

$$\Rightarrow\quad r^2 - 20r + 100 = 0$$

$$\Rightarrow\quad (r - 10)(r - 10) = 0$$

$$\Rightarrow\quad r = 10\ \text{cm}$$

(iii) $\theta = \dfrac{40 - 2(10)}{10} = \dfrac{40 - 20}{10} = 2$ radians

Exercise 3.2

1. (i) 0.7431 (ii) 0.2756 (iii) 0.5407 (iv) 0.7266 (v) 0.5914

2. (i) 48° (ii) 69° (iii) 55° (iv) 78° (v) 42° (vi) 12°

3. (i) $\theta = \sin^{-1}\frac{2}{3} = 41.81^\circ = 42^\circ$

(ii) $\theta = \cos^{-1}\frac{3}{5} = 53.13^\circ = 53^\circ$

(iii) $\theta = \tan^{-1}\frac{7}{8} = 41.186 = 41^\circ$

(iv) $\theta = \sin^{-1}\frac{2}{5} = 23.578 = 24^\circ$

4. (i) $\sin^2 45^\circ + \cos^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

(ii) $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

(iii) $\cos^2 60^\circ + \cos 60^\circ \sin 30^\circ$

$$= \left(\frac{1}{2}\right)^2 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

5. $\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{4} + \sin^2\frac{\pi}{3}$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = \frac{3}{2}$$

6. $|XZ|$ $\Rightarrow \sin 30^\circ = \dfrac{12}{|XZ|}$

$$\Rightarrow \frac{1}{2} = \frac{12}{|XZ|} \quad \Rightarrow |XZ| = 24$$

$|RZ|$ $\Rightarrow |RZ|^2 + (12)^2 = (24)^2$

$$\Rightarrow\quad |RZ|^2 + 144 = 576$$

$$\Rightarrow\quad |RZ|^2 = 432$$

$$\Rightarrow\quad |RZ| = \sqrt{432} = 12\sqrt{3}$$

$|YR| \quad \Rightarrow |YR|^2 + (12)^2 = (13)^2$

$\Rightarrow \quad |YR|^2 + 144 = 169$

$\Rightarrow \quad |YR|^2 = 25 \quad \Rightarrow |YR| = 5$

Hence, perimeter $= 24 + 13 + 5 + 12\sqrt{3}$

$= 42 + 12\sqrt{3}$

7. (i) $\tan 60° = \dfrac{x}{8}$

$\Rightarrow \quad x = 8 \tan 60°$

$= 13.856$

$= 13.9$

(ii) $\sin A = \dfrac{13.9}{20} = 0.695$

$\Rightarrow A = \sin^{-1} 0.695$

$= 44.02$

$= 44°$

8. (i) $\sin 30° = \dfrac{|RT|}{\sqrt{8}}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{|RT|}{\sqrt{8}}$

$\Rightarrow \quad 2|RT| = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \quad |RT| = \sqrt{2}$

(ii) $|PT|^2 + (\sqrt{2})^2 = (\sqrt{8})^2$

$\Rightarrow |PT|^2 + 2 = 8$

$\Rightarrow \quad |PT|^2 = 6$

$\Rightarrow \quad |PT| = \sqrt{6}$

$\triangle RTQ$ is isosceles $\quad \Rightarrow |RT| = |TQ| = \sqrt{2}$

$\Rightarrow \quad$ Area $\triangle RPQ = \frac{1}{2}(\sqrt{2} + \sqrt{6}) \cdot (\sqrt{2})$

$= \frac{1}{2}(\sqrt{4} + \sqrt{12})$

$= \frac{1}{2}(2 + 2\sqrt{3})$

$= 1 + \sqrt{3}$

Exercise 3.3

1. (i) 0.7660 (ii) -0.7660 (iii) 0.6428 (iv) -0.6428 (v) -0.8192 (vi) 0.5736

2. (i) $0.66913 = 0.6691$ (ii) $-0.84804 = -0.8480$ (iii) $-0.900404 = -0.9004$ (iv) $-0.93358 = -0.9336$

3. (i) $\sin 50°$ (ii) $-\cos 65°$ (iii) $-\tan 20°$ (iv) $-\cos 40°$ (v) $-\sin 70°$ (vi) $-\tan 60°$

4. (i) $\sin 120° = \sin 60° = \dfrac{\sqrt{3}}{2}$

(ii) $\cos 135° = -\cos 45° = -\dfrac{1}{\sqrt{2}}$

(iii) $\sin 240° = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

(iv) $\sin 210° = -\sin 30° = -\dfrac{1}{2}$

(v) $\cos 330° = \cos 30° = \dfrac{\sqrt{3}}{2}$

(vi) $\tan 225° = \tan 45° = 1$

(vii) $\cos 150° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

(viii) $\sin 300° = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

5. (i) 3rd (ii) 1st (iii) 2nd (iv) 1st

6. (i) 124° (ii) 68° (iii) 240° (iv) 345° (v) 75°

7. $\sin A = 0.2167 \Rightarrow A = \sin^{-1}(0.2167)$

$\Rightarrow A = 12.515° = 13°$

2nd quadrant $\Rightarrow A = 180° - 13° = 167°$

8. (i) $\cos A = 0.8428 \Rightarrow A = \cos^{-1}(0.8428)$

$= 32.56° = 33°$ (reference angle)

$\cos A = -0.8428 \Rightarrow$ 2nd quadrant $\Rightarrow A = 180° - 33° = 147°$

and 3rd quadrant $\Rightarrow A = 180° + 33° = 213°$

(ii) $\sin B = 0.6947 \Rightarrow B = \sin^{-1}(0.6947)$

$= 44.003° = 44°$ (reference angle)

$\sin B = -0.6947 \Rightarrow$ 3rd quadrant $\Rightarrow B = 180° + 44° = 224°$

and 4th quadrant $\Rightarrow B = 360° - 44° = 316°$

(iii) $\tan C = 0.9325 \Rightarrow C = \tan^{-1}(0.9325)$

$= 42.99° = 43°$

and 3rd quadrant $\Rightarrow C = 180° + 43° = 223°$

9. $\sin\theta = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\frac{1}{2} = 30°$

and 2nd quadrant $\Rightarrow \theta = 180° - 30° = 150°$

10. $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

and 4th quadrant $\Rightarrow \theta = 360° - 45° = 315°$

$\Rightarrow$ $\tan 45° = 1$ and $\tan 315° = -1$

11. $\tan A = \frac{1}{\sqrt{3}} \Rightarrow A = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

and 3rd quadrant $= 180° + 30° = 210°$

$\Rightarrow$ $\cos 30° = \frac{\sqrt{3}}{2}$ and $\cos 210° = -\frac{\sqrt{3}}{2}$

12. $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\frac{\sqrt{3}}{2} = 60°$ (reference angle)

$\sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow$ 3rd quadrant $\Rightarrow \theta = 180° + 60° = 240°$

and 4th quadrant $\Rightarrow \theta = 360° - 60° = 300°$

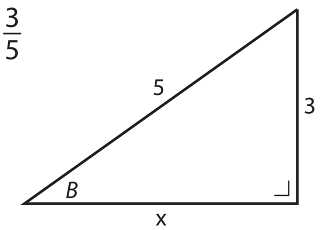
$\Rightarrow \cos 240° = -\frac{1}{2}$ and $\cos 300° = \frac{1}{2}$

13. $\sin A = \frac{4}{5} \Rightarrow A = \sin^{-1}\frac{4}{5} = 53.13 = 53°$ (reference angle)

$\sin < 0$ and $\cos < 0 \Rightarrow$ 3rd quadrant

$\Rightarrow A = 180° + 53° = 233°$

14. $\sin B = \frac{3}{5}$



$$x^2 + 3^2 = 5^2$$
$$x^2 + 9 = 25$$
$$x^2 = 16$$
$$x = 4$$

$\sin > 0$ and $\cos < 0 \Rightarrow$ 2nd quadrant

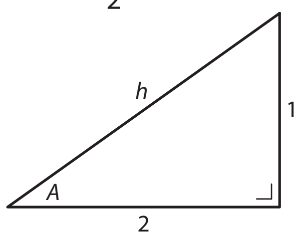
$\Rightarrow \tan B = -\frac{3}{4}$

15. $\tan B = \frac{1}{\sqrt{3}} \quad \Rightarrow B = \tan^{-1}\frac{1}{\sqrt{3}} = 30°$ (reference angle)

$\tan > 0$ and $\sin < 0 \quad \Rightarrow 3^{\text{rd}}$ quadrant

$\Rightarrow \cos B = -\frac{\sqrt{3}}{2}$

16. $\tan A = \frac{1}{2}$



$$h^2 = 1^2 + 2^2$$
$$= 1 + 4$$
$$= 5$$
$$\Rightarrow h = \sqrt{5}$$

$180° < A < 270° \Rightarrow 3^{\text{rd}}$ quadrant

$\Rightarrow \sin A = -\frac{1}{\sqrt{5}}$

17. (i) $\sin 420° = \sin 60° = \frac{\sqrt{3}}{2}$

(ii) $\cos 495° = \cos 135° = -\frac{1}{\sqrt{2}}$

(iii) $\tan(-120°) = \tan 240° = \sqrt{3}$

Exercise 3.4

1. $\frac{c}{\sin 42°} = \frac{8}{\sin 57°}$

$\Rightarrow \quad c = \frac{8 \sin 42°}{\sin 57°} = \frac{5.353}{0.83867} = 6.38 = 6.4$ m

$\frac{b}{\sin 57°} = \frac{14}{\sin 39°}$

$\Rightarrow \quad b = \frac{14 \sin 57°}{\sin 39°} = \frac{11.7414}{0.62932} = 18.657 = 18.7$ m

$\frac{a}{\sin 70°} = \frac{7}{\sin 80°}$

$\Rightarrow \quad a = \frac{7 \sin 70°}{\sin 80°} = \frac{6.5778}{0.9848} = 6.679 = 6.7$ m

2. (i) $\frac{7}{\sin x°} = \frac{10}{\sin 80°} \Rightarrow 10 \sin x° = 7 \sin 80°$

$\Rightarrow \quad \sin x° = \frac{7 \sin 80°}{10}$

$\Rightarrow \quad \sin x° = 0.6894$

$\Rightarrow \quad x° = \sin^{-1}(0.6894)$

$\Rightarrow \quad x° = 43.58°$

$\Rightarrow \quad x° = 44°$

(ii) $\frac{12}{\sin x°} = \frac{14}{\sin 42°} \Rightarrow 14 \sin x° = 12 \sin 42°$

$\Rightarrow \quad \sin x° = \frac{12 \sin 42°}{14}$

$\Rightarrow \quad \sin x° = 0.57354$

$\Rightarrow \quad x° = \sin^{-1}(0.57354)$

$\Rightarrow \quad x° = 34.997$

$\Rightarrow \quad x° = 35°$

(iii) $\dfrac{8}{\sin x°} = \dfrac{9.5}{\sin 51°} \Rightarrow 9.5 \sin x° = 8 \sin 51°$

$$\Rightarrow \quad \sin x° = \frac{8 \sin 51°}{9.5}$$
$$\Rightarrow \quad \sin x° = 0.65444$$
$$\Rightarrow \quad x° = \sin^{-1}(0.6544)$$
$$\Rightarrow \quad x° = 40.877$$
$$\Rightarrow \quad x° = 41°$$

3. (i) $\dfrac{10}{\sin \angle BAC} = \dfrac{8}{\sin 48°} \Rightarrow 8 \sin \angle BAC = 10 \sin 48°$

$$\Rightarrow \quad \sin \angle BAC = \frac{10 \sin 48°}{8}$$
$$\Rightarrow \quad = 0.92893$$
$$\Rightarrow \quad \angle BAC = \sin^{-1}(0.92893)$$
$$\Rightarrow \quad |\angle BAC| = 68.26879°$$

(ii) 3^{rd} angle $\angle ACB = 180° - (48° + 68.26879°)$
$= 63.7312°$

Hence, $\dfrac{|AB|}{\sin 63.7312°} = \dfrac{8}{\sin 48°}$

$$\Rightarrow \quad |AB| = \frac{8 \sin 63.7312°}{\sin 48°} = \frac{7.17382}{0.743} = 9.65$$
$$\Rightarrow |AB| = 9.7$$

(iii) Area $\triangle ABC = \frac{1}{2}(10)(9.7) \sin 48°$
$= 36.04$
$= 36$ sq. units

4. (i) Area $= \frac{1}{2}(8)(10) \sin 45°$
$= 28.28$
$= 28.3 \text{ cm}^2$

(ii) Area $= \frac{1}{2}(3.5)(5.5) \sin 100°$
$= 9.478$
$= 9.5 \text{ cm}^2$

(iii) Area $= \frac{1}{2}(8)(7.5) \sin 80°$
$= 29.544$
$= 29.5 \text{ cm}^2$

5. Area $= \frac{1}{2}(11)(8) \sin A = 25$

$$\Rightarrow \quad \sin A = \frac{25}{44}$$
$$\Rightarrow \quad A = \sin^{-1}\left(\frac{25}{44}\right)$$
$$\Rightarrow \quad A = 34.62°$$
$$\Rightarrow \quad A = 35°$$

Area $= \frac{1}{2}(8)(6.5) \sin B = 26$

$$\Rightarrow \quad \sin B = \left(\frac{26}{26}\right)$$
$$\Rightarrow \quad B = \sin^{-1} 1$$
$$\Rightarrow \quad B = 90°$$

$$\text{Area} = \frac{1}{2}(18)(13)\sin C = 78$$
$$\Rightarrow \quad \sin C = \frac{78}{117}$$
$$\Rightarrow \quad C = \sin^{-1}\left(\frac{78}{117}\right)$$
$$\Rightarrow \quad C = 41.8°$$
$$\Rightarrow \quad C = 42°$$

6. (i) 3rd angle $\angle ACB = 180° - (46° + 71°)$
$= 63°$

Hence, $\dfrac{|BC|}{\sin 71°} = \dfrac{22}{\sin 63°}$

$$\Rightarrow \quad |BC| = \frac{22 \sin 71°}{\sin 63°}$$
$$= \frac{20.8014}{0.891}$$
$$|BC| = 23.34$$
$$= 23 \text{ cm}$$

(ii) $\text{Area } \triangle ABC = \frac{1}{2}(22)(23)\sin 46°$
$= 181.99$
$= 182 \text{ cm}^2$

7. (i) $\dfrac{|RQ|}{\sin 30°} = \dfrac{\sqrt{8}}{\sin 45°}$

$$\Rightarrow |RQ| = \frac{\sqrt{8}\sin 30°}{\sin 45°} = 2$$

(ii) Third angle $\angle PRQ = 180° - (30° + 45°)$
$= 105°$

Hence, area $\triangle PQR = \frac{1}{2}(\sqrt{8})(2)\sin 105°$
$= 2.732$
$= 2.7 \text{ m}^2$

8. $\text{Area } \triangle ABC = \frac{1}{2}(x)(x+2)\sin 150° = 6$

$$\Rightarrow \quad (x^2 + 2x)\left(\frac{1}{2}\right) = 12$$
$$\Rightarrow \quad x^2 + 2x = 24$$
$$\Rightarrow \quad x^2 + 2x - 24 = 0$$
$$\Rightarrow \quad (x+6)(x-4) = 0$$
$$\Rightarrow \quad x = -6 \quad \text{or} \quad x = 4$$

Answer: $x = 4$

9. $\dfrac{5.4}{\sin \angle ACB} = \dfrac{3}{\sin 32°}$

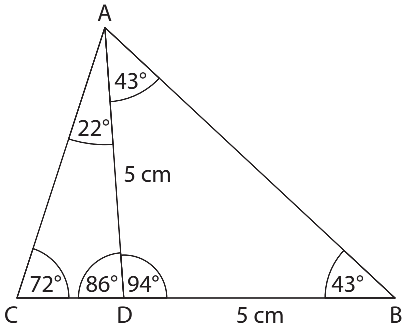
$$\Rightarrow (3)(\sin \angle ACB) = (5.4)(\sin 32°)$$
$$\Rightarrow \quad \sin(\angle ACB) = \frac{(5.4)(\sin 32°)}{3}$$
$$\Rightarrow \quad \sin(\angle ACB) = 0.953855$$
$$\Rightarrow \quad 1^{st} \text{ Quadrant} \Rightarrow \angle AC_1B = 72.526°$$
$$\Rightarrow \quad \angle AC_1B = 72.5°$$
$$\Rightarrow 2^{nd} \text{ Quadrant} \Rightarrow \angle AC_2B = 180° - 72.5°$$
$$= 107.5°$$

10. (i) $|\angle DAB| = 43°$

$\Rightarrow \quad |\angle ADB| = 180° - (43° + 43°) = 94°$

Hence, $\dfrac{|AB|}{\sin 94°} = \dfrac{5}{\sin 43°}$

$\Rightarrow \quad |AB| = \dfrac{5 \sin 94°}{\sin 43°} = 7.31 = 7.3 \text{ cm}$



(ii) $|\angle ADC| = 180° - 94° = 86°$

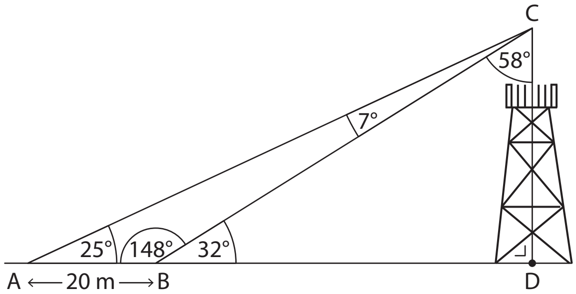
$\Rightarrow \quad |\angle CAD| = 180° - (86° + 72°) = 22°$

Hence, $\dfrac{|CD|}{\sin 22°} = \dfrac{5}{\sin 72°}$

$\Rightarrow \quad |CD| = \dfrac{5 \sin 22°}{\sin 72°} = 1.96 = 2.0 \text{ cm}$

11. $|\angle ABC| = 180° - 32° = 148°$

$\Rightarrow \quad |\angle ACB| = 180° - (25° + 148°) = 7°$



Hence, $\dfrac{|BC|}{\sin 25°} = \dfrac{20}{\sin 7°}$

$\Rightarrow \quad |BC| = \dfrac{(20)(\sin 25°)}{\sin 7°}$

$= 69.356 \text{ m}$

$\triangle BCD$ is right-angled $\triangle$.

$\Rightarrow \quad \sin 32° = \dfrac{|CD|}{69.356}$

$\Rightarrow \quad |CD| = (69.356)(\sin 32°)$

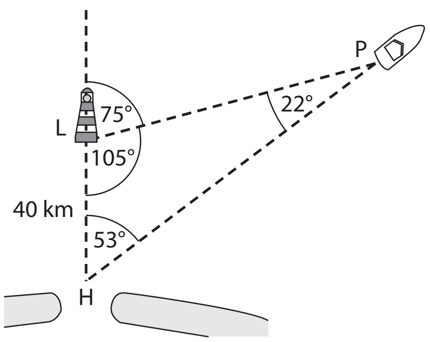
$= 36.75$

$= 36.8 \text{ m}$

12. $|\angle HLP| = 180° - 75° = 105°$

$|\angle HPL| = 180° - (105° + 53°)$

$= 22°$



Hence, $\dfrac{|PH|}{\sin 105°} = \dfrac{40}{\sin 22°}$

$\Rightarrow \quad |PH| = \dfrac{(40)(\sin 105°)}{\sin 22°}$

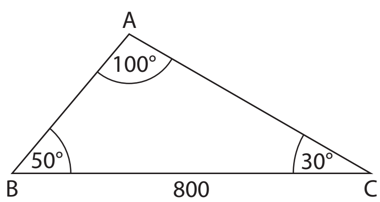
$= 103.14$

$= 103 \text{ km}$

13. $|\angle BAC| = 180° - 30° - 50° = 100°$

$|\angle ABC| = 50°$

$|\angle ACB| = 30°$



Then

(i) $\dfrac{|AB|}{\sin 30°} = \dfrac{800}{\sin 100°}$

$|AB| = \dfrac{800 \sin 30°}{\sin 100°} = 406 \text{ m}$

(ii) $\dfrac{|AC|}{\sin 50°} = \dfrac{800}{\sin 100°}$

$|AC| = \dfrac{800 \sin 50°}{\sin 110°} = 622 \text{ m}$

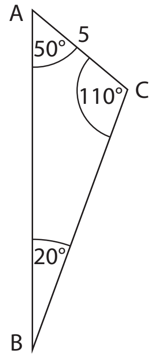
14. $|\angle ACB| = 180° - 50° - 20° = 110°$

Then

$$\frac{|AB|}{\sin 110°} = \frac{5}{\sin 20°}$$

$$|AB| = \frac{5 \sin 110°}{\sin 20°}$$

$$|AB| = 13.7 \text{ km.}$$



Exercise 3.5

1. (i) $x^2 = (8)^2 + (5)^2 - 2(8)(5) \cos 62°$

$\Rightarrow x^2 = 89 - 80(0.46947)$

$\Rightarrow x^2 = 51.4423$

$\Rightarrow x = \sqrt{51.4423}$

$\Rightarrow x = 7.17$

$\Rightarrow x = 7.2$ cm

(ii) $x^2 = (14)^2 + (11)^2 - 2(14)(11) \cos 38°$

$x^2 = 317 - (308)(0.788)$

$x^2 = 74.296$

$\Rightarrow x = \sqrt{74.296}$

$\Rightarrow x = 8.61$

$\Rightarrow x = 8.6$ cm

(iii) $x^2 = (5)^2 + (6.8)^2 - 2(5)(6.8) \cos 105°$

$\Rightarrow x^2 = 71.24 - 68(-0.2588)$

$\Rightarrow x^2 = 88.8397$

$\Rightarrow x = \sqrt{88.8397}$

$\Rightarrow x = 9.42$

$\Rightarrow x = 9.4$ cm

2. $(12)^2 = (7)^2 + (8)^2 - 2(7)(8) \cos A$

$\Rightarrow 144 = 49 + 64 - 112 \cos A$

$\Rightarrow 112 \cos A = 113 - 144$

$\Rightarrow \cos A = -\frac{31}{112} = -0.2768$

$\Rightarrow A = 106.07°$

$\Rightarrow A = 106°$

$(14)^2 = (20)^2 + (16)^2 - 2(20)(16) \cos B$

$\Rightarrow 196 = 400 + 256 - 640 \cos B$

$\Rightarrow 640 \cos B = 656 - 196 = 460$

$\Rightarrow \cos B = \frac{460}{640} = 0.71875$

$\Rightarrow B = \cos^{-1}(0.71875)$

$\Rightarrow B = 44.05$

$\Rightarrow B = 44°$

$(18)^2 = (9)^2 + (13)^2 - 2(9)(13) \cos C$

$\Rightarrow 324 = 81 + 169 - (234) \cos C$

$\Rightarrow 234 \cos C = 250 - 324$

$$\Rightarrow \quad \cos C = -\frac{74}{234} = -0.31624$$
$$\Rightarrow \quad C = \cos^{-1}(-0.31624)$$
$$\Rightarrow \quad C = 108.4°$$
$$\Rightarrow \quad C = 108°$$

3. $(7)^2 = (3)^2 + (5)^2 - 2(3)(5)\cos A$
$$\Rightarrow \quad 49 = 9 + 25 - 30\cos A$$
$$\Rightarrow \quad 30\cos A = 34 - 49 = -15$$
$$\Rightarrow \quad \cos A = -\frac{15}{30} = -0.5$$
$$\Rightarrow \quad A = \cos^{-1}(-0.5) = 120°$$

4. $(10)^2 = (4)^2 + (8)^2 - 2(4)(8)\cos A$
$$\Rightarrow \quad 100 = 16 + 64 - 64\cos A$$
$$\Rightarrow \quad 64\cos A = 80 - 100 = -20$$
$$\Rightarrow \quad \cos A = -\frac{20}{64} = -0.3125$$
$$\Rightarrow \quad A = \cos^{-1}(-0.3125) = 108.21°$$
$$\Rightarrow \quad \text{Area} = \frac{1}{2}(4)(8)\sin(108.21°)$$
$$= 16(0.9499) = 15.19 = 15.2 \text{ sq. units}$$

5. (i) $|\angle QRS| = 180° - (30° + 52°) = 98°$

Hence, $\dfrac{|QS|}{\sin 98°} = \dfrac{2}{\sin 30°}$
$$\Rightarrow \quad |QS| = \frac{2(\sin 98°)}{\sin 30°} = 3.96 = 4.0 \text{ units}$$

(ii) $\angle PQS \Rightarrow (6.5)^2 = (3.5)^2 + (4.0)^2 - 2(3.5)(4.0)\cos\angle PQS$
$$\Rightarrow \quad 42.25 = 12.25 + 16 - 28\cos\angle PQS$$
$$\Rightarrow \quad 28\cos\angle PQS = 28.25 - 42.25 = -14$$
$$\Rightarrow \quad \cos\angle PQS = -\frac{14}{28} = -0.5$$
$$\Rightarrow \quad |\angle PQS| = \cos^{-1}(-0.5) = 120°$$

6. $|QR|^2 = (42)^2 + (50)^2 - 2(42)(50)\cos 72°$
$$= 1764 + 2500 - 4200(0.309)$$
$$= 4264 - 1297.87$$
$$= 2966.13$$
$$\Rightarrow \quad |QR| = \sqrt{2966.13} = 54.46 = 54.5 \text{ m}^2$$
$$\Rightarrow \quad \text{Length of rope} = 42 + 50 + 54.5$$
$$= 146.5 \text{ m}$$

7. (i) Area $\triangle ABC = \frac{1}{2}(12)(15)\sin\angle A = 65$
$$\Rightarrow \quad 90\sin\angle A = 65$$
$$\Rightarrow \quad \sin\angle A = \frac{65}{90} = 0.7222$$
$$\Rightarrow \quad \angle A = \sin^{-1}(0.7222)$$
$$\Rightarrow \quad \angle A = 46.2° = 46°$$

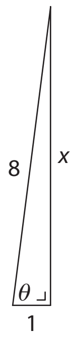
(ii) $|BC|^2 = (12)^2 + (15)^2 - 2(12)(15)\cos 46°$
$$= 144 + 225 - 360(0.69466)$$
$$= 369 - 250.08$$
$$= 118.92$$
$$\Rightarrow \quad |BC| = \sqrt{118.92} = 10.905 = 10.9 \text{ cm}$$

8. (i) $(6)^2 = (4)^2 + (5)^2 - 2(4)(5)\cos\theta$

$\Rightarrow \quad 36 = 16 + 25 - 40\cos\theta$

$\Rightarrow \quad 40\cos\theta = 41 - 36 = 5$

$\Rightarrow \quad \cos\theta = \frac{5}{40} = \frac{1}{8}$



(ii) $x^2 + (1)^2 = (8)^2$

$\Rightarrow \quad x^2 = 64 - 1 = 63$

$\Rightarrow \quad x = \sqrt{63} = \sqrt{9}\sqrt{7} = 3\sqrt{7}$

Hence, $\sin\theta = \frac{3\sqrt{7}}{8} \quad \Rightarrow a = 3, b = 8.$

9. Area $\triangle = \frac{1}{2}(2)(\sqrt{2})\sin A = 1$

$\Rightarrow \quad \sqrt{2}\sin A = 1$

$\Rightarrow \quad \sin A = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad A = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

Hence, $a^2 = (2)^2 + (\sqrt{2})^2 - 2(2)(\sqrt{2})\cos 45°$

$= 4 + 2 - 4\sqrt{2}\cdot\frac{1}{\sqrt{2}}$

$= 2$

$\Rightarrow a = \sqrt{2}$

Pythagoras' th. $\Rightarrow \quad 2^2 = (\sqrt{2})^2 + (\sqrt{2})^2$

$\Rightarrow \quad 4 = 2 + 2$

True; hence $\triangle$ is right-angled.

2 sides have length $\sqrt{2}$ each; hence $\triangle$ is isosceles.

10. Area $\triangle ABC = \left(\frac{1}{2}\right)(3.2)(8.4)\sin\angle B = 10$

$\Rightarrow 13.44\sin\angle B = 10$

$\Rightarrow \sin\angle B = \frac{10}{13.44} = 0.744$

$\Rightarrow \angle B = \sin^{-1}(0.744) = 48.07°$

$|AC|^2 = (3.2)^2 + (8.4)^2 - 2(3.2)(8.4)\cos 48.07°$

$= 10.24 + 70.56 - 53.76(0.668)$

$= 44.888$

$\Rightarrow |AC| = \sqrt{44.888} = 6.69 = 6.7$

Perimeter $= 6.7 + 3.2 + 8.4 = 18.3$ cm

11. (i) $(2x - 1)^2 = (x - 1)^2 + (x + 1)^2 - 2(x - 1)(x + 1)\cos 120°$

$\Rightarrow \quad 4x^2 - 4x + 1 = x^2 - 2x + 1 + x^2 + 2x + 1 - 2(x^2 - 1)\left(-\frac{1}{2}\right)$

$\Rightarrow \quad 4x^2 - 4x + 1 = 2x^2 + 2 + x^2 - 1$

$\Rightarrow \quad x^2 - 4x = 0$

$\Rightarrow \quad x(x - 4) = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad x = 4$

$\Rightarrow \quad$ Answer: $\quad x = 4$

(ii) Area $\triangle = \frac{1}{2}(3)(5)\sin 120°$

$= \left(\frac{15}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{15\sqrt{3}}{4}$

12. (i) $|FG|^2 = (50)^2 + (60)^2 - 2(50)(60)\cos 20°$

$= 2500 + 3600 - 6000(0.9397)$

$= 6100 - 5638.2$

$= 461.8$

$\Rightarrow |FG| = \sqrt{461.8} = 21.4895 = 21.5$ cm

(ii) $\cos 20° = \dfrac{120}{|CB|}$

$\Rightarrow \quad |CB| = \dfrac{120}{\cos 20°} = 127.7$ cm

$\Rightarrow \quad |AC| = 127.7$ cm

$\tan 20° = \dfrac{|CE|}{120}$

$\Rightarrow \quad |CE| = 120(\tan 20°) = 43.676 = 43.7$

$\Rightarrow$ Total length $= 4(60) + 2(127.7) + 2(21.5) + 43.7$

$= 582.1 = 582$ cm

13. $|PR|^2 = (8)^2 + (6)^2 = 64 + 36 = 100$

$\Rightarrow \quad |PR| = \sqrt{100} = 10$

$|PQ|^2 = (8)^2 + (5)^2 = 64 + 25 = 89$

$\Rightarrow \quad |PQ| = \sqrt{89}$

$|RQ|^2 = (6)^2 + (5)^2 = 36 + 25 = 61$

$\Rightarrow \quad |RQ| = \sqrt{61}$

Hence, $(10)^2 = (\sqrt{89})^2 + (\sqrt{61})^2 - 2\sqrt{89}\sqrt{61}\cos\angle Q$

$\Rightarrow \quad 100 = 89 + 61 - (147.36)(\cos\angle Q)$

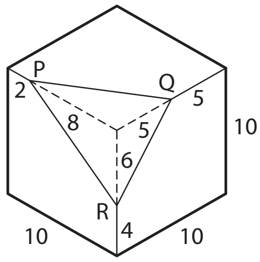
$\Rightarrow \quad (147.36)(\cos\angle Q) = 150 - 100 = 50$

$\Rightarrow \quad \cos(\angle Q) = \dfrac{50}{147.36} = 0.339$

$\Rightarrow \quad \angle PQR = \cos^{-1}(0.339)$

$= 70.18°$

$= 70°$



14. $3^2 = 2^2 + 2^2 - 2(2)(2)\cos A$

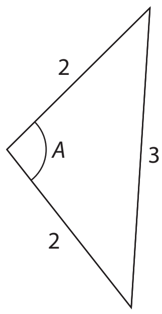
$9 = 8 - 8\cos A$

$8\cos A = -1$

$\cos A = -0.125$

$A = \cos^{-1}(-0.125)$

$A = 97.2°$



15. (i) For triangle ACD,

$|AC|^2 = 26.4^2 + 9.8^2 - 2(26.4)(9.8)\cos 56°$

$|AC|^2 = 503.65$

$|AC| = 22.44$ cm

For triangle ABC, let $\theta = |\angle ABC|$. Then

$22.44^2 = 8.4^2 + 16.3^2 - 2(8.4)(16.3)\cos\theta$

$273.84\cos\theta = -167.3036$

$\cos\theta = -0.61095$

$\theta = \cos^{-1}(-0.61095)$

$\theta = 127.66°$

(ii) Area $\triangle ADC = \frac{1}{2}(26.4)(9.8)\sin 56°$

$= 107.24$

Area $\triangle ABC = \frac{1}{2}(8.4)(16.3)\sin 127.66°$

$= 54.20$

Area quadrilateral $= 107.24 + 54.20$

$= 161.44\text{ cm}^2$

16. $6^2 = 5.5^2 + 1.2^2 - 2(5.5)(1.2)\cos\theta$

$13.2\cos\theta = -4.31$

$\cos\theta = -0.3265$

$\theta = \cos^{-1}(-0.3265)$

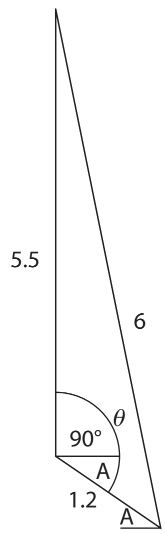
$\theta = 109.06°$

Then

$\theta = 90° + A$

$A = 109.06° - 90°$

$A = 19.06°$.



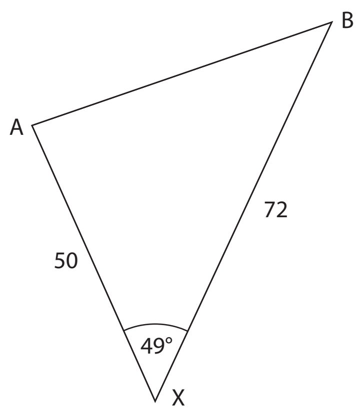
17. Let the two aircraft be A and B.

Then

$|AB|^2 = 50^2 + 72^2 - 2(50)(72)\cos 49°$

$|AB|^2 = 2960.37$

$|AB| = 54.4\text{ km}$

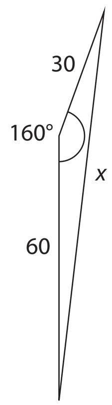


18. (i) Let x km be the distance from the port at 4 pm. Then

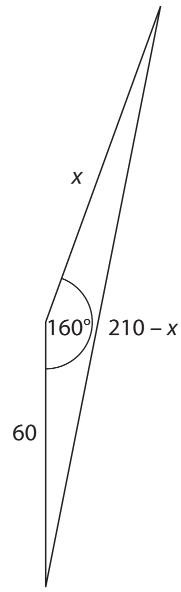
$x^2 = 60^2 + 30^2 - 2(60)(30)\cos 160°$

$x^2 = 7882.89$

$x = 88.79\text{ km}$



(ii) All three sides of the triangle must add up to 270. Let x be the distance travelled between 3 pm and the time to turn back. Then the distance back to the port must be $270 - 60 - x = 210 - x$.



Then

$(210 - x)^2 = 60^2 + x^2 - 2(60)(x)\cos 160°$

$44\,100 - 420x + x^2 = 3600 + x^2 + 112.76x$

$40\,500 = 532.76x$

$x = 76.02$

The time after 3 pm

$= \dfrac{76.02}{30}$

$= 2.53$ hours

$= 2$ hours 32 min

The latest time to return is then 2 hours and 32 min after 3 pm, i.e.
5:32 pm or 17:32.

Exercise 3.6

1. (i) $|HB|^2 = (5)^2 + (10)^2 = 25 + 100 = 125$

$\Rightarrow |HB| = \sqrt{125} = 5\sqrt{5}$

Hence, $|GH|^2 = (5\sqrt{5})^2 + (4)^2 = 125 + 16 = 141$

$\Rightarrow |GH| = \sqrt{141} = 11.8749 = 11.87$ cm

(ii) $\sin(\angle GHB) = \dfrac{4}{11.87} = 0.337$

$\Rightarrow |\angle GHB| = \sin^{-1}(0.337) = 19.69° = 19.7°$

2. (i) $\tan 25° = \dfrac{12}{|AB|}$

$\Rightarrow |AB| = \dfrac{12}{\tan 25°} = 25.73 = 25.7$ m

(ii) $\tan(\angle TCB) = \dfrac{12}{15} = 0.8$

$\Rightarrow |\angle TCB| = \tan^{-1}(0.8) = 38.657° = 38.7°$

(iii) $|DB|^2 = (15)^2 + (25.7)^2 = 225 + 660.49 = 885.79$

$\Rightarrow |DB| = \sqrt{885.79} = 29.757 = 29.8$ m

(iv) $\tan(\angle TDB) = \dfrac{12}{29.8} = 0.4027$

$\Rightarrow |\angle TDB| = \tan^{-1}(0.4027)$

$= 21.93° = 21.9°$

3. (i) $\tan 48° = \dfrac{200}{|PY|} \Rightarrow |PY| = \dfrac{200}{\tan 48°} = 180.08 = 180$ m

$\tan 34° = \dfrac{200}{|QY|} \Rightarrow |QY| = \dfrac{200}{\tan 34°} = 296.5 = 297$ m

(ii) $|PQ|^2 = (180)^2 + (297)^2 - 2(180)(297)\cos 84°$

$= 32\,400 + 88\,209 - 106\,920(0.104528)$

$= 120\,608 - 11176.18$

$= 109431.82$

$\Rightarrow |PQ| = \sqrt{109431.82} = 330.8 = 331$ m

4. (i) $\tan(\angle ABF) = \dfrac{9}{27} = \dfrac{1}{3} = 0.3333$

$\Rightarrow |\angle ABF| = \tan^{-1}(0.3333) = 18.43 = 18.4°$

(ii) $|AC|^2 = (17)^2 + (27)^2$
$= 289 + 729$
$= 1018$
$\Rightarrow |AC| = \sqrt{1018} = 31.906 = 31.9$ cm

(iii) $\tan(\angle ACF) = \dfrac{9}{31.9} = 0.2821$
$\Rightarrow |\angle ACF| = \tan^{-1}(0.2821) = 15.75° = 15.8°$

5. (i) $|PR|^2 = (3)^2 + (5)^2 - 2(3)(5)\cos 120°$
$= 9 + 25 - 30(-0.5)$
$= 34 + 15$
$= 49$
$\Rightarrow |PR| = \sqrt{49} = 7$ m

(ii) $\tan 60° = \dfrac{|DP|}{7}$
$\Rightarrow |DP| = 7\tan 60° = 12.12$
Hence, $|DQ|^2 = (12.12)^2 + 5^2$
$= 146.89 + 25$
$= 171.89$
$\Rightarrow |DQ| = \sqrt{171.89} = 13.11 = 13$ m

6. (i) $|AO| = |OB| = x$
$\Rightarrow x^2 + x^2 = 3^2$
$\Rightarrow 2x^2 = 9$
$\Rightarrow x^2 = \dfrac{9}{2} = 4.5$
$\Rightarrow x = \sqrt{4.5}$
Hence, $|AE|^2 = (\sqrt{4.5})^2 + (2.5)^2$
$= 4.5 + 6.25 = 10.75$
$\Rightarrow |AE| = \sqrt{10.75} = 3.278 = 3.3$ m

(ii) $(3)^2 = (3.3)^2 + (3.3)^2 - 2(3.3)(3.3)(\cos\angle AEB)$
$\Rightarrow 9 = 10.89 + 10.89 - 21.78(\cos\angle AEB)$
$\Rightarrow 21.78(\cos\angle AEB) = 21.78 - 9 = 12.78$
$\cos\angle AEB = \dfrac{12.78}{21.78} = 0.58677$
$\Rightarrow \angle AEB = \cos^{-1}(0.58677)$
$= 54.07°$

Hence, area $\triangle AEB = \dfrac{1}{2}(3.3)(3.3)\sin 54.07°$
$= (5.445)(0.8097)$
$= 4.4088$

Total Area $= 4(4.4088)$
$= 17.6352$
$= 17.6\text{ m}^2$

7. $\tan 60° = \dfrac{36}{|AD|}$
$\Rightarrow AD = \dfrac{36}{\tan 60°} = 20.7846 = 20.78$ m
$\triangle BCD$ is isosceles $\Rightarrow |BD| = |CD| = 36$ m

Hence, $|AB|^2 + (20.78)^2 = (36)^2$
$\Rightarrow |AB|^2 = 1296 - 431.8 = 864.2$
$\Rightarrow |AB| = \sqrt{864.2} = 29.39 = 29$ m

8. (i) $|DB|^2 = (7)^2 + (12)^2 = 49 + 144 = 193$

$\Rightarrow \quad |DB| = \sqrt{193}$

Hence, $|DF|^2 = (\sqrt{193})^2 + (5)^2 = 193 + 25 = 218$

$\Rightarrow \quad |DF| = \sqrt{218} = 14.76 = 14.8$ cm

(ii) $\sin \angle BDF = \dfrac{5}{14.8} = 0.3378$

$\Rightarrow |\angle BDF| = \sin^{-1}(0.3378)$

$= 19.745°$

$= 19.7°$

(iii) $|MB|^2 = (6)^2 + (7)^2 = 36 + 49 = 85$

$\Rightarrow \quad |MB| = \sqrt{85} = 9.2195$

Hence, $\tan \angle FMB = \dfrac{5}{9.2195} = 0.5423$

$\Rightarrow \quad |\angle FMB| = \tan^{-1}(0.5423)$

$= 28.47°$

$= 28.5°$

9. $|OA| = |OB| = y$

Hence, $y^2 + y^2 = (2x)^2$

$\Rightarrow \quad 2y^2 = 4x^2$

$\Rightarrow \quad y^2 = 2x^2$

$\Rightarrow \quad y = \sqrt{2} \cdot x$

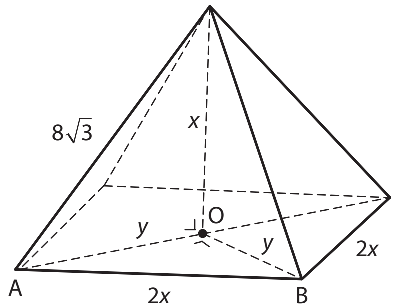
$\triangle$AOE is right-angled $\Rightarrow \quad y^2 + x^2 = (8\sqrt{3})^2$

$\Rightarrow \quad 2x^2 + x^2 = 192$

$\Rightarrow \quad 3x^2 = 192$

$\Rightarrow \quad x^2 = 64$

$\Rightarrow \quad x = 8$



10. $\triangle$BCD is right-angled.

$\Rightarrow \quad |DB|^2 = (5)^2 + (10)^2$

$= 25 + 100$

$= 125$

$\Rightarrow \quad |DB| = \sqrt{125} = 5\sqrt{5}$

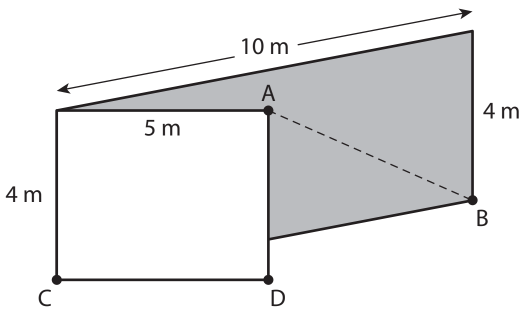
$\triangle$ADB is right-angled.

$\Rightarrow \quad |AB|^2 = (5\sqrt{5})^2 + (4)^2$

$= 125 + 16$

$= 141$

$\Rightarrow \quad |AB| = \sqrt{141} = 11.87 = 11.9$ m



11. $\tan 30° = \dfrac{4}{|SO|}$

$\Rightarrow \quad |SO| = \dfrac{4}{\tan 30°} = \dfrac{4}{\frac{1}{\sqrt{3}}} = 4\sqrt{3}$

$\tan 60° = \dfrac{4}{|OV|}$

$\Rightarrow \quad |OV| = \dfrac{4}{\tan 60°} = \dfrac{4}{\sqrt{3}}$

$|SV|^2 = (4\sqrt{3})^2 + \left(\dfrac{4}{\sqrt{3}}\right)^2 - 2(4\sqrt{3}) \cdot \left(\dfrac{4}{\sqrt{3}}\right) \cos 60°$

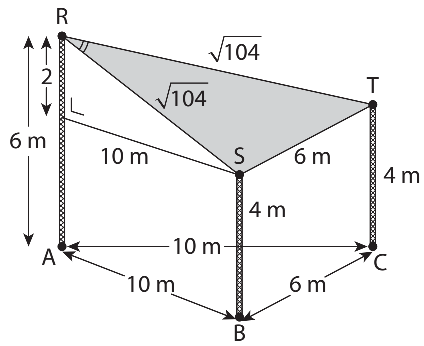
$= 48 + \dfrac{16}{3} - 2 \cdot (4\sqrt{3}) \cdot \left(\dfrac{4}{\sqrt{3}}\right) \cdot \dfrac{1}{2}$

$= 53\dfrac{1}{3} - 16$

$= 37\dfrac{1}{3}$

$\Rightarrow \quad |SV| = \sqrt{37\frac{1}{3}}$
$= 6.11$
$= 6.1$ m

12. (i) $\angle BAC \quad \Rightarrow \quad (6)^2 = (10)^2 + (10)^2 - 2(10)(10)\cos\angle BAC$
$\Rightarrow \quad 36 = 100 + 100 - 200\cos\angle BAC$
$\Rightarrow \quad 200\cos\angle BAC = 200 - 36 = 164$
$\Rightarrow \quad \cos\angle BAC = \frac{164}{200} = 0.82$
$\Rightarrow \quad |\angle BAC| = \cos^{-1}(0.82) = 34.9152°$



Hence, area ABC $= \frac{1}{2}(10)(10)\sin(34.9152°)$
$= 50(0.57236) = 28.618 = 28.6$ m^2

(ii) $|RS|^2 = (10)^2 + (2)^2 = 100 + 4 = 104$
$\Rightarrow \quad |RS| = \sqrt{104} \quad \Rightarrow |RT| = \sqrt{104}$
Hence, $(6)^2 = (\sqrt{104})^2 + (\sqrt{104})^2 - 2(\sqrt{104})(\sqrt{104})\cos\angle SRT$
$\Rightarrow \quad 36 = 104 + 104 - 208\cos\angle SRT$
$\Rightarrow \quad 208\cos\angle SRT = 208 - 36 = 172$
$\Rightarrow \quad \cos\angle SRT = \frac{172}{208} = 0.8269$
$\Rightarrow \quad |\angle SRT| = \cos^{-1}(0.8269) = 34.2184°$
Hence, area $\triangle RST = \frac{1}{2}(\sqrt{104})(\sqrt{104})(\sin 34.2184°)$
$= (52)(0.5623) = 29.23 = 29.2$ m^2

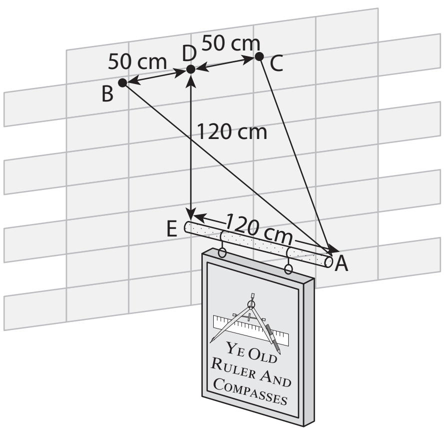
13. (i) $\sin 41° = \frac{105}{|TA|}$
$\Rightarrow \quad |TA| = \frac{105}{\sin 41°} = 160.04 = 160$ m

(ii) $|BT|^2 = (300)^2 + (160)^2 = 90\,000 + 25\,600 = 115\,600$
$\Rightarrow \quad |BT| = \sqrt{115\,600} = 340$ m

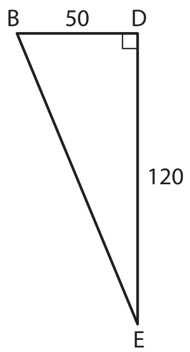
(iii) Vertical $= 105$ m
Length of Road $= 5 \times 105$
$= 525$ m

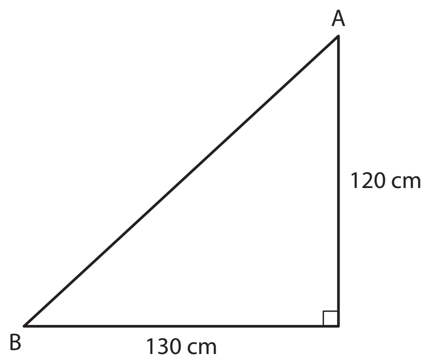
(iv) $|AC|^2 = (525)^2 - (160)^2$
$= 275\,625 - 25\,600$
$= 250\,028$
$\Rightarrow \quad |AC| = \sqrt{250\,028}$
$= 500$ m
Hence, $|BC| = 500 - 300$
$= 200$ m

14. (i) $|DA|^2 = (120)^2 + (120)^2$
$= 14\,400 + 14\,400$
$= 28\,800$
$\Rightarrow \quad |DA| = \sqrt{28\,800} = 169.7$ cm

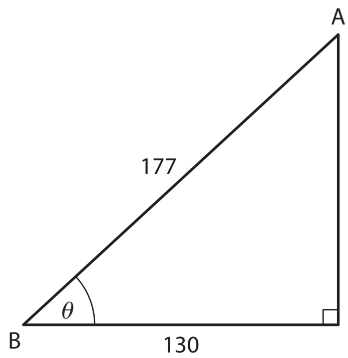


(ii) $|BE|^2 = \sqrt{120^2 + 50^2}$
$= \sqrt{16\,900}$
$|BE| = 130$ cm
$|AB|^2 = \sqrt{120^2 + 130^2}$
$= \sqrt{31\,300}$
$|AB| = 176.9$
$= 177$ cm





(iii) $\cos\theta = \dfrac{130}{177}$
$= 0.7344$
$\theta = 42.7°$
$\theta = 43°$
$\therefore$ the wire makes an angle of 43° with the wall.



Exercise 3.7

1. 4

2.

x	0	45	90	135	180	225	270	315	360	405	450	495	540
$\sin x$	0	0.7	1	0.7	0	−0.7	−1	−0.7	0	0.7	1	0.7	0

3. (i) (540°, 0) (ii) Period $= \pi$

4. (i) Period $= 2\pi$; Range $= [-3, 3]$
(ii) Period $= \pi$; Range $= [-2, 2]$
(iii) Period $= \dfrac{2\pi}{3}$; Range $= [-4, 4]$

5. Period $= \pi$; Range $= [-3, 3]$; $y = 3\cos 2x$

6. (i) Period $= \pi$; Range $= [-1, 1]$; $y = \cos 2x$
(ii) Period $= \pi$; Range $= [-2, 2]$; $y = 2\sin 2x$
(iii) Period $= \dfrac{\pi}{2}$; Range $= \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$; $y = \dfrac{1}{2}\sin 4x$
(iv) Period $= \pi$; Range $= [-4, 4]$; $y = 4\cos 2x$

7. (i) 1 (iii) 1 (v) -1; $\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$
(ii) 0 (iv) -1

Exercise 3.8

1. (i) $y = 3\sin x$
Amplitude $= |3| = 3$; period $= \dfrac{2\pi}{1} = 2\pi$
(ii) $y = -2\cos x$
Amplitude $= |-2| = 2$; period $= \dfrac{2\pi}{1} = 2\pi$
(iii) $y = 4\sin 4x$
Amplitude $= |4| = 4$; period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

(iv) $y = \frac{1}{4} \cos 2x$

Amplitude $= \left|\frac{1}{4}\right| = \frac{1}{4}$; period $= \frac{2\pi}{2} = \pi$

2. (i) $y = 3 \cos 2x$

Period $= \frac{2\pi}{2} = \pi$

(ii) $y = -4 \sin 3x$

Period $= \frac{2\pi}{3}$

(iii) $y = 5 \cos \frac{x}{4}$

Period $= \frac{2\pi}{\frac{1}{4}} = 8\pi$

(iv) $y = \frac{1}{4} \cos 6x$

Period $= \frac{2\pi}{6} = \frac{\pi}{3}$

3. (i) $y = 2 \sin 4x$

Range $= [-2, 2]$

(ii) $y = -5 \cos 2x$

Range $= [-5, 5]$

(iii) $y = 8 \sin \frac{x}{8}$

Range $= [-8, 8]$

(iv) $y = 6 \cos x$

Range $= [-6, 6]$

4. (i) Range of $[-3, 3]$. Thus $a = \pm 3$.

Period $= \pi$. Thus $\frac{2\pi}{b} = \pi$ and so $b = 2$.

(ii) Amplitude $= 4$. Thus $a = \pm 4$.

Period $= 4\pi$. Thus $\frac{2\pi}{b} = 4\pi$ and so $b = \frac{1}{2}$.

5. From the graph the amplitude is 3. Hence $a = 3$.

Also the period is π. Hence $\frac{2\pi}{b} = \pi$ and so $b = 2$.

6. (i) The curve is a cosine curve, $y = a \cos bx$.

As the period is π, $\frac{2\pi}{b} = \pi$. Hence $b = 2$.

As the range is $[-1, 1]$, $a = 1$.

Thus the curve is $y = \cos 2x$.

(ii) The curve is a sine curve, $y = a \sin bx$.

As the period is 4π, $\frac{2\pi}{b} = 4\pi$. Thus $b = \frac{1}{2}$.

As the range is $[-2, 2]$, $a = 2$.

Thus the curve is $y = 2 \sin \frac{1}{2}x$.

(iii) The curve is a sine curve, $y = a \sin bx$.

As the period is $\frac{\pi}{2}$, $\frac{2\pi}{b} = \frac{\pi}{2}$. Thus $b = 4$.

As the range is $[-0.5, 0.5]$, $a = 0.5$.

Thus the curve is $y = 0.5 \sin 4x$.

(iv) The curve is a cosine curve, $y = a \cos bx$.

As the period is π, $\frac{2\pi}{b} = \pi$. Thus $b = 2$.

As the range is $[-4, 4]$, $a = 4$.

Thus the curve is $y = 4 \cos 2x$.

7. (i) As $f(x)$ has an amplitude of 3, $f(x) = 3 \cos 2x$.

As $g(x)$ has an amplitude of 2, $g(x) = 2 \cos 3x$.

As $h(x)$ has an amplitude of 1, $h(x) = \cos 3x$.

(ii) g and h both have a period of $\frac{2\pi}{3}$. A is one quarter of this, i.e.

$\frac{1}{4} \times \frac{2\pi}{3} = \frac{\pi}{6}$. Thus A $= \left(\frac{\pi}{6}, 0\right)$.

B is three quarters of the period of g and h, i.e. $\frac{3}{4} \times \frac{2\pi}{3} = \frac{\pi}{2}$. Thus B $= \left(\frac{\pi}{2}, 0\right)$.

C is five quarters of the period of g and h, i.e. $\frac{5}{4} \times \frac{2\pi}{3} = \frac{5\pi}{6}$. Thus C $= \left(\frac{5\pi}{6}, 0\right)$.

f has a period of $\frac{2\pi}{2} = \pi$. D is seven quarters of this, i.e. $\frac{7}{4} \times \pi = \frac{7\pi}{4}$.

Thus D $= \left(\frac{7\pi}{4}, 0\right)$.

8. (i) $y = 4 + 2 \sin x$
(a) Mid-line equation: $y = 4$
(b) Range $= [4 - 2, 4 + 2] = [2, 6]$
(c) Period $= \frac{2\pi}{1} = 2\pi$

(ii) $y = -1 + 2 \cos 3x$
(a) Mid-line equation: $y = -1$
(b) Range $= [-1 - 2, -1 + 2] = [-3, 1]$
(c) Period $= \frac{2\pi}{3}$

(iii) $y = -2 - \cos 4x$
(a) Mid-line equation: $y = -2$
(b) Range $= [-2 - 1, -2 + 1] = [-3, -1]$
(c) Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

(iv) $y = 2 + 5 \cos \frac{x}{4}$
(a) Mid-line equation: $y = 2$
(b) Range $= [2 - 5, 2 + 5] = [-3, 7]$
(c) Period $= \frac{2\pi}{\frac{1}{4}} = 8\pi$.

9. (i) (a) The middle of the graph is at $\frac{0 + 4}{2} = 2$. Thus the mid-line is $y = 2$.
(b) The period is π.
The curve is a normal cosine curve, $y = a + b \cos cx$, where $b > 0$.
As the mid-line is $y = 2$, $a = 2$. As the amplitude is $4 - 2 = 2$, $b = 2$.

As the period is π, $\frac{2\pi}{c} = \pi$, and so $c = 2$.

Thus the equation of the curve is $y = 2 + 2 \cos 2x$.

(ii) (a) The middle of the graph is $\frac{-4 + 2}{2} = -1$. Thus the mid-line is $y = -1$.
(b) The period is 4π.
The curve is an inverted cosine curve, $y = a + b \cos cx$, where $b < 0$.
As the mid-line is $y = -1$, $a = -1$.
As the amplitude is $2 - (-1) = 3$, $b = -3$.

As the period is 4π, $\frac{2\pi}{c} = 4\pi$, and so $c = \frac{1}{2}$.

Thus the equation of the curve is $y = -1 - 3 \cos \frac{1}{2}x$.

(iii) (a) The middle of the graph is $\frac{3+1}{2} = 2$. Thus the mid-line is $y = 2$.
(b) The period is 2π.
The curve is a normal sine graph, $y = a + b \sin cx$, with $b > 0$.
As the mid-line is $y = 2$, $a = 2$.
As the amplitude is $3 - 2 = 1$, $b = 1$.
As the period is 2π, $\frac{2\pi}{c} = 2\pi$, and so $c = 1$.
Thus the equation of the curve is $y = 2 + \sin x$.

(iv) (a) The middle of the graph is $\frac{-5+1}{2} = -2$. Thus the mid-line is $y = -2$.
(b) From the graph, $\frac{3}{2} \times$ period $= \pi$. Thus period $= \frac{2\pi}{3}$.
The curve is an inverted sine graph, $y = a + b \sin cx$, with $b < 0$.
As the mid-line is $y = -2$, $a = -2$.
As the amplitude is $1 - (-2) = 3$, $b = -3$.
As the period is $\frac{2\pi}{3}$, $\frac{2\pi}{c} = \frac{2\pi}{3}$ and so $c = 3$.
Thus the equation of the curve is $y = -2 - 3 \sin 3x$.

12. (i) Graph is a normal sine graph, $y = a + b \sin cx$, with $b > 0$.
As the mid-line is $y = 0$, $a = 0$.
As the amplitude is 1, $b = 1$.
As the period is 4π, $\frac{2\pi}{c} = 4\pi$ and so $c = \frac{1}{2}$.
The equation of the curve is $y = \sin \frac{1}{2}x$.

(ii) Graph is a normal cosine graph, $y = a + b \cos cx$, with $b > 0$.
As the mid-line is $y = 0$, $a = 0$.
As the amplitude is 1, $b = 1$.
As the period is π, $\frac{2\pi}{c} = \pi$, and so $c = 2$.
The equation of the curve is $y = \cos 2x$.

As the LCM of 4π and π is 4π, and the graphs have coinciding maxima at $x = \pi$, the next coinciding maxima will occur at $\pi + 4\pi = 5\pi$. The point is $(5\pi, 0)$.

13. (i) $N = a + b \cos ct$, where $b > 0$ as the graph is a normal cosine graph.
The middle of the graph is $\frac{40 + 10}{2} = 25$. As this is in thousands, $a = 25\,000$.
The amplitude is $40 - 25 = 15$. As this is in thousands, $b = 15\,000$.
The period is 12. Thus $\frac{2\pi}{c} = 12$, and so $c = \frac{1}{6}$.
The equation of the graph is $N = 25\,000 + 15\,000 \cos \frac{\pi}{6}t$.

(ii) In a one year period the population is exactly 25 000 twice. Thus over a three year period the population is exactly 25 000 six times.

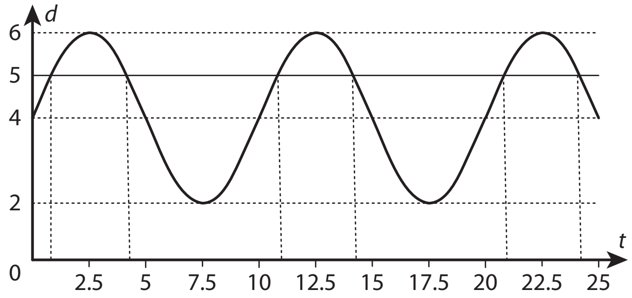
(iii) The horizontal line $N = 17\,500$ cuts the graph at $t = 4$ and at $t = 8$ in the first year. Thus the population is below 17 500 for $8 - 4 = 4$ months each year.

14. (i) $d = 4 + 2 \sin \frac{\pi t}{5}$
The range of d is $[4 - 2, 4 + 2] = [2, 6]$. Thus the low tide is at 2 m and the high tide is at 6 m.

(ii) The table is completed below.

t	0	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25
d	4	6	4	2	4	6	4	2	4	6	4

The graph is shown below.



(iii) At 2 p.m., $t = 14$ as 2 p.m. is 14 hours after midnight. Then

$d = 4 + 2\sin\dfrac{14\pi}{5}$

$d = 5.18$ m

(iv) From the graph, the line $d = 5$ cuts the graph at approx $t = 0.8$ (time 00:50) and at $t = 4.2$ (time 04:10). Between these times would suit.
Also this line cuts the graph at approx $t = 10.8$ (time 10:10) and at $t = 14.2$ (time 14:10). Between these times will also suit.
Finally, $d = 5$ cuts the graph at approx $t = 20.8$ (time 20:50) and at $t = 24.2$ (time 24:10). Between these times will also suit.

Exercise 3.9

1. $x = 30°$ and $150°$

2. $x = 30°$ and $330°$

3. $\theta = \dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$

4. $2\theta = 2n\pi + \dfrac{\pi}{6}$ or $2\theta = 2n\pi + \dfrac{5\pi}{6}$

$\Rightarrow \theta = n\pi + \dfrac{\pi}{12}$ $\Rightarrow \theta = n\pi + \dfrac{5\pi}{12}$

5. $3\theta = 30° + n(360°)$ or $3\theta = 330° + n(360°)$

$\Rightarrow \theta = 10° + n(120°)$ $\Rightarrow \theta = 110° + n(120°)$

6. $3\theta = \dfrac{4\pi}{3} + 2n\pi$ or $3\theta = \dfrac{5\pi}{3} + 2n\pi$

$\Rightarrow \theta = \dfrac{4\pi}{9} + \dfrac{2n\pi}{3}$ $\Rightarrow \theta = \dfrac{5\pi}{9} + \dfrac{2n\pi}{3}$

7. $4\theta = \dfrac{\pi}{3} + 2n\pi$ or $4\theta = \dfrac{5\pi}{3} + 2n\pi$

$\Rightarrow \theta = \dfrac{\pi}{12} + \dfrac{n\pi}{2}$ $\Rightarrow \theta = \dfrac{5\pi}{12} + \dfrac{n\pi}{2}$

8. (i) $2x = \dfrac{5\pi}{6} + 2n\pi$ or $2x = \dfrac{7\pi}{6} + 2n\pi$

$x = \dfrac{5\pi}{12} + n\pi$ or $x = \dfrac{7\pi}{12} + n\pi$

(ii) $n = 0$: $x = \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

$n = 1$: $x = \dfrac{17\pi}{12}, \dfrac{19\pi}{12}$

$n = 2$: $\quad x = \frac{29\pi}{12}, \frac{31\pi}{12}$

$n = 3$: $\quad x = \frac{41\pi}{12}, \frac{43\pi}{12}$

9. $3x = 225° + n(360°)$ or $3x = 315° + n(360°)$
$\Rightarrow x = 75° + n(120°)$ $\Rightarrow x = 105° + n(120°)$
$\Rightarrow x = 75°, 195°, 315°$ $\Rightarrow x = 105°, 225°, 345°$

10. $2\theta = 150° + n(360°)$ or $2\theta = 210° + n(360°)$
$\Rightarrow \theta = 75° + n(180°)$ $\Rightarrow \theta = 105° + n(180°)$
$\Rightarrow \theta = 75°, 255°$ $\Rightarrow \theta = 105°, 285°$

11. (i) $2\cos 3x - \sqrt{2} = 0$

$\cos 3x = \frac{\sqrt{2}}{2}$

$\cos 3x = \frac{1}{\sqrt{2}}$

$3x = \frac{\pi}{4} + 2n\pi$ or $3x = \frac{7\pi}{4} + 2n\pi$

$x = \frac{\pi}{12} + \frac{2n\pi}{3}$ or $x = \frac{7\pi}{12} + \frac{2n\pi}{3}$

(ii) $n = 0$: $\quad x = \frac{\pi}{12}, \frac{7\pi}{12}$

$n = 1$: $\quad x = \frac{3\pi}{4}$

12. $4\theta = \frac{\pi}{6} + 2n\pi$ or $4\theta = \frac{11\pi}{6} + 2n\pi$

$\Rightarrow \theta = \frac{\pi}{24} + \frac{n\pi}{2}$ $\Rightarrow \theta = \frac{11\pi}{24} + \frac{n\pi}{2}$

13. $3\theta = 120° + n(360°)$ or $3\theta = 240° + n(360°)$
$\Rightarrow \theta = 40° + n(120°)$ $\Rightarrow \theta = 80° + n(120°)$
$\Rightarrow \theta = 40°, 160°, 280°$ $\Rightarrow \theta = 80°, 200°, 320°$

14. $3\theta = 51° + n(360°)$ or $3\theta = 129° + n(360°)$
$\Rightarrow \theta = 17° + n(120°)$ $\Rightarrow \theta = 43° + n(120°)$
$\Rightarrow \theta = 17°, 137°, 257°$ $\Rightarrow \theta = 43°, 163°, 283°$

Revision Exercise 3 (Core)

1. Area $= \frac{1}{2}(8)(9)\sin 40°$
$= 36(0.6428)$
$= 23.14$
$= 23.1 \text{ cm}^2$

2. $\theta = 150°, 330°$

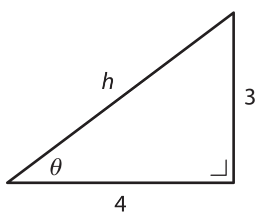
3. (i) Area $= \frac{1}{2}(20)^2\theta = 240$

$\Rightarrow 200\theta = 240$

$\Rightarrow \theta = \frac{240}{200} = \frac{6}{5}$ radians

(ii) Arc length $= (20)\left(\frac{6}{5}\right) = 24$ cm

4. $\tan\theta = \frac{3}{4}$



$$h^2 = 3^2 + 4^2$$
$$= 9 + 16 = 25$$
$$\Rightarrow \quad h = \sqrt{25} = 5 \quad \Rightarrow \sin\theta = \frac{3}{5}$$

Hence, area $= \frac{1}{2}(8)(7)\sin\theta$

$$= (28)\left(\frac{3}{5}\right)$$
$$= 16\frac{4}{5}\text{ cm}^2$$

5. (i) Period = 180°; Range = [−2, 2]

(ii) $a = 2$, Period $= \frac{360°}{b} = 180°$

$$\Rightarrow 180b = 360$$
$$\Rightarrow \quad b = 2$$

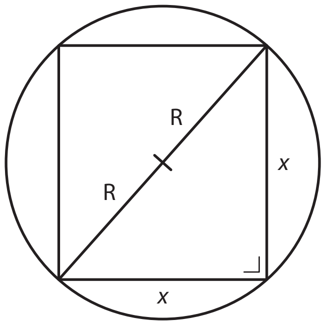
6. Right-angled △ $\Rightarrow x^2 + x^2 = (2R)^2$

$$\Rightarrow \quad 2x^2 = 4R^2$$
$$\Rightarrow \quad x^2 = 2R^2$$

Area Circle $= \pi R^2 = \pi$

$$\Rightarrow \quad R^2 = 1$$
$$\Rightarrow \quad R = 1$$

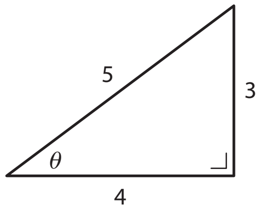
Area Square $= x^2 = 2(1)^2 = 2$ sq. units



7. $\sin < 0$ and $\cos > 0$

$\Rightarrow$ 4th Quadrant

$\Rightarrow \tan\theta = -\frac{3}{4}$



8. Area △PQR $= \frac{1}{2}(10)(8)\sin\angle QPR = 20$

$$\Rightarrow \quad 40\sin\angle QPR = 20$$
$$\Rightarrow \quad \sin\angle QPR = \frac{20}{40} = \frac{1}{2}$$
$$\Rightarrow \quad |\angle QPR| = \sin^{-1}\left(\frac{1}{2}\right) = 30° \text{ or } 150°$$

9. $4\sin\theta = 3$

$$\Rightarrow \quad \sin\theta = \frac{3}{4} = 0.75$$
$$\Rightarrow \quad \theta = \sin^{-1}(0.75)$$
$$\Rightarrow \quad \theta = 48.59° \text{ or } 131.41°$$
$$\Rightarrow \quad \theta = 49° \text{ or } 131°$$

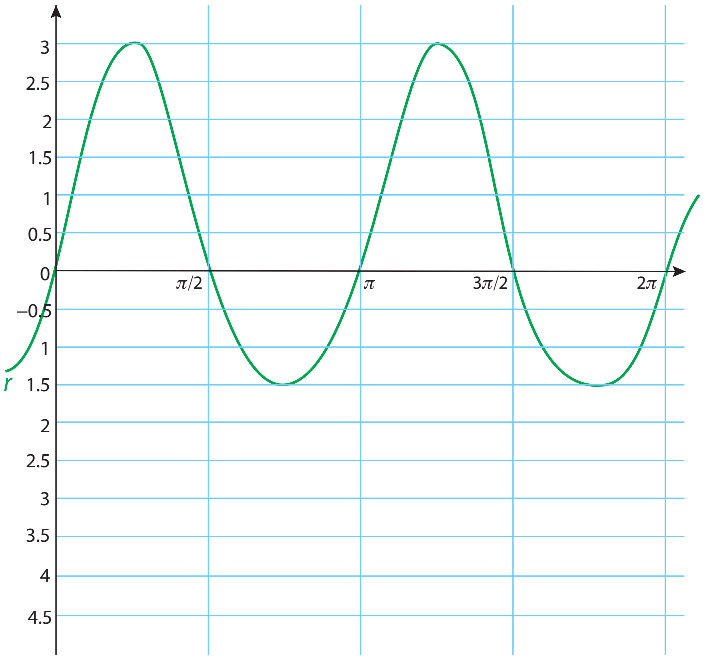
10. Area △ $= \frac{1}{2}(8)(7)\sin\theta = 14\sqrt{3}$

$$\Rightarrow \quad 28\sin\theta = 14\sqrt{3}$$
$$\Rightarrow \quad \sin\theta = \frac{14\sqrt{3}}{28} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \qquad \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$$

$$\Rightarrow \qquad \cos 60° = \frac{1}{2}$$

11.



Revision Exercise 3 (Advanced)

1. (i) $(17)^2 = (8)^2 + (12)^2 - 2(8)(12)\cos A$

$\Rightarrow 289 = 64 + 144 - 192 \cos A$

$\Rightarrow 192\cos A = -81$

$\Rightarrow \cos A = -\dfrac{81}{192} = -0.421875$

$\Rightarrow A = \cos^{-1}(-0.421875)$

$= 114.95° = 115°$

(ii) Area $= \dfrac{1}{2}(8)(12)\sin 115°$

$= 48(0.9063)$

$= 43.502 = 43.5 \text{ cm}^2$

2. (i) $2\theta = 2n\pi + \dfrac{5\pi}{6}$ or $2n\pi + \dfrac{7\pi}{6}$

$\Rightarrow \theta = n\pi + \dfrac{5\pi}{12}$ or $n\pi + \dfrac{7\pi}{12}$

(ii) Area $= \dfrac{1}{2}(4)^2\,\theta = 12$

$\Rightarrow 8\theta = 12$

$\Rightarrow \theta = \dfrac{12}{8} = 1\dfrac{1}{2}$ radians

3. (i) $(20)^2 = (18)^2 + (23)^2 - 2(18)(23)\cos A$

$\Rightarrow 400 = 324 + 529 - (828)\cos A$

$\Rightarrow 828\cos A = 853 - 400 = 453$

$\Rightarrow \cos A = \dfrac{453}{828} = 0.5471$

$\Rightarrow A = \cos^{-1}(0.5471) = 56.83 = 57°$

(ii) $\sin 57° = \dfrac{\text{height}}{47}$

$$\Rightarrow \quad \text{height} = 47(\sin 57°)$$
$$= 47(0.83867)$$
$$= 39.417 = 39 \text{ cm}$$

4. $|OB|^2 = (1600)^2 + (1600)^2$

$= 2\,560\,000 + 2\,560\,000$

$= 5\,120\,000$

$|OB| = \sqrt{5\,120\,000} = 2262.74$

Hence, $\dfrac{|AB|}{\sin 13°} = \dfrac{2262.74}{\sin 122°}$

$\Rightarrow \quad |AB| = \dfrac{(2262.74)\sin 13°}{\sin 122°}$

$= \dfrac{509}{0.848}$

$= 600.236 \text{ m}$

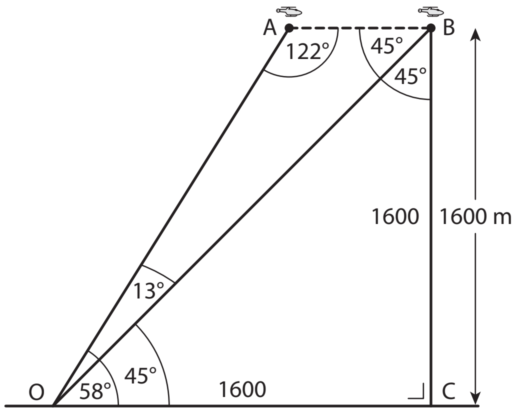
$\Rightarrow \quad \text{Speed} = 600.236 \text{ m per minute}$

$\Rightarrow 600.236 \text{ m} \times 60$

$= 36014.16 \text{ m}$

$= 36.01416 \text{ km} = 36 \text{ km}$

$\Rightarrow \quad \text{Speed} = 36 \text{ km/h}$



5. (i) $(5\sqrt{13})^2 = (10)^2 + (a\sqrt{3})^2 - 2(10)(a\sqrt{3})\cos 150°$

$\Rightarrow \quad 325 = 100 + 3a^2 - 20\sqrt{3}a\left(-\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad 225 = 3a^2 + 30a$

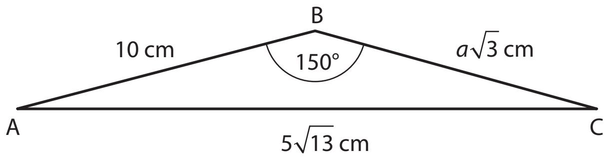
$\Rightarrow \quad 3a^2 + 30a - 225 = 0$

$\Rightarrow \quad a^2 + 10a - 75 = 0$

$\Rightarrow \quad (a + 15)(a - 5) = 0$

$\Rightarrow \quad a = -15 \quad \text{or} \quad a = 5$

Answer: $a = 5$



(ii) Area $\triangle ABC = \dfrac{1}{2}(10)(5\sqrt{3})\sin 150°$

$= 25\sqrt{3}\left(\dfrac{1}{2}\right)$

$= \dfrac{25\sqrt{3}}{2} \text{ cm}^2$

6. $\cos 33° = \dfrac{|AC|}{9}$

$\Rightarrow \quad |AC| = 9\cos 33$

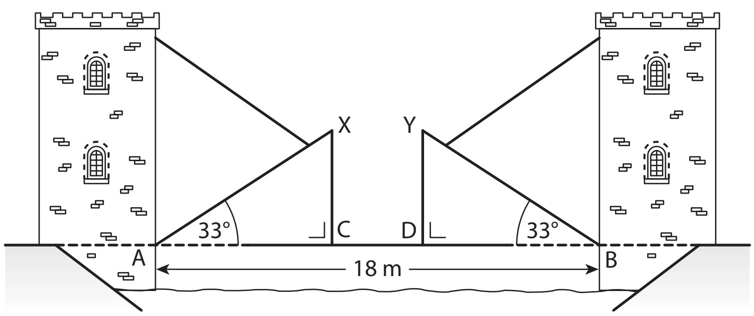
$= 7.548$

Hence, $|DB| = 7.548$

$\Rightarrow \quad |CD| = 18 - 2(7.548)$

$= 2.904 = 2.9 \text{ m}$

Hence, $|XY| = 2.9 \text{ m}$



7. (i) $[-4, 4]$

(ii) π

(iii) -4

(iv) $f(x) = 2\sin 2x$

$g(x) = 4\cos x$

(v) $P = \left(\frac{5\pi}{4}, 2\right)$

8. $\sin 63° = \frac{|OP|}{3}$

$\Rightarrow |OP| = 3\sin 63°$

$= 3(0.891) = 2.673$

$\cos 63° = \frac{|QP|}{1}$

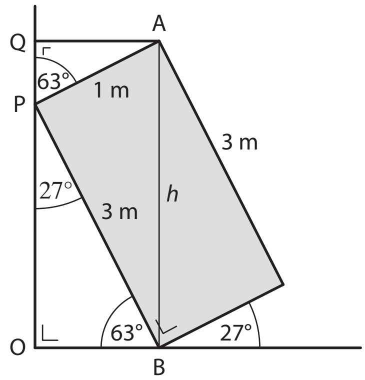
$\Rightarrow |QP| = 0.454$

Hence, $|OQ| = 2.673 + 0.454$

$= 3.127 = 3.13$ m

Since $|OQ| = |AB|$,

hence, $|AB| = 3.13$ m



9. (i) (a) $\tan 23° = \frac{4}{|OA|}$

$\Rightarrow |OA| = \frac{4}{\tan 23°}$

$= 9.423 = 9.4$ m

(b) $\cos \angle OAD = \frac{|DA|}{|OA|} = \frac{7}{9.4} = 0.74468$

$\Rightarrow |\angle OAD| = \cos^{-1}(0.74468)$

$= 41.86° = 42°$

(ii) $|OD|^2 + (7)^2 = (9.4)^2$

$\Rightarrow |OD|^2 + 49 = 88.36$

$\Rightarrow |OD|^2 = 88.36 - 49 = 39.36$

$\Rightarrow |OD| = \sqrt{39.36} = 6.274 = 6.3$

Hence, $\tan \angle COD = \frac{4}{6.3} = 0.6349$

$\Rightarrow |\angle COD| = \tan^{-1}(0.6349)$

$= 32.41° = 32.4°$

No; eye moves 32.4°.

10. (i) $\cos 3\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow 3\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow 3\theta = \frac{\pi}{6}$ (reference angle)

Cosine is positive in the 1st and 4th quadrants

$\Rightarrow 3\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$

Solutions: $3\theta = 2n\pi + \frac{\pi}{6}$ or $3\theta = 2n\pi + \frac{11\pi}{6}$

$\Rightarrow \theta = \frac{2n\pi}{3} + \frac{\pi}{18}$ or $\theta = \frac{2n\pi}{3} + \frac{11\pi}{18}$

(ii) Yellow Shaded Area $= \frac{1}{2}(6)^2\theta - \frac{1}{2}(4)^2\theta$

$$= 18\theta - 8\theta = 10\theta$$

$$\text{Green Shaded Area} = \frac{1}{2}(4)^2(2\pi - \theta)$$

$$= 8(2\pi - \theta)$$

$$= 16\pi - 8\theta$$

$$\text{Hence, } 10\theta = 16\pi - 8\theta$$

$$\Rightarrow \quad 18\theta = 16\pi$$

$$\Rightarrow \quad \theta = \frac{16\pi}{18} = \frac{8\pi}{9} \text{ radians}$$

11. $\tan 35° = \frac{10}{|SO|} \quad \Rightarrow \quad |SO| = \frac{10}{\tan 35°} = 14.28$

$\tan 50° = \frac{10}{|KO|} \quad \Rightarrow \quad |KO| = \frac{10}{\tan 50°} = 8.39$

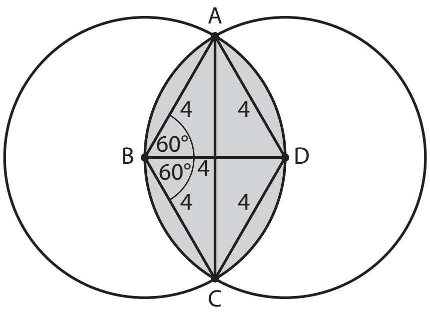
$$\Rightarrow |SK|^2 = (14.28)^2 + (8.39)^2 - 2(14.28)(8.39)\cos 60°$$

$$= 203.9184 + 70.3921 - (239.6184)(0.5)$$

$$= 154.5013$$

$$\Rightarrow \quad |SK| = \sqrt{154.5013} = 12.4 = 12 \text{ m}$$

12. Area sector ABC $= \frac{1}{2}(4)^2\frac{2\pi}{3} = \frac{16\pi}{3}$



Area $\triangle ABC = \frac{1}{2}(4)(4)\sin 120° = 4\sqrt{3}$

$\Rightarrow$ Area segment ADC $= \frac{16\pi}{3} - 4\sqrt{3}$

$\Rightarrow$ Total shaded area $= 2\left(\frac{16\pi}{3} - 4\sqrt{3}\right) = \left(\frac{32\pi}{3} - 8\sqrt{3}\right) \text{cm}^2$

13. (i) $a = (0, 4), b = (0, -4), c = \left(\frac{\pi}{6}, 0\right), d = \left(\frac{\pi}{2}, 0\right)$

$e = \left(\frac{5\pi}{6}, 0\right), f = \left(\frac{4\pi}{3}, 0\right)$

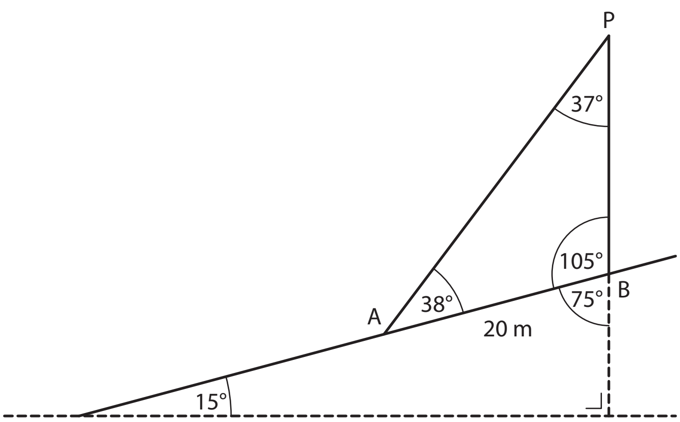
(ii) $\frac{|PB|}{\sin 38°} = \frac{20}{\sin 37°}$

$$\Rightarrow |PB| = \frac{20 \sin 38°}{\sin 37°}$$

$$= \frac{12.313}{0.6018}$$

$$= 20.46$$

$$= 20.5 \text{ m}$$



14. (i) $\cos 2\theta = \frac{\sqrt{3}}{2}$

$$2\theta = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad 2\theta = \frac{11\pi}{6} + 2n\pi$$

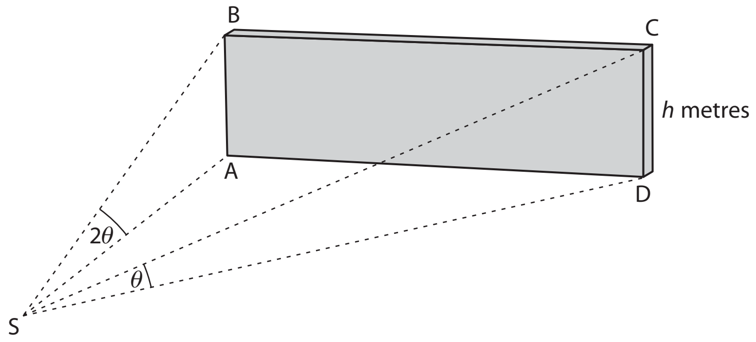
$$\theta = \frac{\pi}{12} + n\pi \quad \text{or} \quad \theta = \frac{11\pi}{12} + n\pi$$

(ii) $n = 0$: $\theta = \frac{\pi}{12}, \frac{11\pi}{12}$

$n = 1$: $\theta = \frac{13\pi}{12}, \frac{23\pi}{12}$

15. $\tan 2\theta = \frac{h}{|SA|} \Rightarrow |SA| = \frac{h}{\tan 2\theta}$

$\tan \theta = \frac{h}{|SD|} \Rightarrow |SD| = \frac{h}{\tan \theta}$



Hence, $|SD| = 5|SA|$

$\Rightarrow \quad \frac{h}{\tan \theta} = 5 \cdot \frac{h}{\tan 2\theta}$

$\Rightarrow \quad \tan 2\theta = 5 \tan \theta$

$\Rightarrow \quad \frac{2 \tan \theta}{1 - \tan^2 \theta} = 5 \tan \theta$

$\Rightarrow \quad 2 \tan \theta = 5 \tan \theta - 5 \tan^3 \theta$

$\Rightarrow \quad 5 \tan^3 \theta - 3 \tan \theta = 0$

$\Rightarrow \quad \tan \theta (5 \tan^2 \theta - 3) = 0$

$\Rightarrow \tan \theta = 0 \quad \text{or} \quad \tan^2 \theta = 0.6$

$\Rightarrow \theta = \tan^{-1} 0 = 0$ (not valid). Hence, $\tan \theta = \sqrt{0.6} = 0.7746$

$\Rightarrow \theta = \tan^{-1} 0.7746$

$\Rightarrow \theta = 37.76° = 38°$

Revision Exercise 3 (Extended-Response Questions)

1. (i) $\tan 18° = \frac{|CF|}{20} \Rightarrow |CF| = 20 \tan 18°$

$= 6.49 = 6.5$ m

(ii) $|DF|^2 = (20)^2 + (6.5)^2$

$= 400 + 42.25 = 442.25$

$|DF| = \sqrt{442.25} = 21.02 = 21.0$ m

(iii) $\tan \angle CAD = \frac{20}{10} = 2$

$\Rightarrow \quad |\angle CAD| = \tan^{-1}(2) = 63.43 = 63.4°$

(iv) $|AF|^2 = (10)^2 + (21.0)^2$

$= 100 + 441 = 541$

$\Rightarrow \quad |AF| = \sqrt{541} = 23.25 = 23.3$ m

(v) $\sin \angle FAC = \frac{|FC|}{|AF|} = \frac{6.5}{23.3} = 0.279$

$\Rightarrow \quad |\angle FAC| = \sin^{-1}(0.279) = 16.2005 = 16.2°$

2. (i) $\tan \theta = \frac{4}{|BP|} \Rightarrow |BP| = \frac{4}{\tan \theta}$

Hence, $|QC| = \frac{4}{\tan \theta}$

$|PQ| = k - \left(\frac{4}{\tan \theta} + \frac{4}{\tan \theta}\right) = k - \frac{8}{\tan \theta}$

(ii) $|PQ| = 12 - \frac{8}{\tan \theta} = 12 - 4\sqrt{3}$

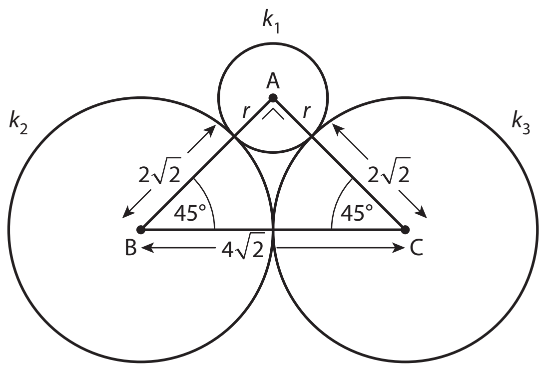
$\Rightarrow \quad \frac{8}{\tan \theta} = 4\sqrt{3}$

$$\Rightarrow \quad 4\sqrt{3}\tan\theta = 8$$

$$\Rightarrow \quad \tan\theta = \frac{8}{4\sqrt{3}} = 1.1547$$

$$\Rightarrow \quad \theta = \tan^{-1}(1.1547) = 49.1° = 49°$$

3. (i) $\sin 45° = \dfrac{r + 2\sqrt{2}}{4\sqrt{2}}$



$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{r + 2\sqrt{2}}{4\sqrt{2}}$$

$$\Rightarrow \quad r + 2\sqrt{2} = 4$$

$$\Rightarrow \quad r = 4 - 2\sqrt{2}$$

(ii) Area sector $k_2 = \frac{1}{2}(2\sqrt{2})^2\left(\frac{\pi}{4}\right)$

$$= \frac{1}{2}(8)\left(\frac{\pi}{4}\right) = \pi$$

Area sector $k_3 = \pi$

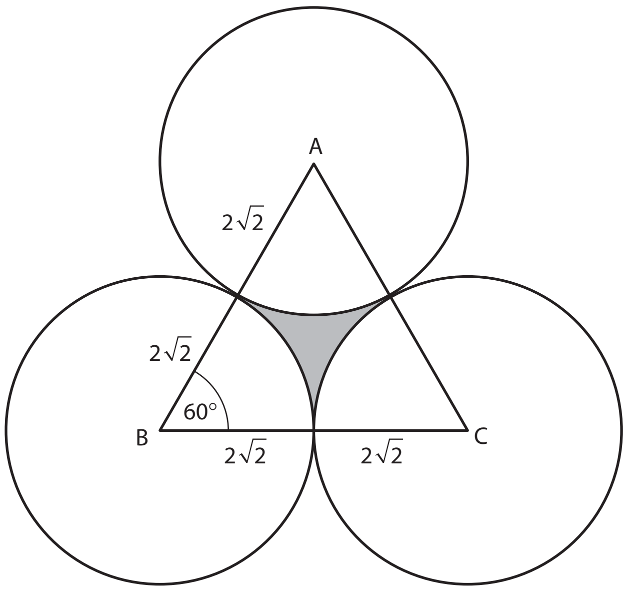
Area sector $k_1 = \frac{1}{2}(4 - 2\sqrt{2})^2\left(\frac{\pi}{2}\right) = \frac{1}{2}(16 - 16\sqrt{2} + 8)\left(\frac{\pi}{2}\right) = 6\pi - 4\sqrt{2}\pi$

Area $\triangle ABC = \frac{1}{2}(4)(4)\sin 45° = \frac{1}{2}16\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 8$

$\Rightarrow$ Shaded area $= 8 - (\pi + \pi + 6\pi - 4\sqrt{2}\pi)$

$$= 8 - 8\pi + 4\sqrt{2}\pi$$

(iii)



ABC is an equilateral triangle with sides of length $4\sqrt{2}$. Thus its area is

$$\frac{1}{2}(4\sqrt{2})(4\sqrt{2})\sin 60°$$

$$= 16\left(\frac{\sqrt{3}}{2}\right) = 8\sqrt{3}$$

Area of one sector $= \frac{1}{6}\left(\pi(2\sqrt{2})^2\right) = \frac{4\pi}{3}$

Area of three sectors $= 3\left(\frac{4\pi}{3}\right) = 4\pi$

Thus the shaded area is $(8\sqrt{3} - 4\pi)$ cm^2.

4. (i) Standard Proof

(ii) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{(a+1)^2 + (a+2)^2 - a^2}{2(a+1)(a+2)}$

$$= \frac{a^2 + 2a + 1 + a^2 + 4a + 4 - a^2}{2(a+1)(a+2)}$$

$$= \frac{a^2 + 6a + 5}{2(a+1)(a+2)} = \frac{(a+1)(a+5)}{2(a+1)(a+2)} = \frac{a+5}{2a+4}$$

5. (i) $\tan 25° = \frac{h}{|AO|} \Rightarrow |AO| = \frac{h}{\tan 25°}$

(ii) $\tan 33° = \frac{h}{|BO|} \Rightarrow |BO| = \frac{h}{\tan 33°}$

(iii) $(60)^2 = \left(\frac{h}{\tan 25°}\right)^2 + \frac{h}{\tan 33°}$

$\Rightarrow \quad 3600 = \frac{h^2}{0.2174} + \frac{h^2}{0.4217}$

$\Rightarrow \quad 3600(0.2174)(0.4217) = h^2(0.4217) + h^2(0.2174)$

$\Rightarrow \quad h^2(0.6391) = 330.04$

$\Rightarrow \quad h^2 = \frac{330.04}{0.6391} = 516.41$

$\Rightarrow \quad h = \sqrt{516.31} = 22.72 = 22.7 \text{ m}$

6. (i) As the period is $9 \times 2 = 18$ hours,

$$\frac{360}{r} = 18$$

$$r = \frac{360}{18} = 20$$

Also, range:

$[p - q, p + q] = [2, 8]$

$p - q = 2$

$p + q = 8$

$2p \quad = 10$

$p = 5$ and $q = 3$.

(iii) From the graph, the first time after low tide that the depth is 3.5 m is at $t = 12$. As low tide is at $t = 9$, the first time is $12 - 9 = 3$ hours after low tide.

7. (i) $s = 6 + 3\sin(3t)$

(a) Greatest distance is $6 + 3 = 9$ m from O.

(b) Least distance is $6 - 3 = 3$ m from O.

(ii) (a) $s = 9$

$6 + 3\sin(3t) = 9$

$3\sin(3t) = 3$

$\sin(3t) = 1$

$3t = \frac{\pi}{2}$

$t = \frac{\pi}{6} = 0.524$ seconds

(b) $s = 3$

$6 + 3\sin(3t) = 3$

$3\sin(3t) = -3$

$\sin(3t) = -1$

$3t = \frac{3\pi}{2}$

$t = \frac{\pi}{2} = 1.571$ seconds.

(iii) (a) Amplitude $= |3| = 3$

(b) Range $= [6 - 3, 6 + 3] = [3, 9]$

(iv) Period $= \frac{2\pi}{3} = 2.1$

(vi) $s = 9$

$6 + 3\sin(3t) = 9$

$3\sin(3t) = 3$

$\sin(3t) = 1$

$3t = \frac{\pi}{2} + 2n\pi$

$t = \frac{\pi}{6} + \frac{2n\pi}{3}$

$n = 0$: $t = \frac{\pi}{6} = 0.524$ s

$n = 1$: $t = \frac{5\pi}{6} = 2.618$ s

$n = 2$: $t = \frac{3\pi}{2} = 4.712$ s

$n = 3$: $t = \frac{13\pi}{6} = 6.807$ s

$n = 4$: $t = \frac{17\pi}{6} = 8.901$ s.

(vii) The motion of the particle is sinusoidal, with a period of $\frac{2\pi}{3}$ and a range of [3, 9].

8. (i) At 10 a.m., $t = 2$ (assumed).

At this time,

$$T_1 = 22 + 3\sin\left(\frac{2\pi}{12}\right) = 23.5°$$

$$T_2 = 19 + 8\sin\left(\frac{2\pi}{12}\right) = 23°$$

(ii) The temperature inside is at its highest when $\sin\left(\frac{\pi t}{12}\right) = 1$. At this time, the temperature outside is $T_2 = 19 + 8 = 27°$.

(iv) The temperatures are the same when

$$T_1 = T_2$$

$$22 + 3\sin\left(\frac{\pi t}{12}\right) = 19 + 8\sin\left(\frac{\pi t}{12}\right)$$

$$3 = 5\sin\left(\frac{\pi t}{12}\right)$$

$$\sin\left(\frac{\pi t}{12}\right) = 0.6$$

$$\frac{\pi t}{12} = 0.6435 + 2n\pi \quad \text{or} \quad \frac{\pi t}{12} = 2.498 + 2n\pi$$

$$\pi t = 7.722 + 24n\pi \quad \text{or} \quad \pi t = 29.976 + 24n\pi$$

$$t = 2.478 + 24n \quad \text{or} \quad t = 9.542 + 24n$$

$n = 0$: $t = 2.478$ or $t = 9.542$

$t = 2.478$ = 2 hours and 29 minutes after 8 a.m., i.e. 10:29 a.m.

$t = 9.542$ = 9 hours and 33 minutes after 8 a.m. i.e. 17:33 or 5:33 p.m.

Chapter 4

Exercise 4.1

1. (i) $|AB| = \sqrt{(3+1)^2 + (-2-3)^2} = \sqrt{16+25} = \sqrt{41}$

(ii) $|BC| = \sqrt{(5-3)^2 + (2+2)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

(iii) Slope AC $= \dfrac{2-3}{5+1} = -\dfrac{1}{6}$

(iv) Midpoint $= \left(\dfrac{3+5}{2}, \dfrac{-2+2}{2}\right) = (4, 0)$

2. Midpoint M $= \left(\dfrac{1-3}{2}, \dfrac{-6+4}{2}\right) = (-1, -1)$

$|AM| = \sqrt{(1+1)^2 + (-6+1)^2} = \sqrt{4+25} = \sqrt{29}$

$|MB| = \sqrt{(-1+3)^2 + (4+1)^2} = \sqrt{4+25} = \sqrt{29}$

3. (i) $\dfrac{3}{4}$ (ii) $\dfrac{-4}{3}$

4. Slope AB $= \dfrac{1-3}{-2-2} = \dfrac{-2}{-4} = \dfrac{1}{2} = m_1$

Slope CD $= \dfrac{1+2}{5+1} = \dfrac{3}{6} = \dfrac{1}{2} = m_2$

$m_1 = m_2 \Rightarrow$ Parallel lines

5. Slope AB $= \dfrac{3-1}{1+1} = \dfrac{2}{2} = 1 = m_1$

Slope CD $= \dfrac{4-2}{4-6} = \dfrac{2}{-2} = -1 = m_2$

$m_1 \cdot m_2 = (1)(-1) = -1 \Rightarrow$ Perpendicular lines

6. Slope $= \dfrac{3-0}{4+2} = \dfrac{3}{6} = \dfrac{1}{2} = m_1$

Slope $= \dfrac{1+1}{k-1} = \dfrac{2}{k-1} = m_2$

Parallel lines $\Rightarrow m_1 = m_2 \Rightarrow \dfrac{1}{2} = \dfrac{2}{k-1} \Rightarrow k - 1 = 4$

$\Rightarrow k = 5$

7. Slope $= \dfrac{13-1}{-P+1} = \dfrac{12}{-P+1} = \dfrac{2}{1}$

$\Rightarrow -2P + 2 = 12$

$-2P = 10$

$\Rightarrow P = -5$

8. (i) Slope AB $= \dfrac{5-3}{2+2} = \dfrac{2}{4} = \dfrac{1}{2}$; Slope BC $= \dfrac{1-5}{4-2} = \dfrac{-4}{2} = -2$; Slope CA $= \dfrac{1-3}{4+2} = \dfrac{-2}{6} = \dfrac{-1}{3}$

(ii) Product of slopes $= \left(\dfrac{1}{2}\right)(-2) = -1 \Rightarrow \angle ABC = 90°$

9. (i) Positive slopes $= a$ and c

(ii) Negative slopes $= b$ and d

10. Slope $a = \frac{2}{3}$

Slope $b = \frac{3}{2}$

Slope $c = \frac{4}{2} = 2$

11. Decreasing slope; Slope $= \frac{-5}{10} = -\frac{1}{2}$

12. $|PQ| = \sqrt{(2-a)^2 + (3-4)^2} = \sqrt{5 - 4a + a^2}$

$|RS| = \sqrt{(-2-3)^2 + (4+1)^2} = \sqrt{25 + 25} = \sqrt{50}$

hence, $5 - 4a + a^2 = 50$

$\Rightarrow a^2 - 4a - 45 = 0$

$\Rightarrow (a+5)(a-9) = 0$

$\Rightarrow a = -5, \quad a = 9$

13. Slope PQ $= \frac{2-6}{k-5} = \frac{-4}{k-5}$

Slope QR $= \frac{-1-2}{9-k} = \frac{-3}{9-k}$

Perpendicular lines $\Rightarrow \frac{-4}{k-5} \cdot \frac{-3}{9-k} = \frac{-1}{1}$

$\Rightarrow \frac{12}{-k^2 + 14k - 45} = \frac{-1}{1}$

$\Rightarrow k^2 - 14k + 45 = 12$

$\Rightarrow k^2 - 14k + 33 = 0$

$\Rightarrow (k-3)(k-11) = 0$

$\Rightarrow k = 3, \quad k = 11$

14. (i) Slope AB $= \frac{2+2}{7+1} = \frac{4}{8} = \frac{1}{2}$

(ii) Slope BC $= \frac{4-2}{k-7} = \frac{2}{k-7} = \frac{-2}{1} \Rightarrow -2k + 14 = 2$

$-2k = -12$

$k = 6$

(iii) $|AB| = \sqrt{(7+1)^2 + (2+2)^2} = \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5}$

(iv) $|BC| = \sqrt{(6-7)^2 + (4-2)^2} = \sqrt{1 + 4} = \sqrt{5}$

Area $\triangle ABC = \frac{1}{2}(4\sqrt{5})(\sqrt{5}) = 2.5 = 10$

15. (i) $|PQ| = \sqrt{(q+2)^2 + (0-2)^2} = \sqrt{q^2 + 4q + 8}$

$|QR| = \sqrt{(5-q)^2 + (3-0)^2} = \sqrt{q^2 - 10q + 34}$

$|PQ| = 2|QR|$

$\Rightarrow \sqrt{q^2 + 4q + 8} = 2\sqrt{q^2 - 10q + 34}$

$\Rightarrow q^2 + 4q + 8 = 4(q^2 - 10q + 34)$

$= 4q^2 - 40q + 136$

$\Rightarrow 3q^2 - 44q + 128 = 0$

$\Rightarrow (3q - 32)(q - 4) = 0$

$\Rightarrow q = \frac{32}{3}, \quad q = 4$

(ii) P(−2, 2) Q(4, 0) R(5, 3)

Slope PQ $= \frac{0-2}{4+2} = \frac{-2}{6} = \frac{-1}{3}$; Slope QR $= \frac{3-0}{5-4} = \frac{3}{1} = 3$

Product of slopes $= \frac{-1}{3} \cdot 3 = -1 \Rightarrow \triangle PQR$ is right-angled

Exercise 4.2

1. (i) Area $= \frac{1}{2}|(2)\cdot(4) - (3)\cdot(1)| = \frac{5}{2}$ sq.units

(ii) Area $= \frac{1}{2}|(5)\cdot(6) - (3)\cdot(1)| = \frac{27}{2}$ sq.units

(iii) Area $= \frac{1}{2}|(-2)\cdot(-4) - (1)\cdot(3)| = \frac{5}{2}$ sq.units

(iv) Area $= \frac{1}{2}|(3)\cdot(-6) - (-2)\cdot(4)| = 5$ sq.units

2. $(2, 3) \to (0, 0)$; $(-5, 1) \to (-7, -2)$; $(3, 1) \to (1, -2)$; Area $= \frac{1}{2}|(-7)(-2) - (1)(-2)| = 8$ sq.units

3. (i) $(2, 3) \to (0, 0)$; $(5, 1) \to (3, -2)$; $(2, 0) \to (0, -3)$; Area $= \frac{1}{2}|(3)(-3) - (0)(-2)| = \frac{9}{2}$ sq.units

(ii) $(-2, 3) \to (0, 0)$; $(4, 0) \to (6, -3)$; $(1, -4) \to (3, -7)$; Area $= \frac{1}{2}|(6)(-7) - (3)(-3)| = \frac{33}{2}$ sq.units

4. $\triangle ABC$; $(0, 0)(4, -1)(2, 3)$; Area $= \frac{1}{2}|(4)(3) - (2)(-1)| = 7$ sq.units

$\triangle ACD$; $(0, 0)(2, 3)(-2, 4)$; Area $= \frac{1}{2}|(2)(4) - (-2)(3)| = 7$ sq.units

Area quadrilateral $= 7 + 7 = 14$ sq. units

5. Area $= \frac{1}{2}|(1)(k + 1) - (-1)(6)| = 7$

$\Rightarrow \quad |k + 7| = 14$

$\Rightarrow \quad k + 7 = 14 \quad$ OR $\quad k + 7 = -14$

$\Rightarrow \quad k = 7 \quad$ OR $\quad k = -21$

6. $(4, 1) \to (0, 0)$; $(-1, -3) \to (-5, -4)$; $(3, k) \to (-1, k - 1)$

Area $= \frac{1}{2}|(-5)(k - 1) - (-1)(-4)| = 12$

$\Rightarrow |-5k + 5 - 4| = 24$

$\Rightarrow |-5k + 1| = 24$

$\Rightarrow -5k + 1 = 24 \quad$ OR $\quad -5k + 1 = -24$

$\Rightarrow \quad -5k = 23 \quad$ OR $\quad -5k = -25$

$\Rightarrow \quad k = \frac{-23}{5} \quad$ OR $\quad k = 5$

7. Area $= \frac{1}{2}|(4)(k) - (6)(3)| = 7$

$\Rightarrow |4k - 18| = 14$

$\Rightarrow 4k - 18 = 14 \quad$ OR $\quad 4k - 18 = -14$

$\Rightarrow \quad 4k = 32 \quad$ OR $\quad 4k = 4$

$\Rightarrow \quad k = 8 \quad$ OR $\quad k = 1$

8. Area $= \frac{1}{2}|(1)(6) - (2)(3)| = 0 \Rightarrow$ Points are collinear

9. $(-2, -1) \to (0, 0)$; $(1, 2) \to (3, 3)$; $(k, 13) \to (k + 2, 14)$

Area $= \frac{1}{2}|(3)(14) - (k + 2)(3)| = 6$

$\Rightarrow |42 - 3k - 6| = 12$

$\Rightarrow \quad |-3k + 36| = 12$

$\Rightarrow \quad -3k + 36 = 12 \quad$ OR $\quad -3k + 36 = -12$

$\Rightarrow \quad -3k = -24 \quad$ OR $\quad -3k = -48$

$\Rightarrow \quad k = 8 \quad\quad k = 16$

10. Slope AB $= \frac{3-1}{b-2} = \frac{2}{b-2}$; Slope BC $= \frac{5-3}{5-b} = \frac{2}{5-b}$

$|\angle ABC| = 90° \Rightarrow \frac{2}{b-2} \cdot \frac{2}{5-b} = -1$

$\Rightarrow \frac{4}{-b^2+7b-10} = \frac{-1}{1}$

$\Rightarrow b^2 - 7b + 10 = 4$

$\Rightarrow b^2 - 7b + 6 = 0$

$\Rightarrow (b-1)(b-6) = 0$

$\Rightarrow b = 1, b = 6$

$b = 6 \Rightarrow A(2, 1) \rightarrow (0, 0); B(6, 3) \rightarrow (4, 2); C(5, 5) \rightarrow (3, 4)$

Area $= \frac{1}{2}|(4)(4) - (2)(3)| = \frac{1}{2}|16 - 6| = 5$ sq. units

11. $(2, -1) \rightarrow (0, 0); (8, k) \rightarrow (6, k+1); (11, 2) \rightarrow (9, 3)$

Area $= \frac{1}{2}|(6)(3) - (k+1)(9)| = 0$

$\Rightarrow |18 - 9k - 9| = 0$

$\Rightarrow |-9k + 9| = 0$

$k = 1$

12. (i) Area $\triangle ABC = \frac{1}{2}bh = \frac{1}{2}(5)(6) = 15$ sq. units

(ii) $|BC| = \sqrt{(1-3)^2 + (7-1)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

(iii) Area $\triangle ABC = \frac{1}{2}|BC| \cdot |AL|$

$= \frac{1}{2} \cdot (2\sqrt{10}) \cdot |AL| = 15$

$|AL| = \frac{15}{\sqrt{10}} = \frac{15\sqrt{10}}{\sqrt{10}\sqrt{10}} = \frac{3\sqrt{10}}{2}$

Exercise 4.3

1. (i) $y + 1 = 3(x - 4)$

$\Rightarrow 3x - y - 13 = 0$

(ii) $y + 2 = -2(x + 5)$

$\Rightarrow 2x + y + 12 = 0$

2. $y - 1 = \frac{2}{3}(x + 3)$

$\Rightarrow 3y - 3 = 2x + 6$

$\Rightarrow 2x - 3y + 9 = 0$

3. (i) Slope $l = \frac{-a}{b} = -\frac{1}{-3} = \frac{1}{3}$

(ii) $y + 4 = \frac{1}{3}(x - 3)$

$\Rightarrow 3y + 12 = x - 3$

$\Rightarrow x - 3y - 15 = 0$

4. (i) Slope AB $= \frac{5+1}{4-3} = 6$

(ii) Perpendicular slope $= -\frac{1}{6}$; $C(-2, 1)$

Equation: $y - 1 = -\frac{1}{6}(x + 2)$

$\Rightarrow 6y - 6 = -x - 2$

$\Rightarrow x + 6y - 4 = 0$

5. $m_1 = -\frac{2}{-3} = \frac{2}{3}$; $m_2 = -\frac{3}{k}$

Perpendicular lines $\Rightarrow \frac{2}{3} \cdot \frac{-3}{k} = \frac{-1}{1}$

$\Rightarrow \quad -3k = -6$

$\Rightarrow \quad k = 2$

6. $m_1 = \frac{-3}{4}$; $m_2 = -\frac{-t}{2} = \frac{t}{2}$

Perpendicular lines $\Rightarrow \frac{-3}{4} \cdot \frac{t}{2} = \frac{-1}{1}$

$\Rightarrow \quad -3t = -8$

$\Rightarrow \quad t = \frac{8}{3}$

7. $2(3) + k(1) - 8 = 0$

$6 + k - 8 = 0$

$k = 2$

8. x-axis $\Rightarrow y = 0 \Rightarrow x - 6 = 0$ $\quad$ y-axis $\Rightarrow x = 0 \Rightarrow -3y - 6 = 0$

$x = 6; (6, 0)$ $\quad$ $y = -2; (0, -2)$

9. Slope $h = \frac{10 + 2}{-4 - 6} = \frac{12}{-10} = \frac{-6}{5}$; Slope $k = \frac{-a}{6}$

Perpendicular lines $\Rightarrow \frac{-6}{5} \cdot \frac{-a}{6} = -1$

$\Rightarrow \quad \frac{6a}{30} = \frac{-1}{1}$

$\Rightarrow \quad 6a = -30$

$\Rightarrow \quad a = -5$

10. (i) x-axis $\Rightarrow y = 0 \Rightarrow 2x + 6 = 0$

$\Rightarrow x = -3 \quad \Rightarrow C(-3, 0) \quad \Rightarrow x = -3$

(ii) Slope $= -\frac{2}{-3} = \frac{2}{3} \Rightarrow$ Perpendicular slope $= \frac{-3}{2}$

$\Rightarrow$ Equation: $y - 0 = \frac{-3}{2}(x + 3)$

$\Rightarrow 2y = -3x - 9$

$\Rightarrow 3x + 2y + 9 = 0$

11. Slope $= \frac{-2}{5}$, Point $= (-2, 3)$

Equation: $y - 3 = \frac{-2}{5}(x + 2)$

$\Rightarrow 5y - 15 = -2x - 4$

$\Rightarrow 2x + 5y - 11 = 0$

12. (i) $m_1 = 3, m_2 = \frac{1}{3}$; Neither

(ii) $m_1 = -2, m_2 = \frac{1}{2}$; Perpendicular

(iii) $m_1 = \frac{-2}{3}, m_2 = \frac{3}{2}$; Perpendicular

(iv) $m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}$; Parallel

(v) $m_1 = 2, m_2 = 2$; Parallel

(vi) $m_1 = -\frac{1}{3}, m_2 = 3$; Perpendicular

13.
$$\begin{array}{r} x + 2y = 1 \\ 2x + 3y = 4 \\ \hline 2x + 4y = 2 \\ 2x + 3y = 4 \\ \hline \end{array}$$
Subtract: $y = -2 \Rightarrow x = 5; (5, -2)$

14.
$$\begin{array}{rr} & x + y = 5 \\ & 2x - y = 1 \\ \hline \text{Add} \Rightarrow & 3x = 6 \end{array}$$
$x = 2 \Rightarrow y = 3 \quad (2, 3)$

Equation: $y - 3 = \frac{2}{3}(x - 2)$
$\Rightarrow 3y - 9 = 2x - 4$
$\Rightarrow 2x - 3y + 5 = 0$

15.
$$\begin{array}{rl} & 2x + 3y = 12 \\ & 3x - 4y = 1 \\ \hline & 8x + 12y = 48 \\ & 9x - 12y = 3 \\ \hline \text{Add} \Rightarrow & 17x = 51 \end{array}$$
$x = 3 \Rightarrow y = 2 \qquad (3, 2)$

Slope $= -\frac{3}{-1} = 3$

Equation: $y - 2 = 3(x - 3)$
$\Rightarrow y - 2 = 3x - 9$
$\Rightarrow 3x - y - 7 = 0$

16. (i) $2 - 2(6) + 10 = 0$...True
(ii) $2(3) + k(2) - 12 = 0$
$\Rightarrow 2k = 6$
$\Rightarrow k = 3$

17.
$$\begin{array}{rr} & 3x - 2y = -7 \\ & 5x + y = -3 \\ \hline & 3x - 2y = -7 \\ & 10x + 2y = -6 \\ \hline \text{Add} \Rightarrow & 13x = -13 \end{array}$$
$x = -1 \Rightarrow y = 2\ (-1, 2)$

Slope $l_2 = \frac{-5}{1} \Rightarrow$ Perpendicular slope $= \frac{1}{5}$

Equation: $y - 2 = \frac{1}{5}(x + 1)$
$\Rightarrow 5y - 10 = x + 1$
$\Rightarrow x - 5y + 11 = 0$

18. x-axis $\Rightarrow y = 0 \Rightarrow 3x = k$
$\Rightarrow x = \frac{k}{3} \qquad \left(\frac{k}{3}, 0\right)$

y-axis $\Rightarrow x = 0 \Rightarrow 4y = k$
$\Rightarrow y = \frac{k}{4} \qquad \left(0, \frac{k}{4}\right)$

Area $= \frac{1}{2}\left|\left(\frac{k}{3}\right)\left(\frac{k}{4}\right) - (0)(0)\right| = 24$

$\Rightarrow \frac{k^2}{12} = 48$

$\Rightarrow \quad k^2 = 576$

$\Rightarrow \quad k = \pm 24 \quad \Rightarrow k = 24$

19. Parallel line : $\quad 2x - 3y + c = 0$

Point $(4, 2) \Rightarrow 2(4) - 3(2) + c = 0$

$\Rightarrow \quad c = -2 \quad \Rightarrow \quad 2x - 3y - 2 = 0$

20. Parallel line: $4x + y = k$

x-axis $\Rightarrow y = 0 \Rightarrow 4x = k$

$\Rightarrow \quad x = \dfrac{k}{4} \qquad \left(\dfrac{k}{4}, 0\right)$

y-axis $\Rightarrow x = 0 \Rightarrow y = k \qquad (0, k)$

$$\text{Area} = \frac{1}{2}\left|\left(\frac{k}{4}\right)(k) - (0)(0)\right| = 18$$

$\Rightarrow \quad \dfrac{k^2}{4} = 36$

$\Rightarrow \quad k^2 = 144$

$\Rightarrow \quad k = \pm 12 \quad \Rightarrow \quad k = 12$

$\Rightarrow \quad 4x + y - 12 = 0$

21. (a) The line AB: $\quad$ A(4, −75) $\quad$ B(36, −4)

$$m = \frac{-4 + 75}{36 - 4} = \frac{71}{32}$$

Equation of AB:

$$(y + 75) = \frac{71}{32}(x - 4)$$

$32y + 2400 = 71x - 284$

$2684 = 71x - 32y$

or $\quad 71x - 32y = 2684$

School: $\quad$ C(20, −36):

$71(20) - 32(-36) = 2684$

$2572 = 2684$

As this is false, the flight path does not pass over the school.

(b) At the school, $y = -36$. Then, along the flight path:

$71x - 32(-36) = 2684$

$71x + 1152 = 2684$

$71x = 1532$

$$x = \frac{1532}{71} = 21.58$$

As the x co-ordinate of the school is 20, the path of the aircraft is

$21.58 - 20 = 1.58$

east of C, the school.

Exercise 4.4

1. $\left[\dfrac{4(5) + 1(-3)}{4 + 1}, \dfrac{4(-4) + 1(4)}{4 + 1}\right] = \left(\dfrac{17}{5}, \dfrac{-12}{5}\right)$

2. $\left[\dfrac{3(3) + 1(-5)}{3 + 1}, \dfrac{3(-8) + 1(8)}{3 + 1}\right] = (1, -4)$

3. $(x_1, y_1) = (2, -3), (x_2, y_2) = (4, 6)$ and $a : b = 5 : 2$. The point is

$$\left(\frac{2(2) + 5(4)}{2 + 5}, \frac{2(-3) + 5(6)}{2 + 5}\right) = \left(\frac{24}{7}, \frac{24}{7}\right)$$

4. $(x_1, y_1) = A(5, 0)$, $(x_2, y_2) = B(1, -2)$ and $a : b = 3 : 2$. The point is

$$\left(\frac{2(5) + 3(1)}{2 + 3}, \frac{2(0) + 3(-2)}{2 + 3}\right) = \left(\frac{13}{5}, \frac{-6}{5}\right)$$

5. $(x_1, y_1) = A(2, 3)$, $(x_2, y_2) = B(5, 7)$ and $a : b = 3 : 1$.

$$(x, y) = \left(\frac{1(2) + 3(5)}{1 + 3}, \frac{1(3) + 3(7)}{1 + 3}\right) = \left(\frac{17}{4}, 6\right).$$

6. $$C = \left[\frac{4(3) - 1(-2)}{4 - 1}, \frac{4(4) - 1(-1)}{4 - 1}\right] = \left(\frac{14}{3}, \frac{17}{3}\right)$$

7. $(x_1, y_1) = A(2, -3)$, $(x_2, y_2) = B(x, y)$ and $a : b = 2 : 1$.

$$\left(\frac{1(2) + 2(x)}{1 + 2}, \frac{1(-3) + 2(y)}{1 + 2}\right) = (6, 1)$$

$$\left(\frac{2 + 2x}{3}, \frac{-3 + 2y}{3}\right) = (6, 1)$$

Thus

$$\begin{aligned} \frac{2 + 2x}{3} &= 6 \quad \text{and} \quad \frac{-3 + 2y}{3} = 1 \\ 2 + 2x &= 18 \quad \text{and} \quad -3 + 2y = 3 \\ 2x &= 16 \quad \text{and} \quad 2y = 6 \\ x &= 8 \quad \text{and} \quad y = 3 \end{aligned}$$

Also, the translation $\overrightarrow{AB}$ is $\frac{3}{2}\overrightarrow{AP}$. Thus translation $A(2, -3) \xrightarrow{4,\,4} P(6, 1)$ and so the translation $\overrightarrow{AB}$ has components $\frac{3}{2}(4, 4) = (6, 6)$.

Hence $B(x, y)$ is

$$A(2, -3) \xrightarrow{6,\,6} (8, 3) = B(x, y)$$

Again $x = 8$ and $y = 3$.

8. $$P = \left[\frac{1(x) + 3(-10)}{1 + 3}, \frac{1(-5) + 3(7)}{1 + 3}\right] = (-6, y)$$

$\Rightarrow x - 30 = -24, \quad -5 + 21 = 4y$

$x = 6 \qquad y = 4$

9. $$C = \left[\frac{4(0) + 3(x)}{4 + 3}, \frac{4(y) + 3(0)}{4 + 3}\right] = (9, -8)$$

$\Rightarrow 0 + 3x = 63, \quad 4y + 0 = -56$

$x = 21 \qquad y = -14$

10. $$P = \left[\frac{h(-2) + k(4)}{h + k}, \frac{h(0) + k(-3)}{h + k}\right] = (2, -2)$$

$$\begin{aligned} -2h + 4k &= 2h + 2k \\ 4h &= 2k \\ 2h &= 1k \\ \frac{h}{k} &= \frac{1}{2} \Rightarrow \text{Ratio } h : k = 1 : 2 \end{aligned}$$

Exercise 4.5

1. (i) Centroid $= \left(\frac{2 + 4 - 3}{3}, \frac{-3 + 0 + 9}{3}\right) = (1, 2)$

(ii) Centroid $= \left(\frac{1 + 6 + 5}{3}, \frac{3 + 2 - 2}{3}\right) = (4, 1)$

2. Midpoint $= (2, 0)$; Slope x-axis $= 0 \Rightarrow$ Perpendicular slope is undefined

$\Rightarrow$ Equation: $x = 2$

Midpoint $= \left(\frac{1}{2}, 1\frac{1}{2}\right)$; Slope $= \frac{3-0}{1-0} = 3 \Rightarrow$ Perpendicular slope $= \frac{-1}{3}$

$\Rightarrow$ Equation: $y - 1\frac{1}{2} = -\frac{1}{3}\left(x - \frac{1}{2}\right)$

$\Rightarrow 3y - 4\frac{1}{2} = -x + \frac{1}{2}$

$\Rightarrow \quad x + 3y = 5$

$x = 2 \quad \Rightarrow \quad 2 + 3y = 5$

$\Rightarrow \quad 3y = 3$

$\Rightarrow \quad y = 1 \Rightarrow$ Circumcentre $= (2, 1)$

3. Midpoint $= \left(\frac{3+6}{2}, \frac{-7+2}{2}\right) = \left(4\frac{1}{2}, -2\frac{1}{2}\right)$

Slope $= \frac{2+7}{6-3} = \frac{9}{3} = 3 \Rightarrow$ Perpendicular slope $= -\frac{1}{3}$

Equation: $y + 2\frac{1}{2} = -\frac{1}{3}\left(x - 4\frac{1}{2}\right)$

$\Rightarrow \quad 3y + 7\frac{1}{2} = -x + 4\frac{1}{2}$

$\Rightarrow \quad x + 3y = -3$

Midpoint $= \left(\frac{6+8}{2}, \frac{2-2}{2}\right) = (7, 0)$

Slope $= \frac{2+2}{6-8} = \frac{4}{-2} = -2 \Rightarrow$ Perpendicular slope $= \frac{1}{2}$

Equation: $y - 0 = \frac{1}{2}(x - 7)$

$\Rightarrow \quad 2y = x - 7$

$\Rightarrow \quad x - 2y = 7$

and $\quad x + 3y = -3$

Subtract $\Rightarrow \quad -5y = 10$

$y = -2 \quad \Rightarrow x = 3$

Circumcentre $= (3, -2)$

4. Slope $= \frac{5-2}{-2-4} = \frac{3}{-6} = \frac{-1}{2} \Rightarrow$ Perpendicular slope $= 2$, Point $= (-1, -3)$

Equation: $y + 3 = 2(x + 1)$

$y + 3 = 2x + 2$

$\Rightarrow \quad 2x - y = 1$

Slope $= \frac{2+3}{4+1} = \frac{5}{5} = 1 \Rightarrow$ Perpendicular slope $= -1$, Point $= (-2, 5)$

Equation: $y - 5 = -1(x + 2)$

$\Rightarrow y - 5 = -x - 2$

$\Rightarrow x + y = 3$

and $2x - y = 1$

Add $\Rightarrow \quad 3x = 4$

$x = \frac{4}{3} \quad \Rightarrow y = \frac{5}{3}$

Orthocentre $= \left(\frac{4}{3}, \frac{5}{3}\right)$

5. Slope $= \dfrac{4-0}{4-0} = \dfrac{4}{4} = 1 \;\Rightarrow\;$ Perpendicular slope $= -1$, Point $= (4, -2)$

Equation: $y + 2 = -1(x - 4)$

$y + 2 = -x + 4$

$x + y = 2$

Slope $= \dfrac{-2-0}{4-0} = \dfrac{-2}{4} = \dfrac{-1}{2} \;\Rightarrow\;$ Perpendicular slope $= 2$, Point $= (4, 4)$

Equation: $y - 4 = 2(x - 4)$

$\Rightarrow \quad y - 4 = 2x - 8$

$\Rightarrow \quad 2x - y = 4$

and $\quad x + y = 2$

Add $\;\Rightarrow 3x \quad = 6$

$\Rightarrow x \quad = 2 \qquad \Rightarrow y = 0$

Orthocentre $= (2, 0)$

6. Midpoint $= \left(\dfrac{-2+5}{2}, \dfrac{2-5}{2}\right) = \left(1\tfrac{1}{2}, -1\tfrac{1}{2}\right)$

Slope $= \dfrac{2+5}{-2-5} = \dfrac{7}{-7} = -1 \;\Rightarrow\;$ Perpendicular slope $= 1$

Equation: $y + 1\tfrac{1}{2} = 1\left(x - 1\tfrac{1}{2}\right)$

$\Rightarrow \; y + 1\tfrac{1}{2} = x - 1\tfrac{1}{2}$

$\Rightarrow \; x - y = 3$

Midpoint $= \left(\dfrac{-4+5}{2}, \dfrac{-2-5}{2}\right) = \left(\tfrac{1}{2}, -3\tfrac{1}{2}\right)$

Slope $= \dfrac{-2+5}{-4-5} = \dfrac{3}{-9} = \dfrac{-1}{3} \;\Rightarrow\;$ Perpendicular slope $= 3$

Equation: $y + 3\tfrac{1}{2} = 3\left(x - \tfrac{1}{2}\right)$

$\Rightarrow \; y + 3\tfrac{1}{2} = 3x - 1\tfrac{1}{2}$

$\Rightarrow \; 3x - y = 5$

and $\quad x - y = 3$

Subtract $\Rightarrow \; 2x \quad = 2$

$\Rightarrow \quad x = 1 \qquad \Rightarrow y = -2$

Circumcentre $= (1, -2)$

7. Centroid $= (-1, 3)$

$\Rightarrow \dfrac{4 - 2 + k}{3} = -1$

$\Rightarrow \quad 2 + k = -3$

$\Rightarrow \quad k = -5$

Exercise 4.6

1. $|PD| = \dfrac{|3(2) - 4(-4) - 17|}{\sqrt{(3)^2 + (-4)^2}} = \dfrac{|22 - 17|}{\sqrt{25}} = \dfrac{|5|}{5} = 1$

2. $|PD| = \dfrac{|3(1) + 4(1) - 12|}{\sqrt{(3)^2 + (4)^2}} = \dfrac{|7 - 12|}{\sqrt{25}} = \dfrac{|-5|}{5} = 1$

$|PD| = \dfrac{|5(1) - 12(1) + 20|}{\sqrt{(5)^2 + (-12)^2}} = \dfrac{|13|}{\sqrt{169}} = \dfrac{|13|}{13} = 1$

3. $|PD| = \dfrac{|5(6) - 3(2) + 10|}{\sqrt{(5)^2 + (-3)^2}} = \dfrac{|40 - 6|}{\sqrt{34}} = \dfrac{|34|}{\sqrt{34}} = \sqrt{34}$

4. $|PD| = \dfrac{|1(5) - 2(-5) + 10|}{\sqrt{(1)^2 + (-2)^2}} = \dfrac{|25|}{\sqrt{5}} = 5\sqrt{5}$

$|PD| = \dfrac{|2(5) + 1(-5) - 30|}{\sqrt{(2)^2 + (1)^2}} = \dfrac{|-25|}{\sqrt{5}} = 5\sqrt{5}$

5. $|PD| = \dfrac{|4(3) + 3(1) + c|}{\sqrt{(4)^2 + (3)^2}} = \dfrac{|15 + c|}{\sqrt{25}} = \dfrac{|15 + c|}{5}$

$\Rightarrow \dfrac{15 + c}{5} = +5 \quad \text{OR} \quad \dfrac{15 + c}{5} = -5$

$\Rightarrow 15 + c = 25 \quad \text{OR} \quad 15 + c = -25$

$\Rightarrow c = 10 \quad \text{OR} \quad c = -40$

6. $3(2) - (2) - 4 = 6 - 6 = 0$. True Statement.

Perpendicular distance from $(2, 2)$ to line $6x - 2y + 7 = 0$:

$|PD| = \dfrac{|6(2) - 2(2) + 7|}{\sqrt{(6)^2 + (-2)^2}} = \dfrac{|15|}{\sqrt{40}} = \dfrac{15}{2\sqrt{10}} = \dfrac{15\sqrt{10}}{20} = \dfrac{3\sqrt{10}}{4}$

7. $|PD| = \dfrac{|1(1) + 7(1) - 3|}{\sqrt{(1)^2 + (7)^2}} = \dfrac{|5|}{\sqrt{50}} = \dfrac{|5|}{5\sqrt{2}} = \dfrac{|1|}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$|PD| = \dfrac{|1(1) - 1(1) + 1|}{\sqrt{(1)^2 + (-1)^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \quad \Rightarrow$ YES.

8. $|PD| = \dfrac{|a(-2) + 1(3) - 7|}{\sqrt{(a)^2 + (1)^2}} = \dfrac{|-2a - 4|}{\sqrt{a^2 + 1}} = \dfrac{\sqrt{10}}{1}$

$\Rightarrow 2a + 4 = \sqrt{10} \cdot \sqrt{a^2 + 1}$

$\Rightarrow 4a^2 + 16a + 16 = 10(a^2 + 1)$

$= 10a^2 + 10$

$\Rightarrow 6a^2 - 16a - 6 = 0$

$\Rightarrow 3a^2 - 8a - 3 = 0$

$\Rightarrow (3a + 1)(a - 3) = 0$

$\Rightarrow a = -\dfrac{1}{3}, a = 3$

9. $|PD| = \dfrac{|4(-2) + 3(a) - 3|}{\sqrt{(4)^2 + (3)^2}} = \dfrac{|3a - 11|}{\sqrt{25}} = \dfrac{3a - 11}{5}$

$|PD| = \dfrac{|12(-2) + 5(a) - 13|}{\sqrt{(12)^2 + (5)^2}} = \dfrac{|5a - 37|}{\sqrt{169}} = \dfrac{5a - 37}{13}$

$\Rightarrow \dfrac{3a - 11}{5} = \dfrac{5a - 37}{13}$

$\Rightarrow 39a - 143 = 25a - 185$

$\Rightarrow 14a = -42$

$\Rightarrow a = -3$

10. $|PD| = \dfrac{|3(-2) + 2(6) - 7|}{\sqrt{(3)^2 + (2)^2}} = \dfrac{|-13 + 12|}{\sqrt{13}} = \dfrac{|-1|}{\sqrt{13}}$ Negative side

$|PD| = \dfrac{|3(0) + 2(0) - 7|}{\sqrt{(3)^2 + (2)^2}} = \dfrac{|-7|}{\sqrt{13}}$ Negative side

11. $|PD| = \dfrac{|3(3) + 4(4) - 36|}{\sqrt{(3)^2 + (4)^2}} = \dfrac{|25 - 36|}{\sqrt{25}} = \dfrac{|-11|}{5}$ Negative side

$|PD| = \dfrac{|3(9) + 4(3) - 36|}{\sqrt{(3)^2 + (4)^2}} = \dfrac{|39 - 36|}{\sqrt{25}} = \dfrac{|3|}{5}$ Positive side

12. $|PD| = \dfrac{|2(-3) - 3(1) + 7|}{\sqrt{(2)^2 + (-3)^2}} = \dfrac{|-9 + 7|}{\sqrt{13}} = \dfrac{|-2|}{\sqrt{13}}$ Negative side

$|PD| = \dfrac{|2(3) - 3(-4) + 7|}{\sqrt{(2)^2 + (-3)^2}} = \dfrac{|6 + 12 + 7|}{\sqrt{13}} = \dfrac{|25|}{\sqrt{13}}$ Positive side

No; Points are not on the same side of the line.

13. Parallel line is $4x + 3y + c = 0$.
A point on the line $4x + 3y + 1 = 0$ is $(-1, 1)$.

$|PD| = \dfrac{|4(-1) + 3(1) + c|}{\sqrt{(4)^2 + (3)^2}} = 2$

$\Rightarrow \dfrac{|-1 + c|}{5} = 2$

$\Rightarrow -1 + c = 10$ OR $-1 + c = -10$

$\Rightarrow c = 11$ OR $c = -9$

$\Rightarrow 4x + 3y + 11 = 0$ OR $4x + 3y - 9 = 0$

14. Perpendicular line is $4x + 3y + k = 0$.

$|PD| = \dfrac{|4(1) + 3(1) + k|}{\sqrt{(4)^2 + (3)^2}} = 4$

$\Rightarrow \dfrac{|7 + k|}{5} = 4$

$\Rightarrow 7 + k = 20$ OR $7 + k = -20$

$\Rightarrow k = 13$ OR $k = -27$

$\Rightarrow 4x + 3y + 13 = 0$ OR $4x + 3y - 27 = 0$

15. Point $(-4, 2)$, Slope $= m$
Equation: $y - 2 = m(x + 4)$

$\Rightarrow y - 2 = mx + 4m$

$\Rightarrow mx - y + 4m + 2 = 0$

$|PD| = \dfrac{|m(0) - 1(0) + 4m + 2|}{\sqrt{(m)^2 + (-1)^2}} = 2$

$\Rightarrow |4m + 2| = 2\sqrt{m^2 + 1}$

$\Rightarrow 16m^2 + 16m + 4 = 4(m^2 + 1)$

$\Rightarrow 16m^2 + 16m + 4 = 4m^2 + 4$

$\Rightarrow 12m^2 + 16m = 0$

$\Rightarrow 3m^2 + 4m = 0$

$\Rightarrow m(3m + 4) = 0$

$\Rightarrow m = 0$ OR $m = -\dfrac{4}{3}$

$\Rightarrow x(0) - y + 4(0) + 2 = 0$ OR $x\left(\dfrac{-4}{3}\right) - y + 4\left(\dfrac{-4}{3}\right) + 2 = 0$

$\Rightarrow y - 2 = 0$ OR $4x + 3y + 10 = 0$

16. Point $(3, 5)$, slope $= m$
Equation: $y - 5 = m(x - 3)$

$\Rightarrow y - 5 = mx - 3m$

$\Rightarrow mx - y - 3m + 5 = 0$

$$|PD| = \frac{|m(0) - 1(0) - 3m + 5|}{\sqrt{(m)^2 + (-1)^2}} = 5$$

$\Rightarrow |-3m + 5| = 5\sqrt{m^2 + 1}$

$\Rightarrow 9m^2 - 30m + 25 = 25(m^2 + 1)$

$\Rightarrow 9m^2 - 30m + 25 = 25m^2 + 25$

$\Rightarrow 16m^2 + 30m = 0$

$\Rightarrow 8m^2 + 15m = 0$

$\Rightarrow m(8m + 15) = 0$

$\Rightarrow m = 0 \quad \text{OR} \quad m = \frac{-15}{8}$

$\Rightarrow x(0) - y - 3(0) + 5 = 0 \quad \text{OR} \quad x\left(\frac{-15}{8}\right) - y - 3\left(\frac{-15}{8}\right) + 5 = 0$

$\Rightarrow y - 5 = 0 \quad \text{OR} \quad 15x + 8y - 85 = 0$

17. (i) Slope BC $= \frac{1 + 2}{3 + 1} = \frac{3}{4}$

Equation BC: $y + 2 = \frac{3}{4}(x + 1)$

$\Rightarrow 4y + 8 = 3x + 3$

$\Rightarrow 3x - 4y - 5 = 0$

$$|AN| = \frac{|3(1) - 4(2) - 5|}{\sqrt{(3)^2 + (-4)^2}} = \frac{|-10|}{\sqrt{25}} = \frac{|-10|}{5} = 2$$

(ii) $|BC| = \sqrt{(3 + 1)^2 + (1 + 2)^2}$

$= \sqrt{(4)^2 + (3)^2}$

$= \sqrt{25}$

$= 5$

Area $\triangle ABC = \frac{1}{2}|BC| \cdot |AN|$

$= \frac{1}{2}(5)(2)$

$= 5$ sq. units

18. (a) Red flight path: $(0, 1), (-2, 4)$

$m = \frac{4 - 1}{-2 - 0} = \frac{-3}{2}$

Equation: $(y - 1) = \frac{-3}{2}(x - 0)$

$2y - 2 = -3x$

$3x + 2y - 2 = 0$

Blue flight path: $m = \frac{-3}{2}$ and point $(0, -1)$

Equation: $(y + 1) = \frac{-3}{2}(x - 0)$

$2y + 2 = -3x$

$3x + 2y + 2 = 0$

(b) One point on the blue flight path is $(0, -1)$. The perpendicular distance from this point to the red flight path is

$$\frac{|3(0) + 2(-1) - 2|}{\sqrt{(3)^2 + (2)^2}} = \frac{|-4|}{\sqrt{13}} = \frac{4}{\sqrt{13}}$$

19. River: $4x - 5y + 6 = 0$

$(-5, -3)$: $4(-5) - 5(-3) + 6 = 1 > 0$

$(4, 4)$: $4(4) - 5(4) + 6 = 2 > 0$

As both are positive, the friends live on the same side of the river.

20. (a) Pipes: $4x - 5y + 2 = 0$ and $4x - 5y + 1 = 0$

Let (x, y) be a point on the third pipe. Then

$$\frac{|4x - 5y + 2|}{\sqrt{4^2 + (-5)^2}} = \frac{|4x - 5y + 1|}{\sqrt{4^2 + (-5)^2}}$$

$$|4x - 5y + 2| = |4x - 5y + 1|$$

$$4x - 5y + 2 = -(4x - 5y + 1) \quad \text{... as + is not possible}$$

$$4x - 5y + 2 = -4x + 5y - 1$$

$$8x - 10y + 3 = 0$$

(b) One point on the pipe $4x - 5y + 1 = 0$ is $(0, 0.2)$. Then the shortest distance between the original pipes is

$$\frac{|4(0) - 5(0.2) + 2|}{\sqrt{4^2 + (-5)^2}} = \frac{1}{\sqrt{41}}.$$

Exercise 4.7

1. (i) Slope of $x + 2y + 4 = 0 \Rightarrow m_1 = -\frac{1}{2}$

Slope of $x - 3y + 2 = 0 \Rightarrow m_2 = -\frac{1}{-3} = \frac{1}{3}$

$$\Rightarrow \tan\theta = \pm\frac{-\frac{1}{2} - \frac{1}{3}}{1 + \left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \pm\frac{-\frac{5}{6}}{1 - \frac{1}{6}} = \pm\frac{-\frac{5}{6}}{\frac{5}{6}} = +1$$

(ii) Slope of $2x + 3y - 1 = 0 \Rightarrow m_1 = \frac{-2}{3}$

Slope of $x - 2y + 3 = 0 \Rightarrow m_2 = -\frac{1}{-2} = \frac{1}{2}$

$$\Rightarrow \tan\theta = \pm\frac{-\frac{2}{3} - \frac{1}{2}}{1 + \left(\frac{-2}{3}\right)\left(\frac{1}{2}\right)} = \pm\frac{-\frac{7}{6}}{1 - \frac{1}{3}} = \pm\frac{-\frac{7}{6}}{\frac{2}{3}} = +\frac{7}{4}$$

(iii) Slope of $2x + y - 6 = 0 \Rightarrow m_1 = \frac{-2}{1} = -2$

Slope of $2x - 3y + 5 = 0 \Rightarrow m_2 = \frac{-2}{-3} = \frac{2}{3}$

$$\Rightarrow \tan\theta = \pm\frac{-2 - \frac{2}{3}}{1 + (-2)\left(\frac{2}{3}\right)} = \pm\frac{\frac{-8}{3}}{1 - \frac{4}{3}} = \pm\frac{-\frac{8}{3}}{-\frac{1}{3}} = +8$$

2. Slope of $y = 2x + 5 \Rightarrow m_1 = 2$

Slope of $3x + y = 7 \Rightarrow m_2 = \frac{-3}{1} = -3$

$$\Rightarrow \tan\theta = \pm\frac{2 - (-3)}{1 + (2)(-3)} = \pm\frac{2 + 3}{1 - 6} = \pm\frac{5}{-5} = +1$$

$$\Rightarrow \theta = \tan^{-1}(1) = 45°$$

3. Slope of $x - 2y - 1 = 0 \Rightarrow m_1 = -\frac{1}{-2} = \frac{1}{2}$

Slope of $3x - y + 2 = 0 \Rightarrow m_2 = \frac{-3}{-1} = 3$

$$\Rightarrow \tan\theta = \pm\frac{\frac{1}{2} - 3}{1 + \frac{1}{2}(3)} = \pm\frac{-2\frac{1}{2}}{2\frac{1}{2}} = -1$$

$$\Rightarrow \theta = \tan^{-1}(-1) = 135°$$

4. Slope of $x - 3y + 4 = 0 \Rightarrow m_1 = -\frac{1}{-3} = \frac{1}{3}$

Slope of $2x + y - 5 = 0 \Rightarrow m_2 = -\frac{2}{1} = -2$

$$\Rightarrow \tan\theta = \pm\frac{\frac{1}{3}+2}{1+\left(\frac{1}{3}\right)(-2)} = \pm\frac{\frac{7}{3}}{1-\frac{2}{3}} = \pm\frac{\frac{7}{3}}{\frac{1}{3}} = +7$$

$$\Rightarrow \theta = \tan^{-1}(7) = 81.87° = 82°$$

5. Slope of $x - 2y + 7 = 0 \Rightarrow m_1 = -\frac{1}{-2} = \frac{1}{2}$

Slope of $3x - y + 2 = 0 \Rightarrow m_2 = -\frac{3}{-1} = 3$

$$\Rightarrow \tan\theta = \pm\frac{\frac{1}{2}-3}{1+\left(\frac{1}{2}\right)\cdot(3)} = \pm\frac{-2\frac{1}{2}}{2\frac{1}{2}} = \pm 1 = -1$$

$$\Rightarrow \theta = \tan^{-1}(-1) = 135°$$

6. Slope of $x - \sqrt{3}y + 4 = 0 \Rightarrow m_1 = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$

Slope of $\sqrt{3}x - y - 7 = 0 \Rightarrow m_2 = -\frac{\sqrt{3}}{-1} = \sqrt{3}$

$$\Rightarrow \tan\theta = \pm\frac{\frac{1}{\sqrt{3}}-\sqrt{3}}{1+\left(\frac{1}{\sqrt{3}}\right)(\sqrt{3})} = \pm\frac{\frac{1-3}{\sqrt{3}}}{2} = \pm\frac{-2}{2\sqrt{3}} = \pm\frac{1}{\sqrt{3}} = +\frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

7. Slope of $2x - 3y + 1 = 0 \Rightarrow m_1 = \frac{-2}{-3} = \frac{2}{3}$

$$\tan 45° = \pm\frac{\frac{2}{3}-m_2}{1+\frac{2}{3}m_2} = \pm\frac{\frac{2-3m_2}{3}}{\frac{3+2m_2}{3}} = \pm\frac{2-3m_2}{3+2m_2}$$

$\Rightarrow +\frac{2-3m_2}{3+2m_2} = \frac{1}{1}$ OR $\frac{-2+3m_2}{3+2m_2} = \frac{1}{1}$

$\Rightarrow 3 + 2m_2 = 2 - 3m_2$ OR $-2 + 3m_2 = 3 + 2m_2$

$\Rightarrow 5m_2 = -1$ OR $m_2 = 5$

$\Rightarrow m_2 = -\frac{1}{5}$

8. Slope of $2x + 3y - 4 = 0 \Rightarrow m_1 = -\frac{2}{3}$

$$\Rightarrow \tan 45° = \pm\frac{\frac{-2}{3}-m_2}{1+\left(\frac{-2}{3}\right)(m_2)} = \pm\frac{\frac{-2-3m_2}{3}}{\frac{3-2m_2}{3}} = \pm\frac{-2-3m_2}{3-2m_2}$$

$\Rightarrow \frac{-2-3m_2}{3-2m_2} = \frac{1}{1}$ OR $\frac{2+3m_2}{3-2m_2} = \frac{1}{1}$

$\Rightarrow 3 - 2m_2 = -2 - 3m_2$ OR $2 + 3m_2 = 3 - 2m_2$

$\Rightarrow m_2 = -5$ OR $5m_2 = 1$

$m_2 = \frac{1}{5}$

Equation: $y - 0 = -5(x - 0)$ OR $y - 0 = \frac{1}{5}(x - 0)$

$\Rightarrow y = -5x$ OR $\Rightarrow 5y = x$

$\Rightarrow 5x + y = 0$ OR $\Rightarrow x - 5y = 0$

9. Slope of $2x + y - 2 = 0 \Rightarrow m_1 = -\frac{2}{1} = -2$

$\Rightarrow \tan 45° = \pm\frac{-2 - m_2}{1 + (-2)m_2} = \pm\frac{-2 - m_2}{1 - 2m_2}$

$\Rightarrow \frac{-2 - m_2}{1 - 2m_2} = \frac{1}{1}$ OR $\frac{2 + m_2}{1 - 2m_2} = \frac{1}{1}$

$\Rightarrow -2 - m_2 = 1 - 2m_2$ OR $2 + m_2 = 1 - 2m_2$

$\Rightarrow m_2 = 3$ OR $3m_2 = -1$

$m_2 = \frac{-1}{3}$

Equation: $y - 1 = 3(x + 1)$ OR $y - 1 = \frac{-1}{3}(x + 1)$

$\Rightarrow y - 1 = 3x + 3$ $\Rightarrow 3y - 3 = -x - 1$

$\Rightarrow 3x - y + 4 = 0$ $\Rightarrow x + 3y - 2 = 0$

10. Slope of $x + y - 2 = 0 \Rightarrow m_1 = -\frac{1}{1} = -1$

$\Rightarrow \tan\theta = \pm\frac{-1 - m_2}{1 + (-1)m_2} = \pm\frac{-1 - m_2}{1 - m_2} = \frac{2}{3}$

$\Rightarrow \frac{-1 - m_2}{1 - m_2} = \frac{2}{3}$ OR $\frac{1 + m_2}{1 - m_2} = \frac{2}{3}$

$\Rightarrow 2 - 2m_2 = -3 - 3m_2$ OR $3 + 3m_2 = 2 - 2m_2$

$\Rightarrow m_2 = -5$ $\Rightarrow 5m_2 = -1$

$m_2 = -\frac{1}{5}$

Equation: $y - 2 = -5(x - 4)$ OR $y - 2 = -\frac{1}{5}(x - 4)$

$y - 2 = -5x + 20$ $\Rightarrow 5y - 10 = -x + 4$

$\Rightarrow 5x + y - 22 = 0$ $\Rightarrow x + 5y - 14 = 0$

11. Line $2x - 3y - 6 = 0$ meets the axes at $(0, -2)$ and $(3, 0)$.

$\Rightarrow$ Axial symmetry in the x-axis has points $(0, 2)$ and $(3, 0)$.

Slope $= \frac{0 - 2}{3 - 0} = \frac{-2}{3}$

Equation: $y - 2 = -\frac{2}{3}(x - 0)$

$\Rightarrow 3y - 6 = -2x$

$\Rightarrow 2x + 3y - 6 = 0$

12. (i) Slope of $tx + y - 7 = 0 \Rightarrow m_1 = -\frac{t}{1} = -t$

(ii) Slope of $y = 2x + 5 = 0 \Rightarrow m_2 = 2$

$\Rightarrow \tan 45° = \pm\frac{-t - 2}{1 + (-t)(2)} = \pm\frac{-t - 2}{1 - 2t}$

$\Rightarrow \frac{-t - 2}{1 - 2t} = \frac{1}{1}$ OR $\frac{+t + 2}{1 - 2t} = \frac{1}{1}$

$\Rightarrow -t - 2 = 1 - 2t$ $\Rightarrow t + 2 = 1 - 2t$

$\Rightarrow t = 3$ $\Rightarrow 3t = -1$

$\Rightarrow t = -\frac{1}{3}$

Exercise 4.8

1. (i) (a) 95°F (b) 58°F (c) 10°C (d) 38°C

(ii) (50, 10) (95, 35)

$$\text{Slope} = \frac{35-10}{95-50} = \frac{25}{45} = \frac{5}{9}$$

$$\text{Equation:} \quad y-10 = \frac{5}{9}(x-50)$$

$$\Rightarrow 9y - 90 = 5x - 250$$

$$\Rightarrow 5x - 9y - 160 = 0$$

(iii) $y = 95 \Rightarrow 5x - 9(95) - 160 = 0$

$$\Rightarrow 5x - 855 - 160 = 0$$

$$\Rightarrow 5x = 1015$$

$$\Rightarrow x = 203°\text{F}$$

2. $C = 20 + 4M$

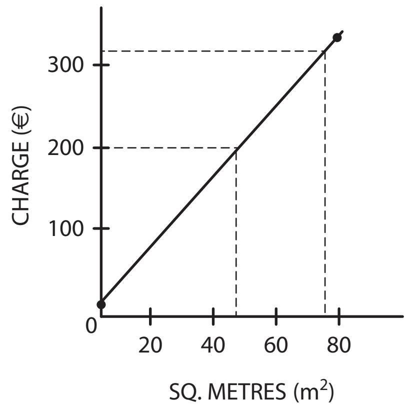
$M = 0 \Rightarrow C = 20 + 0 = 20 \quad (0, 20)$

$M = 80 \Rightarrow C = 20 + 4(80) = 340 \quad (80, 340)$

(i) $M = 75 \Rightarrow C = €320$

(ii) $C = 200 \Rightarrow M = 45\ \text{m}^2$

(iii) $M = 105 \Rightarrow C = 20 + 4(105) = €440$



3. (i) $T = 1 \quad \Rightarrow I = 5000\left(\frac{8}{100}\right)(1) = 400(1) = €400$

$T = 2 \quad \Rightarrow I = 5000\left(\frac{8}{100}\right)(2) = 400(2) = €800$

$T = 3 \quad \Rightarrow I = 5000\left(\frac{8}{100}\right)(3) = 400(3) = €1200$

(ii) $I = 400T$

(iii) $3500 = 400T \quad \Rightarrow T = 8\frac{3}{4}$ years

(iv) $A = 400T + 5000$

4. (i) (60, 100), (100, 50)

(ii) $\text{Slope} = \frac{50-100}{100-60} = \frac{-50}{40} = \frac{-5}{4}$

$$\text{Equation: } N - 100 = \frac{-5}{4}(P - 60)$$

$$\Rightarrow 4N - 400 = -5P + 300$$

$$\Rightarrow 5P + 4N = 700$$

(iii) $N = 88 \Rightarrow 5P + 4(88) = 700$

$$\Rightarrow 5P + 352 = 700$$

$$\Rightarrow 5P = 348$$

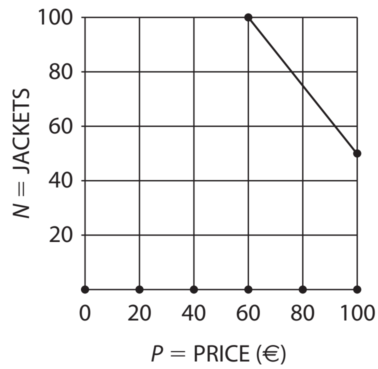
$$\Rightarrow P = €69.60$$

(iv) $P = 72 \Rightarrow 5(72) + 4N = 700$

$$\Rightarrow 360 + 4N = 700$$

$$\Rightarrow 4N = 340$$

$$\Rightarrow N = 85$$



5. (i) A : $P = 5 + 2D$ (0, 5)(10, 25)
B : $P = 2.2D$ (0, 0)(10, 22)

(ii) Line A: Slope $= \dfrac{25-5}{10-0} = \dfrac{20}{10} = 2$

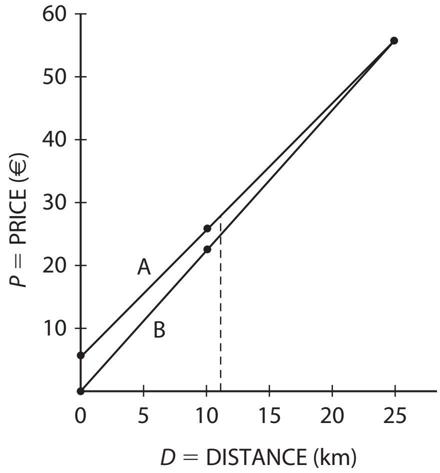
Equation: $P - 5 = 2(D - 0)$
$\Rightarrow P - 5 = 2D$
$\Rightarrow \quad P = 5 + 2D$

Line B: Slope $= \dfrac{22-0}{10-0} = 2.2$

Equation: $P - 0 = 2.2(D - 0)$
$\Rightarrow \quad P = 2.2D$

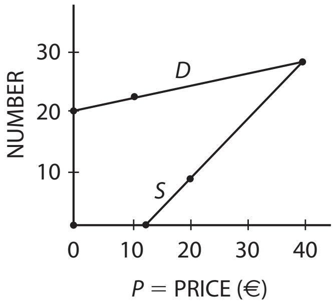
(iii) $D = 25$ km

(iv) Firm B



6. (i) $D = 20 + 0.2p$ (0, 20)(10, 22)
$S = -12 + p$ (12, 0)(20,8)

(ii) $p =$ €40 and 28 articles



Revision Exercise 4 (Core)

1. Slope $l = \dfrac{-3}{-2} = \dfrac{3}{2} \quad \Rightarrow \quad$ Perpendicular slope $= -\dfrac{2}{3}$

Equation: $y - 4 = -\dfrac{2}{3}(x + 1)$
$\Rightarrow 3y - 12 = -2x - 2$
$\Rightarrow 2x + 3y - 10 = 0$

2. Area $= \dfrac{1}{2}|(3)(4) - (-2)(-2)|$
$= \dfrac{1}{2}|12 - 4|$
$= \dfrac{1}{2}(8)$
$= 4$

3. Slope $= \dfrac{6+2}{1-3} = \dfrac{8}{-2} = -4 = m_1$

Slope $= \dfrac{-2}{a} = m_2$

Perpendicular lines $\Rightarrow m_1 \cdot m_2 = -4 \cdot \dfrac{-2}{a} = \dfrac{8}{a} = \dfrac{-1}{1}$
$\Rightarrow -a = 8$
$\Rightarrow \quad a = -8$

4. Slope $= \dfrac{6-a}{-3-6} = \dfrac{6-a}{-9}$

$\Rightarrow \quad \dfrac{6-a}{-9} = \dfrac{1}{3}$
$\Rightarrow \quad 18 - 3a = -9$
$\Rightarrow \quad -3a = -27$
$\Rightarrow \quad a = 9$

5. (i) Slope $= \frac{3}{2}$

(ii) x-axis $\Rightarrow$ $y = 0 \Rightarrow \frac{3}{2}x - 2 = 0$

$\Rightarrow 3x - 4 = 0$

$\Rightarrow 3x = 4$

$\Rightarrow x = \frac{4}{3}$ $\quad$ $P\left(\frac{4}{3}, 0\right)$

y-axis $\Rightarrow$ $x = 0 \Rightarrow y = -2$ $\quad$ $Q(0, -2)$

(iii) Area $= \frac{1}{2}\left|\left(\frac{4}{3}\right)(-2) - (0)(0)\right|$

$= \frac{1}{2}\left|\frac{-8}{3}\right| = \frac{4}{3}$

6. (1, 1) (6, 3)

Slope $l = \frac{3-1}{6-1} = \frac{2}{5}$

Equation: $y - 1 = \frac{2}{5}(x - 1)$

$\Rightarrow 5y - 5 = 2x - 2$

$\Rightarrow 2x - 5y + 3 = 0$

7. (i) Slope $= k^2 = m_1$

Slope $= \frac{4}{2k} = \frac{2}{k} = m_2$ $\quad$ Perpendicular lines $\Rightarrow m_1 \cdot m_2 = -1$

$\Rightarrow k^2 \cdot \frac{2}{k} = -1$

$\Rightarrow k = \frac{-1}{2}$

(ii) $l_1 : y = \frac{1}{4}x + 12$

$l_2 : 2\left(-\frac{1}{2}\right)y = 4x + 5 \Rightarrow -y = 4x + 5$

$\Rightarrow \quad y = -4x - 5$

$l_1 \cap l_2 \Rightarrow \frac{1}{4}x + 12 = -4x - 5$

$\Rightarrow x + 48 = -16x - 20$

$\Rightarrow \quad 17x = -68$

$\Rightarrow \quad x = -4$

$\Rightarrow y = \frac{1}{4}(-4) + 12 = 11$ $\quad$ $(-4, 11)$

8. (a) Slope $m_1 = \frac{-2}{1} = -2$

Slope $m_2 = -\frac{1}{-2} = \frac{1}{2}$ $\quad \Rightarrow m_1 \cdot m_2 = (-2)\left(\frac{1}{2}\right) = -1$: YES

(b) Slope $m_1 = 3$

Slope $m_2 = -\frac{1}{3}$ $\quad \Rightarrow m_1 \cdot m_2 = (3)\left(-\frac{1}{3}\right) = -1$: YES

(c) Slope $m_1 = \frac{-2}{1} = -2$

Slope $m_2 = \frac{1}{2}$ $\quad \Rightarrow m_1 \cdot m_2 = (-2)\left(\frac{1}{2}\right) = -1$: YES

(d) Slope $m_1 = -\frac{1}{3}$

Slope $m_2 = -\frac{3}{1} = -3$ $\quad \Rightarrow m_1 \cdot m_2 = \left(-\frac{1}{3}\right)(-3) = 1$ (i.e. $\neq -1$) : NO

9. Slope $= -\frac{1}{2}$ $\Rightarrow$ Perpendicular slope $= 2$

Equation: $y - 2 = 2(x - 5)$

$\Rightarrow y - 2 = 2x - 10$

$\Rightarrow 2x - y - 8 = 0$

10. Midpoint $= \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$

Slope $= \frac{4-2}{5-1} = \frac{2}{4} = \frac{1}{2}$ $\Rightarrow$ Perpendicular slope $= -2$

Equation: $y - 3 = -2(x - 3)$

$\Rightarrow y - 3 = -2x + 6$

$\Rightarrow 2x + y = 9$

Point $(0, k) \Rightarrow 2(0) + k = 9$

$0 + k = 9$

$k = 9$

Revision Exercise 4 (Advanced)

1. (i) $|PD| = \frac{|3(-1) - 4(-5) - 2|}{\sqrt{(3)^2 + (-4)^2}} = \frac{|-3 + 20 - 2|}{\sqrt{25}} = \frac{15}{5} = 3$

(ii) $|PD| = \frac{|3(-1) - 4(-5) + k|}{\sqrt{(3)^2 + (-4)^2}} = \frac{|-3 + 20 + k|}{\sqrt{25}} = \frac{|17 + k|}{5}$

$\Rightarrow \frac{17+k}{5} = \frac{3}{1}$ OR $\frac{17+k}{5} = -3$

$\Rightarrow 17 + k = 15$ $\qquad \Rightarrow 17 + k = -15$

$\Rightarrow k = -2$ (not valid) $\qquad \Rightarrow k = -32$

2. (i) Point $= \left[\frac{2(8) + 3(-7)}{2+3}, \frac{2(-2) + 3(3)}{2+3}\right]$

$= \left(-\frac{5}{5}, \frac{5}{5}\right) = (-1, 1)$

(ii) (a) x-axis $\Rightarrow y = 0 \Rightarrow 2x = 6$

$\Rightarrow x = 3$ $\qquad (3, 0)$

y-axis $\Rightarrow x = 0 \Rightarrow ky = 6$

$\Rightarrow y = \frac{6}{k}$ $\qquad \left(0, \frac{6}{k}\right)$

(b) Area $= \frac{1}{2}\left|(3)\left(\frac{6}{k}\right) - (0)(0)\right| = k$

$\Rightarrow \frac{18}{k} = \frac{2k}{1}$

$\Rightarrow 2k^2 = 18$

$\Rightarrow k^2 = 9$

$\Rightarrow k = 3$

3. (i) Slope $= \frac{-2}{1} = -2$ $\qquad$ Point $(2, 5)$

Equation: $y - 5 = -2(x - 2)$

$\Rightarrow y - 5 = -2x + 4$

$\Rightarrow 2x + y - 9 = 0$

(ii) (a) Slope $= \frac{-2}{1} = -2$ $\Rightarrow$ Perpendicular slope $= \frac{1}{2}$ $\qquad$ Point $(1, k)$

Equation: $y - k = \frac{1}{2}(x - 1)$

$\Rightarrow 2y - 2k = x - 1$

$\Rightarrow x - 2y + 2k - 1 = 0$

(b) Origin (0, 0) $\Rightarrow 0 - 2(0) + 2k - 1 = 0$

$$\Rightarrow 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2}$$

4. Centroid $= \dfrac{4 - 1 + h}{3} = 2, \ \dfrac{2 + 7 + k}{3} = 4$

$\Rightarrow \quad 3 + h = 6 \qquad \Rightarrow 9 + k = 12$

$\Rightarrow \quad h = 3 \qquad \Rightarrow \quad k = 3$

5. A(1, 8), B(1, −2), C(7, 1)

Altitude 1: As AB is vertical, the altitude through C is horizontal. Thus its equation is $y = 1$.

Altitude 2: Slope of BC $= \dfrac{1 + 2}{7 - 1} = \dfrac{1}{2}$.

Thus the slope of altitude 2, through A(1, 8), is -2.

The equation of altitude 2 is

$$(y - 8) = -2(x - 1)$$
$$y - 8 = -2x + 2$$
$$2x + y = 10$$

Solving: $y = 1$

$$2x + 1 = 10$$
$$2x = 9$$
$$x = \frac{9}{2}$$

Thus the orthocentre is $\left(\dfrac{9}{2}, 1\right)$.

6. (i) $y - 6 = m(x + 4)$

(ii) x-axis $\Rightarrow y = 0 \Rightarrow -6 = m(x + 4)$

$$-6 = mx + 4m$$

$$\Rightarrow x = \frac{-6 - 4m}{m} \qquad \left(\frac{-6 - 4m}{m}, 0\right)$$

y-axis $\Rightarrow x = 0 \Rightarrow y - 6 = m(4)$

$$\Rightarrow \quad y = 4m + 6 \quad (0, 4m + 6)$$

(iii) Area $= \dfrac{1}{2}\left|\left(\dfrac{-6 - 4m}{m}\right)(4m + 6) - (0)(0)\right| = 54$

$\Rightarrow |-24m - 36 - 16m^2 - 24m| = 108m$

$\Rightarrow 16m^2 + 48m + 36 = 108m$

$\Rightarrow 4m^2 - 15m + 9 = 0$

$\Rightarrow (4m - 3)(m - 3) = 0 \quad \Rightarrow m = \dfrac{3}{4}$ OR $m = 3$

7. (i) $|PD| = \dfrac{|3(3) - 4(k) + 7|}{\sqrt{(3)^2 + (-4)^2}} = 6$

$\Rightarrow \dfrac{|16 - 4k|}{5} = 6$

$\Rightarrow |16 - 4k| = 30$

$\Rightarrow \quad 16 - 4k = 30 \quad$ OR $\quad 16 - 4k = -30$

$\Rightarrow \quad -4k = 14 \qquad \Rightarrow -4k = -46$

$\Rightarrow \quad k = -3\frac{1}{2} \qquad \Rightarrow \quad k = 11\frac{1}{2}$

ANS: $k = -3\frac{1}{2}$

(ii) Slope $= -\frac{3}{-4} = \frac{3}{4}$ Point $= \left(3, -3\frac{1}{2}\right)$

Equation: $y + 3\frac{1}{2} = \frac{3}{4}(x - 3)$

$\Rightarrow 4y + 14 = 3x - 9$

$\Rightarrow 3x - 4y - 23 = 0$

8. Line $4x - 3y + 8 = 0 \quad \Rightarrow \quad$ Equation of parallel lines: $4x - 3y + k = 0$

Perpendicular Distance: $\dfrac{|4(0) - 3(0) + k|}{\sqrt{(4)^2 + (-3)^2}} = 4$

$\Rightarrow \dfrac{|k|}{5} = 4$

$\Rightarrow |k| = 20$

$k = 20 \quad$ OR $\quad k = -20$

ANS: $4x - 3y + 20 = 0 \quad$ OR $\quad 4x - 3y - 20 = 0$

9. Slope $= -\frac{1}{-2} = \frac{1}{2} = m_1$

$\Rightarrow \tan 45° = \pm \dfrac{\frac{1}{2} - m_2}{1 + \left(\frac{1}{2}\right)(m_2)} = \dfrac{1}{1}$

$\Rightarrow \pm\left(\frac{1}{2} - m_2\right) = 1 + \frac{1}{2}m_2$

$\Rightarrow \quad \frac{1}{2} - m_2 = 1 + \frac{1}{2}m_2 \quad$ OR $\quad -\frac{1}{2} + m_2 = 1 + \frac{1}{2}m_2$

$\Rightarrow \quad 1 - 2m_2 = 2 + m_2 \qquad \Rightarrow -1 + 2m_2 = 2 + m_2$

$\Rightarrow \quad -3m_2 = 1 \qquad \Rightarrow \quad m_2 = 3$

$\Rightarrow \quad m_2 = -\frac{1}{3}$

Equation: $y - 4 = -\frac{1}{3}(x - 2) \quad$ OR $\quad y - 4 = 3(x - 2)$

$\Rightarrow 3y - 12 = -x + 2 \qquad \Rightarrow y - 4 = 3x - 6$

$\Rightarrow x + 3y - 14 = 0 \qquad \Rightarrow 3x - y - 2 = 0$

10. (i) $4(2) + 3(-1) - 5 = 0$

$\Rightarrow 8 - 3 - 5 = 0$

$\Rightarrow 8 - 8 = 0$.... True

(ii) $4x + 3y + c = 0$

(iii) $|PD| = \dfrac{|4(2) + 3(-1) + c|}{\sqrt{(4)^2 + (3)^2}} = 2$

$\Rightarrow \dfrac{|5 + c|}{5} = 2$

$\Rightarrow |5 + c| = 10$

$\Rightarrow \quad 5 + c = 10 \quad$ OR $\quad 5 + c = -10$

$\Rightarrow \quad c = 5 \quad$ OR $\quad c = -15$

$\Rightarrow 4x + 3y + 5 = 0 \quad$ OR $\quad 4x + 3y - 15 = 0$

Revision Exercise 4 (Extended-Response Questions)

1. (i) Points on ladder are: (2.5, 0) and (1.5, 2)

Slope $= \dfrac{2 - 0}{1.5 - 2.5} = \dfrac{2}{-1} = -2$

Equation of line of ladder: point $= (2.5, 0)$; Slope $= -2$

$y - 0 = -2(x - 2.5)$

$y = -2x + 5$

$\Rightarrow 2x + y - 5 = 0$ is the equation

(ii) To find the point A, let $x = 0$ in the equation $2x + y - 5 = 0$

$x = 0 \quad \Rightarrow \quad y = 5$

(iii) $\therefore$ the height of point $A = 5$ metres

$$\text{Length} = \sqrt{(2.5 - 0)^2 + (5 - 0)^2}$$
$$= \sqrt{6.25 + 25}$$
$$= \sqrt{31.25}$$
$$= 5.59017 \text{ m}$$
$$= 559 \text{ cm}$$

2. (i) Equation: $y - 5 = m(x - 2)$

$$\Rightarrow y - 5 = mx - 2m$$
$$\Rightarrow mx - y + 5 - 2m = 0$$

(ii) x-axis $\Rightarrow y = 0 \Rightarrow mx = -5 + 2m$

$$\Rightarrow x = \frac{-5 + 2m}{m} \left(\frac{-5 + 2m}{m}, 0\right)$$

y-axis $\Rightarrow x = 0 \Rightarrow -y + 5 - 2m = 0$

$$\Rightarrow y = 5 - 2m \quad (0, 5 - 2m)$$

(iii) $$\text{Area} = \frac{1}{2}\left|\left(\frac{-5 + 2m}{m}\right)(5 - 2m) - (0)(0)\right| = 36$$
$$\Rightarrow |-25 + 10m + 10m - 4m^2| = 72m$$
$$\Rightarrow -4m^2 + 20m - 25 = 72m$$
$$\Rightarrow 4m^2 + 52m + 25 = 0$$
$$\Rightarrow (2m + 1)(2m + 25) = 0$$
$$\Rightarrow m = -\frac{1}{2}, \quad m = \frac{-25}{2}$$

3. (i) $$\text{Slope AB} = \frac{k - 4}{1 - 3} = \frac{k - 4}{-2} = \frac{4 - k}{2}$$

(ii) $$\text{Slope BC} = \frac{-3 - k}{4 - 1} = \frac{-3 - k}{3}$$

$$\text{Perpendicular lines} \Rightarrow \left(\frac{k - 4}{-2}\right) \cdot \left(\frac{-3 - k}{3}\right) = -1$$
$$\Rightarrow -3k + 12 - k^2 + 4k = 6$$
$$\Rightarrow -k^2 + k + 6 = 0$$
$$\Rightarrow k^2 - k - 6 = 0$$
$$\Rightarrow (k + 2)(k - 3) = 0$$
$$\Rightarrow k = -2 \quad \text{OR} \quad k = 3$$

(iii) $A(3, 4) \rightarrow (0, 0) \quad B(1, 3) \rightarrow (-2, -1) \quad C(4, -3) \rightarrow (1, -7)$

$$\text{Area } \triangle = \frac{1}{2}|(-2)(-7) - (1)(-1)|$$
$$= \frac{1}{2}|14 + 1|$$
$$= \frac{15}{2}$$

$$\text{Area rectangle} = 2\left(\frac{15}{2}\right) = 15 \text{ sq.units}$$

4. (i) x-axis $\Rightarrow Q(x_1, 0)$; $\quad$ y-axis $\Rightarrow R(0, y_2)$

$$|PQ| : |PR| = 3 : 1 \Rightarrow \frac{(3)(0) + (1)(x_1)}{3 + 1} = 2$$
$$\Rightarrow x_1 = 8 \quad \Rightarrow Q(8, 0)$$

and $$\frac{(3)(y_2) + 1(0)}{3 + 1} = -9$$
$$\Rightarrow 3y_2 = -36$$
$$\Rightarrow y_2 = -12 \quad \Rightarrow R(0, -12)$$

(ii) x-axis $\Rightarrow y = 0 \Rightarrow 3x = c$

$$\Rightarrow x = \frac{c}{3} \quad P\left(\frac{c}{3}, 0\right)$$

y-axis $\Rightarrow x = 0 \Rightarrow 2y = c$

$$\Rightarrow y = \frac{c}{2} \quad Q\left(0, \frac{c}{2}\right)$$

$$\text{Area} = \frac{1}{2}\left|\left(\frac{c}{3}\right)\left(\frac{c}{2}\right) - (0)(0)\right| = 24$$

$$\Rightarrow \left|\frac{c^2}{6}\right| = 48$$

$$\Rightarrow c^2 = 288$$

$$\Rightarrow c = \pm 12\sqrt{2}$$

5. (i) $\text{Midpoint} = \left(\frac{0-3}{2}, \frac{-9+6}{2}\right) = \left(\frac{-3}{2}, \frac{-3}{2}\right)$

$\text{Slope} = \frac{6+9}{-3-0} = -5 \Rightarrow \text{Perpendicular slope} = \frac{1}{5}$

$\Rightarrow \text{Equation: } y + \frac{3}{2} = \frac{1}{5}\left(x + \frac{3}{2}\right)$

$\Rightarrow 5y + \frac{15}{2} = x + \frac{3}{2}$

$\Rightarrow x - 5y = 6$

$\text{Midpoint} = \left(\frac{-3+8}{2}, \frac{6+3}{2}\right) = \left(\frac{5}{2}, \frac{9}{2}\right)$

$\text{Slope} = \frac{3-6}{8+3} = \frac{-3}{11} \Rightarrow \text{Perpendicular slope} = \frac{11}{3}$

$\Rightarrow \text{Equation: } y - \frac{9}{2} = \frac{11}{3}\left(x - \frac{5}{2}\right)$

$\Rightarrow 3y - \frac{27}{2} = 11x - \frac{55}{2}$

$\Rightarrow 11x - 3y = 14$

and $\quad x - 5y = 6$

$\Rightarrow \quad 55x - 15y = 70$

$\Rightarrow \quad 3x - 15y = 18$

Subtract: $52x \quad = 52$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad 11 - 3y = 14$

$\Rightarrow \quad -3y = 3$

$\Rightarrow \quad y = -1$

Circumcentre $= (1, -1)$

(ii) $\text{Radius} = \sqrt{(1-0)^2 + (-1+9)^2} = \sqrt{1+64} = \sqrt{65}$

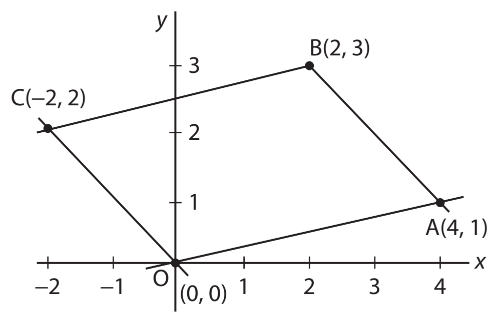
(iii) $\text{Area circle} = \pi\left(\sqrt{65}\right)^2 = 65\pi$ sq.units

6. (i) $\text{Slope OA} = \frac{1}{4} \Rightarrow \text{Slope BC} = \frac{1}{4}$

$\text{Equation BC: } y - 3 = \frac{1}{4}(x - 2)$

$\Rightarrow 4y - 12 = x - 2$

$\Rightarrow x - 4y + 10 = 0$



(ii) Slope OC $= -1$, origin $(0, 0)$

Equation OC: $y - 0 = -1(x - 0)$

$\Rightarrow \quad y = -x$

$\Rightarrow \quad x + y = 0$

and $\quad x - 4y = -10$

Subtract: $\quad 5y = +10$

$\Rightarrow \quad y = +2$

$\Rightarrow \quad x = -2 \qquad C(-2, 2)$

Midpoint OB $= \left(1, 1\frac{1}{2}\right)$

Central symmetry of $C(-2, 2)$ through $\left(1, 1\frac{1}{2}\right)$ is $(4, 1)$

$\Rightarrow A = (4, 1)$

7. (i) $R(-1, -5) \to (0, 0)$; $\quad Q(3, -1) \to (4, 4)$; $\quad U(-2k, 3k) \to (-2k + 1, 3k + 5)$

Area $= \frac{1}{2}|(4)(3k + 5) - (4)(-2k + 1)| = 28$

$\Rightarrow |12k + 20 + 8k - 4| = 56$

$\Rightarrow |20k + 16| = 56$

$\Rightarrow 20k + 16 = 56 \quad$ OR $\quad 20k + 16 = -56$

$\Rightarrow 20k = 40 \qquad \Rightarrow \qquad 20k = -72$

$\Rightarrow \quad k = 2 \qquad \Rightarrow \qquad k = -3.6$

ANS: $k = 2$

(ii) Slope TS $= \frac{-3}{11} \qquad S(13, 9)$

$\Rightarrow$ Equation TS: $y - 9 = \frac{-3}{11}(x - 13)$

$\Rightarrow 11y - 99 = -3x + 39$

$\Rightarrow 3x + 11y = 138$

Slope SR $= \frac{9 + 5}{13 + 1} = \frac{14}{14} = 1 \quad \Rightarrow$ Slope TU $= 1$ and $U(-4, 6)$

$\Rightarrow$ Equation TU: $y - 6 = 1(x + 4)$

$\Rightarrow \quad y - 6 = x + 4$

$\Rightarrow \quad x - y = -10$

and $\quad 3x + 11y = 138$

$\Rightarrow \quad 3x - 3y = -30$

$\quad 3x + 11y = 138$

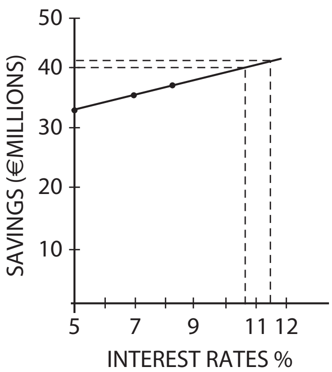
Subtract: $\quad -14y = -168$

$\Rightarrow \quad y = 12$

$\Rightarrow \quad x - 12 = -10$

$\Rightarrow \quad x = 2 \qquad \Rightarrow T(2, 12)$

8. (i) Graph



(ii) Slope $= \frac{37 - 35}{8.5 - 7} = \frac{2}{1.5} = \frac{4}{3}$

Equation: $S - 35 = \frac{4}{3}(I - 7)$

$\Rightarrow 3S - 105 = 4I - 28$

$\Rightarrow 4I - 3S + 77 = 0$

(iii) €41 million

(iv) $10\frac{3}{4}\%$

9. Equation: $y + 4 = m(x - 2)$

$$\Rightarrow y + 4 = mx - 2m$$
$$\Rightarrow mx - y - 2m - 4 = 0$$

x-axis $\Rightarrow y = 0 \quad mx = 2m + 4$

$$\Rightarrow x = \frac{2m+4}{m} \qquad \left(\frac{2m+4}{m}, 0\right)$$

y-axis $\Rightarrow x = 0 \quad -y = 2m + 4$

$$\Rightarrow y = -2m - 4 \qquad (0, -2m - 4)$$

Hence, $\left(\frac{2m+4}{m}\right) + (-2m - 4) = -4$

$$\Rightarrow 2m + 4 - 2m^2 - 4m = -4m$$
$$\Rightarrow 2m^2 - 2m - 4 = 0$$
$$\Rightarrow m^2 - m - 2 = 0$$
$$\Rightarrow (m + 1)(m - 2) = 0$$
$$\Rightarrow m = -1 \quad \text{OR} \quad m = 2$$

$$\tan\theta = \pm\frac{-1-2}{1 + (-1)(2)} = \pm\frac{-3}{-1} = \pm 3$$

Acute $\Rightarrow \tan\theta = 3$

10. (i) Slope $= \frac{-t}{t+2}$

(ii) Slope $m_1 = -\frac{1}{-2} = \frac{1}{2}$

$$\tan 45° = \pm\frac{\frac{1}{2} - \left(\frac{-t}{t+2}\right)}{1 + \left(\frac{1}{2}\right)\left(\frac{-t}{t+2}\right)}$$

$$\Rightarrow \pm\frac{\frac{t+2+t(2)}{2(t+2)}}{\frac{2t+4-t}{2(t+2)}} = 1$$

$$\Rightarrow \frac{3t+2}{t+4} = \pm 1$$

$$\Rightarrow 3t + 2 = \pm(t + 4)$$

$$\Rightarrow 3t + 2 = -t - 4 \quad \text{OR} \quad 3t + 2 = t + 4$$

$$\Rightarrow \quad 4t = -6 \qquad \Rightarrow \quad 2t = 2$$

$$\Rightarrow \quad t = -\frac{3}{2} \qquad \Rightarrow \quad t = 1$$

11. (i) Midpoint of [AB] $= \left(\frac{7+2}{2}, \frac{2+5}{2}\right) = \left(\frac{9}{2}, \frac{7}{2}\right)$.

Slope of AB $= \frac{5-2}{2-7} = \frac{-3}{5}$.

Slope of perpendicular bisector $= \frac{5}{3}$.

Equation of perpendicular bisector:

$$\left(y - \frac{7}{2}\right) = \frac{5}{3}\left(x - \frac{9}{2}\right)$$

$$3y - \frac{21}{2} = 5x - \frac{45}{2}$$

$$-5x + 3y = -12$$

(ii) D is the point of intersection of $-5x + 3y = -12$ and $4x - y = 26$

1: $\quad -5x + 3y = -12$

2 $\times$ 3: $\quad 12x - 3y = 78$

$$7x \quad = 66$$

$$x = \frac{66}{7}$$

1: $-5\left(\frac{66}{7}\right) + 3y = -12$

$3y = \frac{246}{7}$

$y = \frac{82}{7}$

Thus $D = \left(\frac{66}{7}, \frac{82}{7}\right)$.

(iii) As $|\angle ABC| = 90°$, BC is parallel to the perpendicular bisector of [AB].

Thus the slope of BC is $\frac{5}{3}$.

(iv) The equation of BC is

$$(y - 5) = \frac{5}{3}(x - 2).$$

If $(8, c)$ belongs to this line, then

$$c - 5 = \frac{5}{3}(8 - 2)$$

$$c = 5 + 10 = 15$$

(v) Triangle ABC.

$$A(7, 2) \xrightarrow{-7, -2} (0, 0)$$

$$B(2, 5) \xrightarrow{-7, -2} (-5, 3) = (x_1, y_1)$$

$$C(8, 15) \xrightarrow{-7, -2} (1, 13) = (x_2, y_2)$$

$$\text{Area } \triangle ABC = \frac{1}{2}|(-5)(13) - (1)(3)| = \frac{1}{2}|-68| = 34$$

Triangle ADC.

$$A(7, 2) \xrightarrow{-7, -2} (0, 0)$$

$$D\left(\frac{66}{7}, \frac{82}{7}\right) \xrightarrow{-7, -2} \left(\frac{17}{7}, \frac{68}{7}\right) = (x_1, y_1)$$

$$C(8, 15) \xrightarrow{-7, -2} (1, 13) = (x_2, y_2)$$

$$\text{Area } \triangle ADC = \frac{1}{2}\left|\left(\frac{17}{7}\right)(13) - (1)\left(\frac{68}{7}\right)\right| = \frac{1}{2}\left|\frac{153}{7}\right| = \frac{153}{14}$$

Thus the area of the quadrilateral is

$$34 + \frac{153}{14} = \frac{629}{14}.$$

Chapter 5

Exercise 5.1

1. $\boxed{4}\boxed{3}\boxed{5} = 60$ ways

2. $\boxed{6}\boxed{7} = 42$ ways

3. $\boxed{26}\boxed{9}\boxed{8} = 1872$ codes

4. $\boxed{10}\boxed{6}\boxed{4} = 240$ ways

5. $\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 6! = 720$ ways

6. $\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 7! = 5040$ ways

$6! \times 2! = 1440$ ways

7. $\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 5! = 120$ ways

(i) $\boxed{1}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 24$

(ii) $\boxed{1}\boxed{3}\boxed{2}\boxed{1}\boxed{1} = 6$

8. $\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 5040$ arrangements

(i) $\boxed{2}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 1440$

(ii) $6! \times 2! = 1440$

9. (i) $\boxed{4}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 480$ arrangements

(ii) $\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{2} = 240$ arrangements

(iii) $5! \times 2! = 240$ arrangements

10. (i) $\boxed{1}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 720$

(ii) $6! \times 2! = 1440$

11. $\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 7! = 5040$

$5! \times 3! = 720$

12. (i) $5! \times 3! = 720$ arrangements

(ii) $\boxed{4}\boxed{3}\boxed{3}\boxed{2}\boxed{2}\boxed{1}\boxed{1} = 144$ arrangements

13. $\boxed{7}\boxed{6}\boxed{5}\boxed{4}$ or $({}^7P_4) = 840$

14. $\boxed{6}\boxed{5}\boxed{4}\boxed{3}$ or $({}^6P_4) = 360$

15. $\boxed{8}\boxed{7}\boxed{6}$ or $({}^8P_3) = 336$ ways

16. $\boxed{5}\boxed{4}\boxed{9}\boxed{8} = 1440$ codes

17. (i) $\boxed{9}\boxed{8}\boxed{7} = 504$ three-digit numbers

(ii) $\boxed{9}\boxed{9}\boxed{8} = 648$ three-digit numbers

18. $\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 24$ four-digit numbers

(i) $\boxed{1}\boxed{3}\boxed{2}\boxed{1} = 6$

(ii) $\boxed{3}\boxed{2}\boxed{1}\boxed{1} = 6$

(iii) $\boxed{2}\boxed{3}\boxed{2}\boxed{1} = 12$

19. $\boxed{9}\boxed{9}\boxed{8}\boxed{7} = 4536$ four-digit numbers

(i) $\boxed{2}\boxed{9}\boxed{8}\boxed{7} = 1008$

(ii) $\boxed{9}\boxed{8}\boxed{7}\boxed{1} = 504$

20. $\boxed{3}\boxed{4}\boxed{3}\boxed{2} = 72$ four-digit numbers

Start with 5 $\boxed{1}\boxed{3}\boxed{2}\boxed{1} = 6$

Start with 8 $\boxed{1}\boxed{3}\boxed{2}\boxed{2} = 12$

Start with 9 $\boxed{1}\boxed{3}\boxed{2}\boxed{1} = 6$

24 four-digit numbers

21. $\boxed{5}\boxed{5}\boxed{4} = 100$ three-digit numbers

(i) $\boxed{3}\boxed{5}\boxed{4} = 60$

(ii) $\boxed{1}\boxed{5}\boxed{4} = 20$

22. (i) $\boxed{5}\boxed{5}\boxed{4}\boxed{3} = 300$ codes

(ii) $\boxed{1}\boxed{5}\boxed{4}\boxed{3} = 60$ codes

23. $\boxed{3}\boxed{2}\boxed{1}\boxed{3}\boxed{2}\boxed{1}$ or $3! \times 3! = 36$ codes

24. $\boxed{9}\boxed{8}\boxed{7}\boxed{1}\boxed{6}\boxed{5}\boxed{4} = 60\,480$ arrangements

25. $\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 7! = 5040$ arrangements

(i) $6! \times 2! = 1440$

(ii) $5040 - 1440 = 3600$

26. (i) $\boxed{10}\boxed{9}\boxed{8} = 720$ ways

(ii) $\boxed{8}\boxed{7}\boxed{6} = 336$ ways

(iii) $\boxed{1}\boxed{1}\boxed{8} = 8 \times 3! = 48$ ways

Exercise 5.2

1. (i) 15 (iii) 45 (v) 153
(ii) 35 (iv) 66

2. (i) $\binom{12}{9} = 220, \binom{12}{8} = 495, \binom{13}{9} = 715$

Hence, $220 + 495 = 715$

(ii) $\binom{10}{2} = 45 \quad \Rightarrow \quad 8\binom{10}{2} = 360$

$\binom{10}{3} = 120 \quad \Rightarrow \quad 3\binom{10}{3} = 360$

Hence, $8\binom{10}{2} = 3\binom{10}{3}$

3. $\binom{8}{5} = 56$ different selections

4. $\binom{14}{11} = 364$ teams; $\binom{13}{10} = 286$ teams

5. $\binom{9}{5} = 126$ selections

(i) $\binom{8}{4} = 70$

(ii) $\binom{7}{4} = 35$

6. $\binom{9}{5} = 126$ ways; $\binom{8}{4} = 70$ ways

7. $\binom{52}{3} = 22\,100$ hands; $\binom{13}{3} = 286$ hands

8. (i) $\binom{8}{3} = 56$

(ii) $\binom{9}{5} = 126$

(iii) $\binom{7}{3} = 35$

9. $\binom{5}{3} \times \binom{4}{3} = 10 \times 4 = 40$ ways

10. (i) $\binom{10}{3} \times \binom{12}{3} = 120 \times 220 = 26\,400$

(ii) $\binom{10}{2} \times \binom{12}{4} = 45 \times 495 = 22\,275$

11. $\binom{6}{3} = 20$ subsets

(i) $\binom{2}{1} \times \binom{4}{2} = 2 \times 6 = 12$

(ii) No vowels $= \binom{4}{3} = 4 \Rightarrow$ at least one vowel $\Rightarrow 20 - 4 = 16$

12. $\binom{8}{6} = 28$ ways

$\binom{4}{4} \times \binom{4}{2} = 1 \times 6 = 6$

13. (i) $\binom{5}{4} \times \binom{3}{2} = 5 \times 3 = 15$ ways

(ii) 4 men and 2 women $= \binom{5}{4} \times \binom{3}{2} = 5 \times 3 = 15$

or 5 men and 1 woman $= \binom{5}{5} \times \binom{3}{1} = 1 \times 3 = \underline{\ 3\ }$

Total $= 18$ ways

14. $\binom{3}{1} \times \binom{6}{3} \times \binom{4}{2} = 3 \times 20 \times 6 = 360$ teams

15. (i) $\binom{8}{4} = 70$ subcommittees

(ii) $\binom{6}{2} = 15$ subcommittees

(iii) $\binom{6}{4} = 15$ subcommittees

16. $\binom{5}{3} = 10$ triangles

[XY] as one side $= \binom{3}{1} = 3$

17. (i) $\binom{6}{4} = 15$ quadrilaterals

(ii) [AB] as one side $= \binom{4}{2} = 6$

18. (i) $\binom{7}{3} = 35$ ways

(ii) Ann included, Barry excluded $= \binom{7}{4} = 35$

or Barry included, Ann excluded $= \binom{7}{4} = \underline{35}$

Total $= 70$ ways

(iii) No restrictions: 9 people, select 5 $= \binom{9}{5} = 126$

Ann, Barry and Claire excluded $= \binom{6}{5} = 6$

Hence, at least one of Ann, Barry and Claire must be included $= 126 - 6 = 120$ ways

19. 3 section A and 2 section B $\Rightarrow \binom{5}{3} \times \binom{7}{2} = 10 \times 21 = 210$

or 2 section A and 3 section B $\Rightarrow \binom{5}{2} \times \binom{7}{3} = 10 \times 35 = \underline{350}$

Total $= 560$ ways

20.

4	3	4	3	2

$= 288$ registrations

21. (i) $\binom{n}{2} = 10$

$$\Rightarrow \frac{n(n-1)}{2.1} = \frac{10}{1} \quad \Rightarrow \quad n^2 - n = 20$$

$$\Rightarrow \quad n^2 - n - 20 = 0$$

$$\Rightarrow \quad (n-5)(n+4) = 0$$

$$\Rightarrow \quad n = 5 \text{ or } n = -4$$

since $n \in N, \Rightarrow n = 5$

(ii) $\binom{n}{2} = 45$

$$\Rightarrow \frac{n(n-1)}{2.1} = \frac{45}{1} \quad \Rightarrow \quad n^2 - n = 90$$

$$\Rightarrow \quad n^2 - n - 90 = 0$$

$$\Rightarrow \quad (n-10)(n+9) = 0$$

$$\Rightarrow \quad n = 10 \text{ or } n = -9$$

since $n \in N, \Rightarrow n = 10$

(iii) $\binom{n+1}{2} = 28$

$$\Rightarrow \frac{(n+1)(n)}{2.1} = \frac{28}{1} \quad \Rightarrow \quad n^2 + n = 56$$
$$\Rightarrow \quad n^2 + n - 56 = 0$$
$$\Rightarrow \quad (n+8)(n-7) = 0$$
$$\Rightarrow \quad n = -8 \text{ or } n = 7$$
$$\text{since } n \in N, \quad \Rightarrow \quad n = 7$$

Exercise 5.3

1. (i) Impossible
(ii) Very likely
(iii) Very unlikely
(iv) Very unlikely
(v) Even chance
(vi) Certain
(vii) Unlikely

2. (i) 6
(ii) 4
(iii) 0
(iv) 2

3. (i) 6 (ii) 8 (iii) 2

4. (i) $\frac{1}{6}$
(ii) $\frac{2}{6} = \frac{1}{3}$
(iii) $\frac{3}{6} = \frac{1}{2}$
(iv) $\frac{3}{6} = \frac{1}{2}$
(v) $\frac{2}{6} = \frac{1}{3}$
(vi) $\frac{3}{6} = \frac{1}{2}$

5. (i) $\frac{4}{52} = \frac{1}{13}$
(ii) $\frac{13}{52} = \frac{1}{4}$
(iii) $\frac{12}{52} = \frac{3}{13}$
(iv) $\frac{2}{52} = \frac{1}{26}$
(v) $\frac{20}{52} = \frac{5}{13}$

6. (i) $\frac{9}{17}$
(ii) $\frac{8}{17}$
(iii) $\frac{5}{17}$
(iv) $\frac{4}{17}$

7. (i) $\frac{1}{8}$
(ii) $\frac{2}{8} = \frac{1}{4}$
(iii) $\frac{3}{8}$
(iv) $\frac{4}{8} = \frac{1}{2}$

8. (i) $\frac{15}{30} = \frac{1}{2}$
(ii) $\frac{5}{30} = \frac{1}{6}$
(iii) $\frac{20}{30} = \frac{2}{3}$
(iv) $\frac{15}{30} = \frac{1}{2}$

9. (i) $\frac{3}{36} = \frac{1}{12}$
(ii) $\frac{9}{36} = \frac{1}{4}$
(iii) $\frac{6}{36} = \frac{1}{6}$
(iv) $\frac{12}{36} = \frac{1}{3}$

10. (i) (3, 3) gives a score = 9 $\Rightarrow$ $P(9) = \frac{1}{36}$
(ii) (1, 4), (4, 1), (2, 2) gives a score = 4 $\Rightarrow$ $P(4) = \frac{3}{36} = \frac{1}{12}$
(iii) (3, 4) (4, 3) (2, 6) (6, 2) gives a score = 12 $\Rightarrow$ $P(12) = \frac{4}{36} = \frac{1}{9}$

11. Box has 6 counters; 3 of these are green.
One green counter removed; hence, box has 2 green counters left.
$P(\text{green}) = \frac{2}{5}$

12. (i) $P(\text{purple}) = 1 - \frac{2}{5} = \frac{3}{5}$

(ii) 3

(iii) 3

13. Sample space S

$\#S = 12$

(i) $\frac{1}{12}$

(ii) $\frac{2}{12} = \frac{1}{6}$

(iii) $\frac{6}{12} = \frac{1}{2}$

9 occurs most often $\Rightarrow P(9) = \frac{3}{12} = \frac{1}{4}$

	1	**2**	**3**	**4**
5	6	7	8	9
7	8	9	10	11
8	9	10	11	12

14. $\#S = 36$

(i) Total = 7 occurs from (3, 4), (4, 3), (2, 5), (5, 2), (6, 1), (1, 6)

Total = 11 occurs from (5, 6) (6, 5)

$\Rightarrow P(\text{wins}) = \frac{8}{36} = \frac{2}{9}$

(ii) Total = 2 occurs from (1, 1); Total = 3 occurs from (1, 2) (2, 1)

Total = 12 occurs from (6, 6)

$\Rightarrow P(\text{loses}) = \frac{4}{36} = \frac{1}{9}$

15. Sample space (S)

$S = 8$

HHH HTT
HHT THT
HTH TTH
THH TTT

(i) $P(\text{HHH}) = \frac{1}{8}$

(ii) $P(\text{HTH}) = \frac{1}{8}$

(iii) $P(\text{2H and IT}) = \frac{3}{8}$

16. (i) $\frac{25}{50} = \frac{1}{2}$

(ii) $\frac{16}{50} = \frac{8}{25}$

(iii) $\frac{16}{50} = \frac{8}{25}$

25 males $\Rightarrow$ $P(\text{He wears glasses}) = \frac{16}{25}$

17. (i) $P(\text{Bus}) = \frac{60°}{360°} = \frac{1}{6}$

(ii) Walk $= 360° - (90° + 60°) = 210°$

$\Rightarrow$ $P(\text{Walk}) = \frac{210°}{360°} = \frac{7}{12}$

18. (i) No. of ways $= \binom{13}{3} = 286$

(ii) No. of ways $= \binom{52}{3} = 22\,100$

(iii) $P(\text{3 clubs}) = \frac{286}{22\,100} = \frac{11}{850}$.

19. (i) Orders Coffee: M 10, W 20, Total 30
No Coffee: M 15, W 5, Total 20
Total: M 25, W 25, Total 50
(ii) Orders Coffee: M 0.2, W 0.4, Total 0.6
No Coffee: M 0.3, W 0.1, Total 0.4
Total: M 0.5, W 0.5, Total 1
(iii) (a) $P(\text{Man and no coffee}) = 0.3$
(b) $P(\text{Customer and no coffee}) = 0.4$

20. Choose 4 out of 15 trucks: No of ways $= \binom{15}{4} = 1365$

Choose 4 from the 10 good trucks: No of ways $= \binom{10}{4} = 210$

$P(\text{all 4 trucks are good}) = \dfrac{210}{1365} = \dfrac{2}{13}$.

21. (i) No. of outcomes = 36.
Total of 7: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)
No. of ways = 6

$P(\text{total of 7}) = \dfrac{6}{36} = \dfrac{1}{6}$

(ii) At least two 7s: (2 7s) or (3 7s) or (4 7s) (success)
Failure: (0 7s) or (1 7)
Let W be getting a total of 7 and N be not getting a total of 7.

Then $P(W) = \dfrac{1}{6}$ and $P(N) = 1 - \dfrac{1}{6} = \dfrac{5}{6}$.

$$P(0\ 7s) = P(NNNN) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

1 7 can be obtained in 4 ways: $NNNW$, $NNWN$, $NWNN$, $WNNN$, each of which has the same probability.

$$P(NNNW) = \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) = \frac{125}{1296}.$$

Thus

$$P(1\ 7) = 4 \times \frac{125}{1296} = \frac{500}{1296}$$

Hence

$$P(\text{failure}) = \frac{625}{1296} + \frac{500}{1296} = \frac{1125}{1296} = \frac{125}{144}$$

and

$$P(\text{success}) = 1 - \frac{125}{144} = \frac{19}{144} = \frac{171}{1296}.$$

22. (i) (a) Choose 4 from 11. No. of ways $= \binom{11}{4} = 330$

(b) More boys than girls: (3 boys and 1 girl) or (4 boys)

$$\text{No of ways} = \left[\binom{6}{3} \times \binom{5}{1}\right] + \left[\binom{6}{4}\right]$$
$$= (20 \times 5) + (15) = 115$$

(ii) One boy = 1 boy and 3 girls

No. of ways $= \binom{6}{1} \times \binom{5}{3} = 60$

$P(\text{one boy}) = \dfrac{60}{330} = \dfrac{2}{11}$.

Exercise 5.4

1. (i) $P(2) = \frac{1}{6} \Rightarrow$ Expected frequency $= \frac{1}{6} \times 900 = 150$

(ii) $P(6) = \frac{1}{6} \Rightarrow$ Expected frequency $= \frac{1}{6} \times 900 = 150$

(iii) $P(2 \text{ or } 6) = \frac{2}{6} = \frac{1}{3} \Rightarrow$ Expected frequency $= \frac{1}{3} \times 900 = 300$

2. (i) $P(\text{red}) = \frac{4}{8} = \frac{1}{2}$

(ii) (a) Expected frequency $= \frac{1}{2} \times 400 = 200$ times

(b) $P(\text{white}) = \frac{3}{8}$

$\Rightarrow$ Expected frequency $= \frac{3}{8} \times 400 = 150$ times

3. (i) Relative frequency (Heads) $= \frac{34}{100} = \frac{17}{50}$

(ii) No, probably not; as 34 is well below the expected value of 50.

4. (i) (a) Exp. $P(6) = \frac{60}{300} = \frac{1}{5}$

(b) Exp. $P(2) = \frac{40}{300} = \frac{2}{15}$

(ii) (a) $P(6) = \frac{1}{6}$

(b) $P(2) = \frac{1}{6}$

(iii) No; as 60 is well above the expected value of 50, and 40 is well below the expected value of 50.

5. (i) Estimate = relative frequency $= \frac{154}{300} = \frac{77}{150}$

(ii) No; as red is far higher than one would expect (54% higher).

6. $P(\text{red}) = \frac{5}{10} = \frac{1}{2}$

Expected frequency $= \frac{1}{2} \times 300 = 150$

Spinner is almost definitely not fair as red (120 times) should be much closer to 150 times.

7. (i) $x + 0.2 + 0.1 + 0.3 + 0.1 + 0.2 = 1$

$\Rightarrow x + 0.9 = 1$

$\Rightarrow x = 1 - 0.9 = 0.1$

(ii) $P(\text{number} > 3) = 0.3 + 0.1 + 0.2 = 0.6$

(iii) $P(6) = 0.2$

Expected frequency of 6 $= 0.2 \times 1000 = 200$ times

8. $P(\text{Gemma wins}) = \frac{21}{30} = \frac{7}{10}$

9. $P(6) = \frac{165}{1000} = \frac{33}{200}$

Use the largest number of trials

10. (i) Bill's

(ii)

	Results		
Number of spins	0	1	2
Total = 580	187	267	126

Hence, spinner is biased.

(iii) $P(2) = \frac{126}{580} = \frac{63}{290}$

(iv) $P(0) = \frac{187}{580} = 0.322 \quad \Rightarrow \quad \text{Expected frequency} = 0.322 \times 1000$

$= 322$

11. (i) $P(1) = \frac{2}{6} = \frac{1}{3}$

(ii) 1, 2, 2, 3, 3, 4

12. (i) Expected number of home wins $= 10 \times 0.9 = 9$

Expected number of away wins $= 10 \times 0.85 = 8.5$

The expected number of wins is $9 + 8.5 = 17.5 = 17$.

(ii) Expected number of home wins $= 10 \times 0.7 = 7$

Expected number of away wins $= 10 \times 0.7 = 7$

The expected number of wins is $7 + 7 = 14$.

13. (i) Let R be that it rains on a day and N be that it doesn't rain on a day.

$P(RRR) = (0.4)(0.7)(0.7) = 0.196$

(ii) $P(NRR) = (0.6)(0.2)(0.7) = 0.084$

(iii) $P(RRNR) = (0.4)(0.7)(0.3)(0.2) = 0.0168$

Exercise 5.5

1. (i) $\frac{8}{16} = \frac{1}{2}$ (ii) $\frac{4}{16} = \frac{1}{4}$ (iii) $\frac{8+4}{16} = \frac{12}{16} = \frac{3}{4}$

2. (i) $\frac{13}{52} = \frac{1}{4}$ (ii) $\frac{6}{52} = \frac{3}{26}$ (iii) $\frac{13+6}{52} = \frac{19}{52}$

3. (i) $\frac{10}{30} = \frac{1}{3}$

(ii) $\frac{6}{30} = \frac{1}{5}$

Not mutually exclusive as 15 and 30 are multiples of both 3 and 5

$\Rightarrow \quad P(\text{multiple of 3 or 5}) = \frac{10}{30} + \frac{6}{30} - \frac{2}{30} = \frac{14}{30} = \frac{7}{15}$

4. (i) $\frac{6}{12} = \frac{1}{2}$

(ii) $\frac{4}{12} = \frac{1}{3}$

(iii) $\frac{6}{12} + \frac{4}{12} - \frac{2}{12} = \frac{8}{12} = \frac{2}{3}$

5. (i) $\frac{13}{52} = \frac{1}{4}$

(ii) $\frac{4}{52} = \frac{1}{13}$

(iii) $\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

(iv) $\frac{26}{52} = \frac{1}{2}$

(v) $\frac{4}{52} = \frac{1}{13}$

(vi) $\frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$

6. (i) $\frac{6}{36} = \frac{1}{6}$

(ii) Total of 8 $\quad \Rightarrow \quad$ (2, 6) (6, 2) (3, 5) (5, 3) (4, 4)

$\Rightarrow \quad P(\text{total of 8}) = \frac{5}{36}$

(iii) $\frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$

7. $\#S = 3 + 4 + 6 + 8 + 5 + 2 = 28$

(i) $\dfrac{14}{28} = \dfrac{1}{2}$

(ii) $\dfrac{21}{28} = \dfrac{3}{4}$

(iii) $\dfrac{8}{28} = \dfrac{2}{7}$

(iv) $\dfrac{14}{28} + \dfrac{14}{28} - \dfrac{6}{28} = \dfrac{22}{28} = \dfrac{11}{14}$

(v) $\dfrac{21}{28} = \dfrac{3}{4}$

(vi) $\dfrac{0}{28} = 0$

8. $\#S = 68 + 62 + 26 + 32 + 6 + 6 = 200$

(i) $\dfrac{32}{200} = \dfrac{4}{25}$

(ii) $\dfrac{100}{200} = \dfrac{1}{2}$

(iii) $\dfrac{12}{200} + \dfrac{100}{200} - \dfrac{6}{200} = \dfrac{106}{200} = \dfrac{53}{100}$

9. (i) $\dfrac{4}{16} = \dfrac{1}{4}$

(ii) $\dfrac{4}{16} = \dfrac{1}{4}$

(iii) $\dfrac{7}{16}$

(iv) $\dfrac{12}{16} = \dfrac{3}{4}$

(v) $\dfrac{4}{16} = \dfrac{1}{4}$

(vi) $\dfrac{12}{16} + \dfrac{4}{16} - \dfrac{2}{16} = \dfrac{14}{16} = \dfrac{7}{8}$

(vii) $\dfrac{4}{16} + \dfrac{8}{16} - \dfrac{2}{16} = \dfrac{10}{16} = \dfrac{5}{8}$

10. $n =$ number of green beads

$\Rightarrow \quad \#S = 8 + 12 + n = 20 + n$

$$P(\text{green}) = \frac{n}{20 + n} = \frac{1}{5}$$

$\Rightarrow \quad 5n = 20 + n$

$\Rightarrow \quad 4n = 20$

$\Rightarrow \quad n = 5$

11. 40 red, all even

30 blue, all odd

30 green — 20 even, 10 odd

(i) $P(\text{red}) = \dfrac{40}{100} = \dfrac{2}{5}$

(ii) $P(\text{not blue}) = \dfrac{70}{100} = \dfrac{7}{10}$

(iii) $P(\text{green or even}) = \dfrac{30}{100} + \dfrac{60}{100} - \dfrac{20}{100} = \dfrac{70}{100} = \dfrac{7}{10}$

12. 100 people

- 40 male
  - 4 play tennis
  - 36 do not play tennis
- 60 female
  - 9 play tennis
  - 51 do not play tennis

(i) $\dfrac{4}{100} = \dfrac{1}{25}$

(ii) $\dfrac{4 + 9}{100} = \dfrac{13}{100}$

(iii) $\dfrac{60}{100} + \dfrac{13}{100} - \dfrac{9}{100} = \dfrac{64}{100} = \dfrac{16}{25}$

13. A = {20, 21, 22, 23}
B = {20, 25}
C = {23, 29}
D = {21, 24, 27}

(i) (a) No (b) No (c) No (d) Yes (e) Yes

(ii) $\frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$

(iii) No, as these 2 events are not mutually exclusive.

14. $\#S = 6 + 2 + 9 + 3 = 20$

(i) $P(\text{A}) = \frac{8}{20} = \frac{2}{5}$

(ii) $P(\text{B}) = \frac{11}{20}$

(iii) $P(\text{A} \cup \text{B}) = \frac{6 + 2 + 9}{20} = \frac{17}{20}$

$P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B}) = \frac{8}{20} + \frac{11}{20} - \frac{2}{20} = \frac{17}{20}$

Hence, $P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

15. (i) $P(\text{C}) = 0.4 + 0.2 = 0.6$

(ii) $P(\text{D}) = 0.2 + 0.3 = 0.5$

(iii) $P(\text{C} \cup \text{D}) = 0.4 + 0.2 + 0.3 = 0.9$

(iv) $P(\text{C} \cap \text{D}) = 0.2$

$P(\text{C}) + P(\text{D}) - P(\text{C} \cap \text{D}) = 0.6 + 0.5 - 0.2 = 0.9$

Hence, $P(\text{C} \cup \text{D}) = P(\text{C}) + P(\text{D}) - P(\text{C} \cap \text{D})$

16. (i) $25 + 5 + x + 8 = 50$

$\Rightarrow \quad x + 38 = 50$

$\Rightarrow \quad x = 50 - 38 = 12$

(ii) $P(\text{French}) = \frac{25 + 5}{50} = \frac{30}{50} = \frac{3}{5}$

(iii) $P(\text{French and Spanish}) = \frac{5}{50} = \frac{1}{10}$

(iv) $P(\text{French or Spanish}) = \frac{25 + 5 + 12}{50} = \frac{42}{50} = \frac{21}{25}$

(v) $P(\text{one language only}) = \frac{25 + 12}{50} = \frac{37}{50}$

17. (i) $\#S = 5 + 3 + 11 + 1 + 2 + 8 + 7 + 3 = 40$

(ii) $\frac{1 + 2}{40} = \frac{3}{40}$

(iii) $\frac{1 + 2}{5 + 3 + 2 + 1} = \frac{3}{11}$

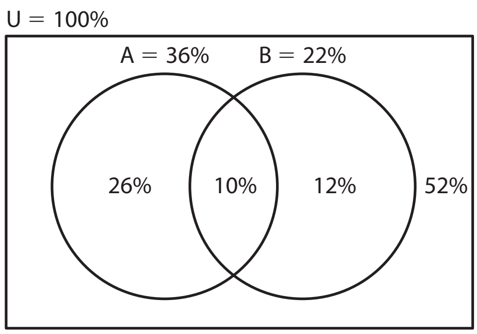
(iv) $\frac{3 + 2}{3 + 11 + 8 + 2} = \frac{5}{24}$

(v) $\frac{2}{1 + 2} = \frac{2}{3}$

18. (i) Venn Diagram

(ii) 52%

(iii) 26%



19. $P(A) + P(B) - P(A \cap B) = P(A \cup B)$

$\Rightarrow \quad \frac{2}{3} + P(B) - \frac{5}{12} = \frac{3}{4}$

$\Rightarrow \quad P(B) + \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \quad P(B) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

20. $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$\Rightarrow \quad \frac{9}{10} = \frac{1}{2} + \frac{3}{5} - P(X \cap Y)$

$\Rightarrow \quad \frac{9}{10} = \frac{11}{10} - P(X \cap Y)$

$\Rightarrow P(X \cap Y) = \frac{11}{10} - \frac{9}{10} = \frac{2}{10} = \frac{1}{5}$

21. $P(C) + P(D) - P(C \cap D) = P(C \cup D)$

$\Rightarrow \quad 0.7 + P(D) - 0.3 = 0.9$

$\Rightarrow \quad P(D) + 0.4 = 0.9$

$\Rightarrow \quad P(D) = 0.9 - 0.4 = 0.5$

22. (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.8 + 0.5 - 0.3$

$= 1$

(ii) $P(A \cup B) = 1$

$P(A) + P(B) - P(A \cap B) = 0.8 + 0.5 - 0.3 = 1$

Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

23. (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{8}{15} + \frac{2}{3} - \frac{1}{3}$

$= \frac{13}{15}$

(ii) No, because $A \cap B \neq \phi$.

24. $P(A \cup B) = P(A) + P(B)$

$= \frac{3}{7} + \frac{1}{5} = \frac{22}{35}$

25. (a) The Venn diagram is completed in the answers at the back of the book.

(b) $A \cap B$ is the set of students who went to the cinema and watched movies at home.

(c) The second Venn diagram is completed at the back of the book.

(d) Student who went to the cinema but did not watch movies at home = $A \backslash B$.

From the Venn diagram,

$P(A \backslash B) = 0.2$.

(e) (i) $P(A) = 0.2 + 0.5 = 0.7$

(ii) $P(B) = 0.5 + 0.1 = 0.6$

Exercise 5.6

1. (i) $P(R, R) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$

(ii) $P(G, G) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$

(iii) $P(Y, Y) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$

(iv) $P(R, G) = \frac{2}{5} \cdot \frac{1}{5} = \frac{2}{25}$

2. (i) $P(6, 6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

(ii) $P(6, \text{Even}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

(iii) $P(\text{Odd, multiple of } 3) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

3. (i) $P(\text{Heads}, 6) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

(ii) $P(\text{Heads, Even}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

4. (i) $P(\text{Black, Black}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

(ii) $P(\text{King, King}) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

(iii) $P(\text{Black Ace, Diamond}) = \frac{1}{26} \cdot \frac{1}{4} = \frac{1}{104}$

5. (i) $P(\text{Red, Red}) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$

(ii) $P(\text{Blue, Red}) = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$

(iii) $P(\text{Red, Blue}) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$

(iv) $P(\text{Blue, Blue}) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$

(v) $P(\text{same colour}) = \frac{4}{25} + \frac{9}{25} = \frac{13}{25}$

6. $P(\text{rain tomorrow, forget umbrella}) = \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$

7. (i) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

(ii) $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

(iii) $\frac{1}{13} \cdot \frac{3}{13} = \frac{3}{169}$

(iv) $\frac{1}{13} \cdot \frac{1}{52} = \frac{1}{676}$

(v) $\frac{1}{52} \cdot \frac{1}{52} = \frac{1}{2704}$

8. $P(\text{rasp berry, rasp berry, rasp berry}) = \frac{4}{12} \cdot \frac{3}{12} \cdot \frac{2}{12} = \frac{24}{1728} = \frac{1}{72}$

9. $P(\text{hit gold area}) = 0.2 \quad \Rightarrow \quad P(\text{miss gold area}) = 0.8$

(i) $P(\text{hit, hit}) = (0.2)(0.2) = 0.04$

(ii) $P(\text{hit, miss}) + P(\text{miss, hit}) = (0.2)(0.8) + (0.8)(0.2) = 0.32$

10. $P(\text{Chris passes}) = 0.8 \quad \Rightarrow \quad P(\text{Chris fails}) = 0.2$

$P(\text{Georgie passes}) = 0.9 \quad \Rightarrow \quad P(\text{Georgie fails}) = 0.1$

$P(\text{Phil passes}) = 0.7 \quad \Rightarrow \quad P(\text{Phil fails}) = 0.3$

(i) $P(\text{all 3 pass}) = (0.8)(0.9)(0.7) = 0.504$

(ii) $P(\text{all 3 fail}) = (0.2)(0.1)(0.3) = 0.006$

(iii) $P(\text{at least one passes})$

$= 1 - P(\text{all 3 fail}) = 1 - 0.006 = 0.994$

11. $P(\text{Alan hits target}) = \frac{1}{2} \quad \Rightarrow \quad P(\text{Alan misses target}) = \frac{1}{2}$

$P(\text{Shane hits target}) = \frac{2}{3} \quad \Rightarrow \quad P(\text{Shane misses target}) = \frac{1}{3}$

(i) $P(\text{both men hit}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$

(ii) $P(\text{both men miss}) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

(iii) $P(\text{only one hits}) = \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$

12. $P(\text{stops at first}) = 0.6 \quad \Rightarrow \quad P(\text{doesn't stop at first}) = 0.4$

$P(\text{stops at second}) = 0.7 \quad \Rightarrow \quad P(\text{doesn't stop at second}) = 0.3$

$P(\text{stops at third}) = 0.8 \quad \Rightarrow \quad P(\text{doesn't stop at third}) = 0.2$

(i) $P(\text{stops at all three}) = (0.6)(0.7)(0.8) = 0.336$

(ii) $P(\text{he is late}) = P(\text{stop, stop, doesn't stop}) + P(\text{stop, doesn't stop, stop})$

$+ P(\text{doesn't stop, stop, stop}) + P(\text{stop, stop, stop})$

$= (0.6)(0.7)(0.2) + (0.6)(0.3)(0.8) + (0.4)(0.7)(0.8) + (0.6)(0.7)(0.8)$

$= 0.084 + 0.144 + 0.224 + 0.336$

$= 0.788$

13. (i) $P(\text{no sixes}) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}$

(ii) $P(\text{at least one six})$

$= 1 - P(\text{no sixes}) = 1 - \frac{125}{216} = \frac{91}{216}$

(iii) $P(\text{exactly one six}) = P(6, \text{other, other}) + P(\text{other}, 6, \text{other}) + P(\text{other, other}, 6)$

$= \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$

$= \frac{25}{72}$

$P(\text{same number}) = \left(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}\right) \times 6 = \frac{1}{36}$

14. (i) $P(\text{both on Monday}) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

(ii) $P(\text{both on same day}) = 1 \cdot \frac{1}{7} = \frac{1}{7}$

(iii) $P(\text{both on different days}) = 1 \cdot \frac{6}{7} = \frac{6}{7}$

(iv) $P(\text{both on Monday}) + P(\text{Monday, other day}) + P(\text{other day, Monday})$

$= \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{6}{7} + \frac{6}{7} \cdot \frac{1}{7} = \frac{13}{49}$

15. (i) $P(\text{none on a Sunday}) = \frac{6}{7} \cdot \frac{6}{7} \cdot \frac{6}{7} = \frac{216}{343}$

(ii) $P(\text{one on a Sunday})$

$= P(\text{Sunday, other, other}) + P(\text{other, Sunday, other}) + P(\text{other, other, Sunday})$

$= \frac{1}{7} \cdot \frac{6}{7} \cdot \frac{6}{7} + \frac{6}{7} \cdot \frac{1}{7} \cdot \frac{6}{7} + \frac{6}{7} \cdot \frac{6}{7} \cdot \frac{1}{7} = \frac{108}{343}$

(iii) $P(\text{at least one on a Sunday}) = 1 - P(\text{none on a Sunday})$

$= 1 - \frac{216}{343} = \frac{127}{343}$

Exercise 5.7

1. (i) $\#S = 26 \Rightarrow P(\text{Spade}) = \frac{13}{26} = \frac{1}{2}$

(ii) $\#S = 26 \Rightarrow P(\text{Queen}) = \frac{2}{26} = \frac{1}{13}$

(iii) $\#S = 12 \Rightarrow P(\text{King}) = \frac{4}{12} = \frac{1}{3}$

2. (i) $\#S = 90 \Rightarrow P(\text{Person can drive}) = \frac{70}{90} = \frac{7}{9}$

(ii) $\#S = 40 \Rightarrow P(\text{Man can drive}) = \frac{32}{40} = \frac{4}{5}$

(iii) $\#S = 50 \Rightarrow P(\text{Female can drive}) = \frac{38}{50} = \frac{19}{25}$

3. (i) $\frac{2}{12} = \frac{1}{6}$

(ii) $\frac{8}{12} = \frac{2}{3}$

4. (i) $\#S = 120 \Rightarrow P(\text{Ordinary}) = \frac{45}{120} = \frac{3}{8}$

(ii) $\#S = 55 \Rightarrow P(\text{Girl, Higher}) = \frac{35}{55} = \frac{7}{11}$

(iii) $\#S = 45 \Rightarrow P(\text{Boy, Ordinary}) = \frac{25}{45} = \frac{5}{9}$

5. (i) $P(\text{Red}) = \frac{5}{8}$

(ii) $P(\text{Red, Red}) = \frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14}$

(iii) $P(\text{Blue, Blue}) = \frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$

(iv) $P(\text{both same colour}) = \frac{5}{14} + \frac{3}{28} = \frac{10}{28} + \frac{3}{28} = \frac{13}{28}$

6. (i) $P(\text{Red, Red}) = \frac{5}{11} \cdot \frac{4}{10} = \frac{2}{11}$

(ii) $P(\text{Red, Black}) = \frac{5}{11} \cdot \frac{6}{10} = \frac{3}{11}$

(iii) $P(\text{Black, Black}) = \frac{6}{11} \cdot \frac{5}{10} = \frac{3}{11}$

(iv) $P(\text{Same colour}) = \frac{2}{11} + \frac{3}{11} = \frac{5}{11}$

(v) $P(\text{Second is Red}) = P(\text{Red, Red}) + P(\text{Black, Red})$

$$= \frac{5}{11} \cdot \frac{4}{10} + \frac{6}{11} \cdot \frac{5}{10} = \frac{20}{110} + \frac{30}{110} = \frac{2}{11} + \frac{3}{11} = \frac{5}{11}$$

7. (i) $P(T, N) = \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$

(ii) $P(E, V) = \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$

(iii) $P(\text{Second is } E) = P(E, E) + P(V, E) + P(N, E) + P(T, E)$

$$= \frac{2}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{2}{4} + \frac{1}{5} \cdot \frac{2}{4} + \frac{1}{5} \cdot \frac{2}{4} = \frac{8}{20} = \frac{4}{10} = \frac{2}{5}$$

8. $P(\text{Same letters}) = P(I, I) + P(M, M)$

$$= \frac{2}{8} \cdot \frac{1}{7} + \frac{2}{8} \cdot \frac{1}{7} = \frac{2}{56} + \frac{2}{56} = \frac{4}{56} = \frac{1}{14}$$

9. (i) $\#S = 33 \Rightarrow P(\text{girl}) = \dfrac{20}{33}$

(ii) $\#S = 13 \Rightarrow P(\text{boy, left-handed}) = \dfrac{4}{13}$

(iii) $\#S = 13 \text{ boys} \Rightarrow P(\text{left-handed}) = \dfrac{4}{13}$

$\#S = 20 \text{ girls} \Rightarrow P(\text{left-handed}) = \dfrac{5}{20} = \dfrac{1}{4}$

$\Rightarrow P(\text{both left-handed}) = \dfrac{4}{13} \cdot \dfrac{1}{4} = \dfrac{4}{52} = \dfrac{1}{13}$

(iv) $\#S = 24 \Rightarrow P(\text{boy, right-handed}) = \dfrac{9}{24} = \dfrac{3}{8}$

10. (i) $(0.8)(0.6) = 0.48$

(ii) $P(\text{fails in at least one task}) = 1 - P(\text{succeeds in both tasks})$

$= 1 - 0.48 = 0.52$

11. $\left(\dfrac{3}{5} \cdot \dfrac{2}{4}\right) \times 2 = \dfrac{3}{5} \left[\text{OR } P(O, E) + P(E, O) \Rightarrow \dfrac{3}{5} \cdot \dfrac{2}{4} + \dfrac{2}{5} \cdot \dfrac{3}{4} = \dfrac{6}{20} + \dfrac{6}{20} = \dfrac{3}{5}\right]$

12. (i) $P(\text{A}) = 0.4 + 0.2 = 0.6$

(ii) $P(\text{A} \cap \text{B}) = 0.2$

(iii) $P(\text{A} \cup \text{B}) = 0.4 + 0.2 + 0.3 = 0.9$

(iv) $P(\text{A}|\text{B}) = \dfrac{P(\text{A} \cap \text{B})}{P(\text{B})} = \dfrac{0.2}{0.5} = 0.4$

(v) $P(\text{B}|\text{A}) = \dfrac{P(\text{B} \cap \text{A})}{P(\text{A})} = \dfrac{0.2}{0.6} = \dfrac{1}{3}$

13. (i) $P(\text{A}) = \dfrac{8 + 4}{30} = \dfrac{12}{30} = \dfrac{2}{5}$

(ii) $P(\text{A} \cap \text{B}) = \dfrac{4}{30} = \dfrac{2}{15}$

(iii) $P(\text{A} \cup \text{B}) = \dfrac{8 + 4 + 12}{30} = \dfrac{24}{30} = \dfrac{4}{5}$

(iv) $P(\text{A}|\text{B}) = \dfrac{P(\text{A} \cap \text{B})}{P(\text{B})} = \dfrac{4}{16} = \dfrac{1}{4}$

$P(\text{B}|\text{A}) = \dfrac{P(\text{B} \cap \text{A})}{P(\text{A})} = \dfrac{4}{12} = \dfrac{1}{3} \neq \dfrac{1}{4}$

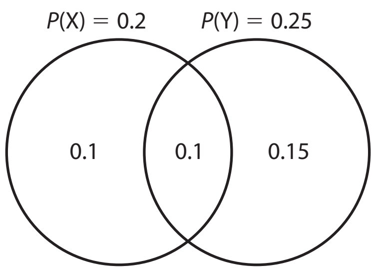
Hence, $P(\text{A}|\text{B}) \neq P(\text{B}|\text{A})$

14. (i) $P(\text{X} \cup \text{Y}) = 0.1 + 0.1 + 0.15$

$= 0.35$

(ii) $P(\text{X}|\text{Y}) = \dfrac{P(\text{X} \cap \text{Y})}{P(\text{Y})} = \dfrac{0.1}{0.25} = 0.4$

(iii) $P(\text{Y}|\text{X}) = \dfrac{P(\text{Y} \cap \text{X})}{P(\text{X})} = \dfrac{0.1}{0.2} = 0.5$



15. (i) $P(\text{A}) = 0.2 + 0.1 = 0.3$

(ii) $P(\text{A} \cup \text{B}) = 0.2 + 0.1 + 0.4 = 0.7$

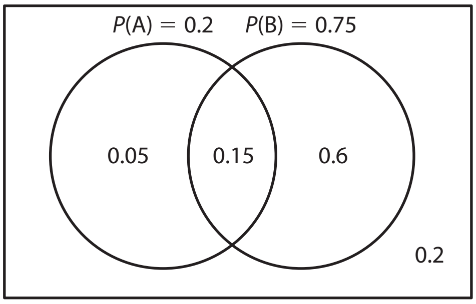
(iii) $P(\text{A}') = 1 - P(\text{A}) = 1 - 0.3 = 0.7$

(iv) $P(\text{A} \cup \text{B})' = 1 - P(\text{A} \cup \text{B}) = 1 - 0.7 = 0.3$

(v) $P(\text{A}' \cap \text{B}) = 0.4$

(vi) $P(\text{B}|\text{A}) = \dfrac{P(\text{B} \cap \text{A})}{P(\text{A})} = \dfrac{0.1}{0.3} = \dfrac{1}{3}$

16. (i) Complete the Venn diagram.



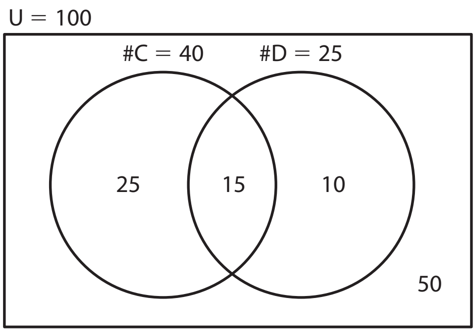
(ii) 0.2

(iii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.15}{0.75} = 0.2$

(iv) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.15}{0.2} = 0.75 \neq 0.2$

Hence, $P(A|B) \neq P(B|A)$

17. (i) $\dfrac{25 + 15 + 10}{100} = \dfrac{50}{100} = 0.5$

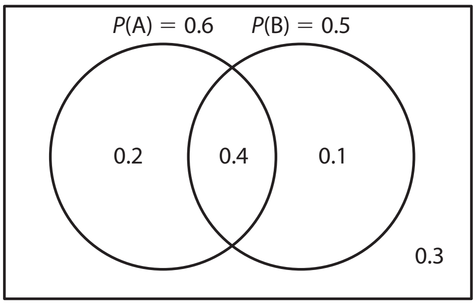


(ii) $\dfrac{25 + 10}{100} = \dfrac{35}{100} = 0.35$

(iii) $P(D|C) = \dfrac{P(D \cap C)}{P(C)} = \dfrac{15}{40} = 0.375$

(iv) $P(C'|D) = \dfrac{P(C' \cap D)}{P(D)} = \dfrac{10}{25} = 0.4$

18. (i) $P(A \cup B) = 0.2 + 0.4 + 0.1 = 0.7$

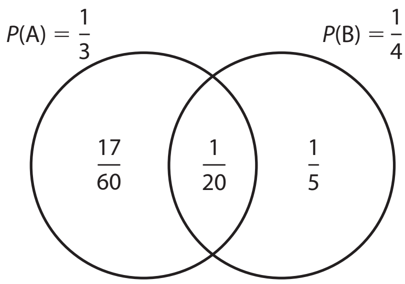


(ii) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.4}{0.6} = \dfrac{2}{3}$

(iii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.4}{0.5} = 0.8$

(iv) $P(B \cap A') = 0.1$

19. (i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$



$\Rightarrow P(A \cap B) = P(A|B) \cdot P(B)$

$= \dfrac{1}{5} \cdot \dfrac{1}{4} = \dfrac{1}{20}$

(ii) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{20}}{\frac{1}{3}} = \dfrac{3}{20}$

20. (i) $P(B) = 0.18 + 0.17 + 0.02 + 0.05 = 0.42$

(ii) $P(A \cap C) = 0.08 + 0.02 = 0.1$

(iii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.2}{0.42} = \dfrac{10}{21}$

(iv) $P(C|B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{0.07}{0.42} = \dfrac{1}{6}$

(v) $P(A \cap C') = 0.3 + 0.18 = 0.48$

(vi) $P[B|(A \cap C)] = \dfrac{P(B \cap A \cap C)}{P(A \cap C)} = \dfrac{0.02}{0.1} = 0.2$

Revision Exercise 5 (Core)

1. $\boxed{5}\boxed{4}\boxed{3} = 60$ three-digit numbers

(i) $\boxed{1}\boxed{4}\boxed{3} = 12$

(ii) $\boxed{3}\boxed{4}\boxed{3} = 36$

2. (i) $\binom{11}{4} = 330$ different groups

(ii) $\binom{\mathbf{5}}{2} \times \binom{\mathbf{6}}{2} = (10)(15) = 150$

3. (i) $\frac{1}{36}$

(ii) $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \times 6 = \frac{6}{36} = \frac{1}{6}$

(iii) $\frac{1}{36} + \frac{6}{36} - \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$

4. (i) $1 - (0.35 + 0.1 + 0.25 + 0.15) = 0.15$

(ii) 1

(iii) $0.25 \times 200 = 50$ times

5. (i) $\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 6! = 720$ arrangements

(ii) $5! \times 2! = 240$

6. (i) (a)

(ii) $P(\text{H or T}) = P(\text{H} \cup \text{T}) = P(\text{H}) + P(\text{T}) = \frac{10}{30} + \frac{12}{30} = \frac{22}{30} = \frac{11}{15}$

7. (i) $P(2) = \frac{2}{6} = \frac{1}{3}$

(ii) $P(\text{2 on first two throws}) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

(iii) $P(\text{first 2 on third throw}) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$

8. $P(\text{blue, red, red or green}) = \frac{6}{13} \cdot \frac{4}{12} \cdot \frac{6}{11} = \frac{144}{1716} = \frac{12}{143}$

9. Event A = {3, 6, 9, 12, 15, 18}
Event B = {4, 8, 12, 16, 20}

(i) $P(\text{A}) = \frac{6}{20} = \frac{3}{10} = 0.3$

(ii) $P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

$= \frac{6}{20} + \frac{5}{20} - \frac{1}{20} = \frac{10}{20} = \frac{1}{2}$

(iii) $P(\text{A} \cap \text{B})' = \frac{19}{20}$

10. (i) $\#S = 25 \quad \Rightarrow \quad P(E) = \frac{6}{25} + \frac{5}{25} = \frac{11}{25}$

(ii) $\#S = 13 \quad \Rightarrow \quad P(E) = \frac{7}{13}$

(iii) $P(E) = \frac{5}{25} \cdot \frac{4}{24} = \frac{20}{600} = \frac{1}{30}$

[Note: Here E = Event]

Revision Exercise 5 (Advanced)

1. (i) $P(\text{6 on first throw}) = \frac{1}{6}$

(ii) $P(\text{first 6 on second throw}) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$

(iii) $P(\text{first 6 on either first or second throw}) = \frac{1}{6} + \frac{5}{36} = \frac{11}{36}$

2. (i) $\binom{8}{4} = 70$ choices

(ii) $\binom{7}{3} = 35$

(iii) A and 6 others, select 3 $= \binom{6}{3} = 20$

B and 6 others, select 3 $= \binom{6}{3} = \underline{20}$

Total $= 40$

3. (i)

7	6	5	4	3	2	1

$= 7! = 5040$ arrangements

(ii)

1	1	5	4	3	2	1

$= 120$

(iii)

4	5	4	3	2	1	3

$= 1440$

4. (i) Score of 6 $\Rightarrow$ $P(2, 2, 2) = \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{9}$

(ii) Score of 9 $\Rightarrow$ $P(3, 3, 3) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{18}$

(iii) Score of 7 $\Rightarrow$ $P(2, 2, 3) + P(2, 3, 2) + P(3, 2, 2)$

$$= \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3}$$
$$= \frac{2}{18} + \frac{4}{18} + \frac{2}{18} = \frac{8}{18} = \frac{4}{9}$$

5. (i) Events L and M cannot happen at the same time.

(ii) (a) $\binom{22}{4} = 7315$ possible selections

(b) 22 students $\Rightarrow$ select Janelle and 3 others $= \binom{21}{3} = 1330$

(c) $P(\text{Janelle included}) = \frac{1330}{7315} = \frac{2}{11}$

6. (i) $P(\text{first 10c, second 5c}) = \frac{4}{6} \cdot \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$

(ii) $P(\text{sum} = 15\text{c}) = P(\text{first 10c, second 5c}) + P(\text{first 5c, second 10c})$

$$= \frac{4}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{4}{5} = \frac{8}{30} + \frac{8}{30} = \frac{16}{30} = \frac{8}{15}$$

(iii) $P(\text{sum} = 20\text{c}) = P(\text{first 10c, second 10c}) = \frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30} = \frac{2}{5}$

7. (i) $0.2 + 0.3 + x + 0.1 = 1$

$\Rightarrow x = 1 - 0.6 = 0.4$

(ii) $P(A) = 0.2 + 0.3 = 0.5$

(iii) $P(A \cup B) = 0.2 + 0.3 + 0.4 = 0.9$

(iv) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.7} = \frac{3}{7}$

(v) $P(A|B) = \frac{3}{7}$ and $\frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.7} = \frac{3}{7}$

Hence, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

8. (i) $P(A) = \frac{20}{35} = \frac{4}{7}$

(ii) $P(B) = \frac{26}{35}$

(iii) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{16}{26} = \frac{8}{13}$

(iv) $P(A \cap B) = \frac{16}{35}$

(v) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{7} + \frac{26}{35} + \frac{16}{35} = \frac{30}{35} = \frac{6}{7}$

$P(B) \cdot P(A|B) = \frac{26}{35} \cdot \frac{8}{13} = \frac{16}{35} = P(A \cap B)$

Events are not mutually exclusive, hence we cannot apply $P(A \cup B) = P(A) + P(B)$.

9. (i) $P(\text{green, green}) = \frac{x}{x+6} \cdot \frac{x-1}{x+5}$

(ii) $\frac{x}{x+6} \cdot \frac{x-1}{x+5} = \frac{4}{13}$

$\Rightarrow 4x^2 + 44x + 120 = 13x^2 - 13x$

$\Rightarrow 9x^2 - 57x - 120 = 0$

$\Rightarrow \quad 3x^2 - 19x - 40 = 0$

$\Rightarrow \quad (3x + 5)(x - 8) = 0$

$\Rightarrow x = -\frac{5}{3}$ or $x = 8$

$\Rightarrow$ valid answer: $x = 8$

$\Rightarrow$ number of discs $= 4 + 2 + 8 = 14$

(iii) $P(\text{not green, not green}) = \frac{6}{14} \cdot \frac{5}{13} = \frac{15}{91}$

10. $P(\text{correct answer}) = \frac{5}{8} + \frac{1}{5} \cdot \frac{3}{8}$

$= \frac{28}{40}$

$= \frac{7}{10}$

Revision Exercise 5 (Extended-Response Questions)

1. (i) $P(\text{red, red, red}) + P(\text{green, green, green})$

$= \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} + \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} = \frac{5}{42} + \frac{1}{21} = \frac{7}{42} = \frac{1}{6}$

(ii) $P(\text{at least one is red}) = 1 - P(\text{none red})$

$= 1 - \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} = 1 - \frac{1}{21} = \frac{20}{21}$

(iii) $P(\text{at most one is green}) = P(G, R, R) + P(R, G, R) + P(R, R, G) + P(R, R, R)$

$= \frac{4}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} + \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} + \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} + \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7}$

$= \frac{25}{42}$

2. (i) $\frac{8}{100} \cdot \frac{1}{10} = 0.8\%$

(ii) $\frac{8}{100} \cdot \frac{9}{10} = 7.2\%$

(iii) $\frac{92}{100} \cdot \frac{1}{10} = 9.2\%$

3. (i) Equally likely outcomes.

(ii) Probability of second event is dependent on the outcome of first.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

(iii) (a) $P(C|D) = \frac{P(C \cap D)}{P(D)} \Rightarrow \frac{P(C \cap D)}{\frac{1}{3}} = \frac{1}{5} \Rightarrow P(C \cap D) = \frac{1}{15}$

(b) $P(C \cup D) = P(C) + P(D) - P(C \cap D)$

$= \frac{8}{15} + \frac{1}{3} - \frac{1}{15} = \frac{12}{15} = \frac{4}{5}$

(c) $P[(C \cup D)'] = 1 - P(C \cup D) = 1 - \frac{4}{5} = \frac{1}{5}$

4. (i) $1 - \frac{1}{5} = \frac{4}{5}$

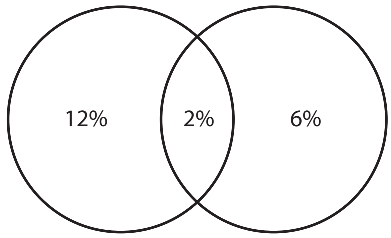
(ii) $P(\text{blue eyes and left-handed}) = \frac{2}{5} \cdot \frac{1}{5} = \frac{2}{25}$

2 people chosen at random $= \binom{2}{1} = 2$

$P(\text{blue eyes and not left-handed}) = \frac{2}{5} \cdot \frac{4}{5} = \frac{8}{25}$

Hence, $P(E) = 2 \cdot \frac{2}{25} \cdot \frac{8}{25} = \frac{32}{625}$

5. (i) Venn diagram



(ii) $P(\text{drive illegally})$

$= 12\% + 2\% + 6\% = 20\% = \frac{1}{5}$

(iii) $300 \times 12\% =$ Roughly 36

6. (i) $\frac{5}{12} \cdot \frac{5}{12} \cdot \frac{5}{12} \cdot \frac{5}{12} = \frac{625}{20\,736} = 0.03014 = 0.030$

(ii) $\left(\frac{7}{12}\right)^4 + \left(\frac{2}{12}\right)^4 = 0.11578 + 0.03014 = 0.14592 = 0.146$

(iii) 0

7. (i) $P(\text{shaded square first throw}) = \frac{2}{6} = \frac{1}{3}$

(ii) $\frac{2}{6} = \frac{1}{3}$

(iii) (a) {3, 4, 5} or {5, 4, 3} or {2, 6, 4} etc (i.e. see below)

(b) {3, 5, 4} or {1, 6, 5} or {2, 5, 5} or {3, 6, 3} or {5, 2, 5} or {5, 3, 4} or {5, 6, 1}

$\Rightarrow$ Total = 10 ways

(iv) (a) 3 throws

(b) $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216}$

8. (i) $P(B) = 0.4 \Rightarrow 0.1 + 0.05 + 0.05 + x = 0.4$

$\Rightarrow 0.2 + x = 0.4$

$\Rightarrow x = 0.2$

$P(C) = 0.35 \Rightarrow 0.05 + 0.05 + 0.05 + y = 0.35$

$\Rightarrow 0.15 + y = 0.35$

$\Rightarrow y = 0.2$

$0.3 + 0.1 + 0.2 + 0.05 + 0.05 + 0.05 + 0.2 + z = 1$

$\Rightarrow 0.95 + z = 1$

$\Rightarrow z = 0.05$

(ii) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.4} = \frac{3}{8}$

(iii) $P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{0.1}{0.35} = \frac{2}{7}$

(iv) $P[(A \cup B)'] = 0.2 + 0.05 = 0.25$

(v) $P(A \cup B \cup C) = 0.95$

$P(A|B) = \frac{3}{8}$ and $\frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.4} = \frac{3}{8}$

9. (i) $P(\text{at least one } 6) = 1 - P(\text{on } 6)$

$$= 1 - \frac{5}{6} \cdot \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36} \quad [= P(A)]$$

(ii) $P(\text{sum is } 8) \Rightarrow \text{possibilities} = \{(2, 6)(6, 2)(3, 5)(5, 3)(4, 4)\} \quad [= P(E)]$

$P(E) = \frac{5}{36}$

(iii) $P(A \cap E) = \frac{2}{36} = \frac{1}{18}$

(iv) $P(A \cup E) = \frac{11}{36} + \frac{5}{36} - \frac{1}{18} = \frac{14}{36} = \frac{7}{18}$

(v) $P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{1}{18}}{\frac{5}{36}} = \frac{2}{5}$

10. (i) $P(E) = 0.6 + (0.4)(0.6) + (0.4)(0.4)(0.6) = 0.936$

(ii) $P(\text{not successful at } 1.70 \text{ m}) = 1 - P(\text{successful at } 1.70 \text{ m})$

$$= 1 - [(0.2) + (0.8)(0.2) + (0.8)(0.8)(0.2)]$$
$$= 1 - 0.488 = 0.512$$

(iii) $1 - 0.936 = 0.064$

(iv) $(0.936) \cdot (0.512) = 0.479232 = 0.479$

11. (i) $n(6) = 360$

$n = 60$

(ii) $P(\text{win}) = \frac{5}{60} = \frac{1}{12}$

(iii) Success: at least one win

Failure: no win (two losses)

$P(\text{loss}) = 1 - \frac{1}{12} = \frac{11}{12}$

$P(\text{Failure}) = \frac{11}{12} \times \frac{11}{12} = \frac{121}{144}$

$P(\text{Success}) = 1 - \frac{121}{144} = \frac{23}{144}$

(iv) (a) Expected number of wins $= 36 \times \frac{1}{12} = 3$

(b) Winnings $= 3 \times €20 = €60$

(c) Cost to play $= 36 \times €2 = €72$

Loss $= €72 - €60 = €12$

(d) Fraction of money kept by charity $= \frac{12}{72} = \frac{1}{6}$

(e) Let €x be the charge for each play. Then

(i) if the player wins, the charity loses $20 - x$, or wins $x - 20$, with a probability of $\frac{1}{12}$,

(ii) if the player loses, the charity wins x, with a probability of $\frac{11}{12}$.

Thus the expected win for the charity is

$$(x - 20)\left(\frac{1}{12}\right) + (x)\left(\frac{11}{12}\right)$$
$$= \frac{x}{12} - \frac{5}{3} + \frac{11x}{12}$$
$$= x - \frac{5}{3}$$

To make €400 on 300 plays, the expected win per play should be

$\frac{400}{300} = \frac{4}{3}$. Thus

$x - \frac{5}{3} = \frac{4}{3}$

$x = 3.$

The charity should charge €3 per play.

Chapter 6

Exercise 6.1

1. $a = 46°$, $\quad b = 180° - 46° = 134°$
$c = 80°$, $\quad d = 180° - 80° = 100°$
$e = 180° - 115° = 65°$

2. $a = 71° + 40° = 111°$, $\quad b = 32° + 42° = 74°$
$\frac{180° - 44°}{2} = \frac{136°}{2} = 68° \Rightarrow c = 68° + 44° = 112°$
$\frac{180° - 50°}{2} = \frac{130°}{2} = 65° = d$, $\quad e = 180° - (70° + 70°) = 40°$
$f + f = 105° \Rightarrow 2f = 105° \Rightarrow f = 52\frac{1°}{2}$
$g + g = 180° - 60° \Rightarrow 2g = 120° \Rightarrow g = 60°$
$h + h = 60° \quad \Rightarrow 2h = 60° \Rightarrow h = 30°$

3. $a + a = 124° \Rightarrow 2a = 124° \Rightarrow a = 62°$
$b + 35° + 35° = 180° \Rightarrow b = 180° - 70° = 110°$
$c + 55° = 110° \Rightarrow c = 110° - 55° = 55°$
Exterior angle of $\triangle$ = sum of 2 interior opposite angles $\Rightarrow 30° + 43° = 73°$.
Hence, $d + 73° + 73° = 180° \Rightarrow d = 180° - 146° = 34°$

4. (i) $x = 55°, y = 180° - (55° + 80°) = 45°$
(ii) $x + 32° + 32° = 180° \Rightarrow x = 180° - 64° = 116°$
$180° - 116° = 64° \quad \Rightarrow y + 64° + 64° = 180°$
$\Rightarrow y = 180° - 128° = 52°$
(iii) $x + 50° + 50° = 180° \Rightarrow x = 180° - 100° = 80°$
$y + 50° = 80° \Rightarrow y = 80° - 50° = 30°$

5. (i) Horizontal line $= y \quad \Rightarrow y^2 + (3)^2 = (7)^2$
$\Rightarrow y^2 + 9 = 49$
$\Rightarrow y^2 = 49 - 9 = 40$
$\Rightarrow y = \sqrt{40}$

Hence, $x^2 = (6)^2 + (\sqrt{40})^2$
$= 36 + 40 = 76$
$\Rightarrow x = \sqrt{76} = 2\sqrt{19}$

(ii) Vertical line $= y \Rightarrow y^2 + (3)^2 = (5)^2$
$\Rightarrow y^2 + 9 = 25$
$\Rightarrow y^2 = 25 - 9 = 16$
$\Rightarrow y = \sqrt{16} = 4$

Hence, $x^2 = (4)^2 + (4)^2$
$\Rightarrow x^2 = 16 + 16 = 32$
$\Rightarrow x = \sqrt{32} = 4\sqrt{2}$

(iii) Vertical line $= y \quad \Rightarrow y^2 + (6)^2 = (8)^2$
$\Rightarrow y^2 + 36 = 64$
$\Rightarrow y^2 = 64 - 36 = 28$
$\Rightarrow y = \sqrt{28}$

Hence, $x^2 = (\sqrt{28})^2 + (4)^2$
$= 28 + 16$
$= 44$
$\Rightarrow x = \sqrt{44} = 2\sqrt{11}$

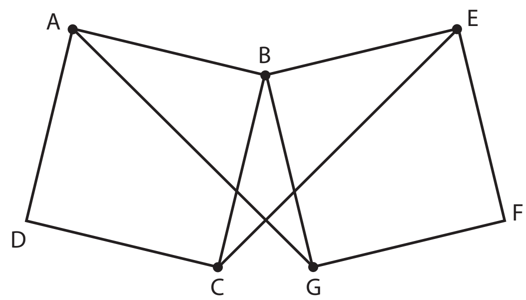
6. $x^2 + (\sqrt{44})^2 = (12)^2$
$\Rightarrow x^2 + 44 = 144$
$\Rightarrow x^2 = 144 - 44 = 100$
$\Rightarrow x = \sqrt{100} = 10$
Hence, $y^2 + (6)^2 = (10)^2$
$\Rightarrow y^2 + 36 = 100$
$\Rightarrow y^2 = 100 - 36 = 64$
$\Rightarrow y = \sqrt{64} = 8$

7. $|AC| = |AD| \Rightarrow |\angle ACD| = |\angle ADC|$
$\Rightarrow 180° - |\angle ACD| = 180° - |\angle ADC|$
$\Rightarrow |\angle ACB| = |\angle ADE|$
$|BD| = |CE| \Rightarrow |BC| + |CD| = |CD| + |DE|$
$\Rightarrow |BC| = |DE|$
In $\triangle$s ABC, ADE
$|AC| = |AD|$
$|\angle ACB| = |\angle ADE|$
$|BC| = |DE|$
$\Rightarrow \triangle$s are congruent by S.A.S.

8. (i) Alternate angles
(ii) In $\triangle$s AMD, MBP
$|\angle DAM| = |\angle MBP|$
$|AM| = |MB|$
$|\angle AMD| = |\angle BMP|$
$\Rightarrow \triangle$s are congruent by A.S.A.
(iii) $\triangle$s AMD, MBP are congruent $\Rightarrow |AD| = |PB|$
ABCD is a parallelogram $\Rightarrow |AD| = |BC|$
$\Rightarrow |PB| = |BC|$
$\Rightarrow$ B is the midpoint of [CP]

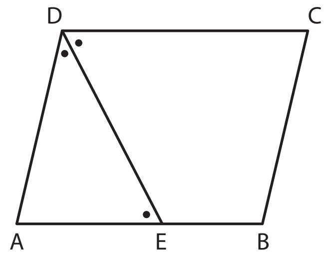
9. (i) Lengths of sides may be different
(ii) In $\triangle$s PTQ, STR
$|\angle TPQ| = |\angle TSR|$
$|PQ| = |RS|$
$|\angle PQT| = |\angle TRS|$
$\Rightarrow \triangle$s are congruent by A.S.A.

10. (i) $90° + |\angle CBG| = 90° + |\angle CBG|$
$\Rightarrow |\angle ABG| = |\angle CBE|$
(ii) In $\triangle$s ABG, CBE
$|AB| = |BE|$
$|\angle ABG| = |\angle CBE|$
$|BG| = |BC|$
$\Rightarrow \triangle$s are congruent by S.A.S.
$\Rightarrow |AG| = |CE|$



11. $|ED|^2 = (4)^2 + (8)^2$
$= 16 + 64$
$= 80$
$\Rightarrow |ED| = \sqrt{80}$
$|HD|^2 = |HE|^2 + |ED|^2 = (5)^2 + (\sqrt{80})^2$
$= 25 + 80$
$= 105$
$\Rightarrow |HD| = \sqrt{105}$

12. $|\angle ADE| = |\angle EDC|$
$|\angle DEA| = |\angle EDC|$
$\Rightarrow |\angle ADE| = |\angle DEA|$
$\Rightarrow |AD| = |AE|$
ABCD is a parallelogram $\Rightarrow |AD| = |BC|$
$\Rightarrow |AE| = |BC|$



13. As $|CX| = |DX|$, triangle XCD is isosceles.
Hence $|\angle BCF| = |\angle ADE|$.
Also
$|CE| = |FD|$ ... given
$|CE| + |EF| = |EF| + |FD|$
$|CF| = |ED|$
Also
$|\angle XAB| = |\angle EDA|$ ... alternate angles
$= |\angle FCB|$ ... from above
$= |\angle XBA|$ ... alternate angles
Thus $\triangle XAB$ is isosceles.
$|XB| = |XA|$
$|CX| + |XB| = |DX| + |XA|$ ... given
$|CB| = |DA|$
Hence $\triangle BCF$ and ΔADE are congruent, by S.A.S.

Exercise 6.2

1. (i) Area $\triangle ABC = \frac{1}{2}(9)(8) = 36\text{ cm}^2$

(ii) Area $\triangle ABC = \frac{1}{2}(12)(h) = 36$
$\Rightarrow 6h = 36$
$\Rightarrow h = \frac{36}{6} = 6\text{ cm}$

2. (i) Area $\triangle = \frac{1}{2}(6)(8) = 24$

Area $\triangle = \frac{1}{2}(10)(h) = 24$
$\Rightarrow 5h = 24$
$\Rightarrow h = \frac{24}{5} = 4.8\text{ cm}$

(ii) Area $\triangle = \frac{1}{2}(12)(14) = 84$

Area $\triangle\ \frac{1}{2} = (16)(h) = 84$
$\Rightarrow 8h = 84$
$\Rightarrow h = \frac{84}{8} = 10.5\text{ cm}$

(iii) Area $\triangle = \frac{1}{2}(24)(18) = 216$

Area $\triangle = \frac{1}{2}(20)(h) = 216$

$\Rightarrow 10h = 216$

$\Rightarrow h = \frac{216}{10} = 21.6$ cm

3. (i) Area parallelogram $= (12)(8) = 96$ cm^2

(ii) Area parallelogram $= (14)(9) = 126$ cm^2

(iii) Area parallelogram $= (13)(11) = 143$ cm^2

4. Area ABCD $= (22)(14) = 308$ cm^2

Area ABCD $= |BC| \cdot (18) = 308$

$\Rightarrow |BC| = \frac{308}{18} = 17\frac{1}{9}$ cm

5. (i) Largest is $\angle BAC$; Smallest is $\angle ACB$

(ii) $|AC| > 5$ cm and <15 cm

6. (i) Area ABCD $= 80$

$\Rightarrow$ Area triangle ADC $= 40$

$\Rightarrow \frac{1}{2}|AC| \cdot |DE| = 40$

$\Rightarrow \frac{1}{2}(16)|DE| = 40$

$\Rightarrow 8|DE| = 40$

$\Rightarrow |DE| = \frac{40}{8} = 5$ cm

(ii) Area $\triangle ADC =$ Area $\triangle ABC$

$\Rightarrow \frac{1}{2}|AC| \cdot |DE| = \frac{1}{2}|AC| \cdot |BF|$

$\Rightarrow |DE| = |BF|$

(iii) Area ABCD $= |AB| \cdot h = 80$

$\Rightarrow (10) \cdot h = 80$

$\Rightarrow h = \frac{80}{10} = 8$ cm

7. (i) Area ABCD $= 2$ Area $\triangle DCB = 2(15) = 30$ cm^2

(ii) Area ADBE $= 2(15) = 30$ cm^2

(iii) Area ADCE $= 3(15) = 45$ cm^2

(iv) Area ABCD $= |DC| \cdot h = 30$

$\Rightarrow (7.5)h = 30$

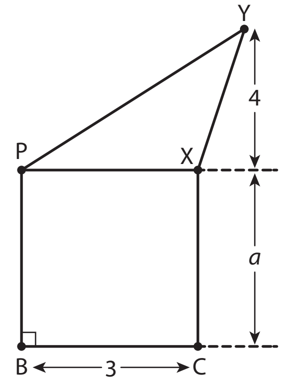
$\Rightarrow h = \frac{30}{7.5} = 4$ cm

8. (i) [AF] bisects area ABFE.

Since area $\triangle AFE = 30$ sq.units $\Rightarrow$ Area $\triangle AFB = 30$ sq.units

(ii) Area ABCDE $= 30 + 30 + 30 + 60 = 150$ sq.units

9. (i) Area $\triangle PXY = \frac{1}{2}|PX| \cdot h = \frac{1}{2}(3)(4) = 6$



Area Rectangle BPXC $= (3)(a) = 3a$

$\Rightarrow$ Total area $= 3a + 6 = 3(a + 2)$

Shaded area $=$ Area $\triangle DEG -$ Area $\triangle EFG$

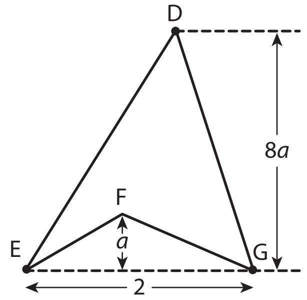
$= \frac{1}{2}(2)(8a) - \frac{1}{2}(2)(a)$

$= 8a - a = 7a$

(ii) Hence, $7a = 3a + 6$

$\Rightarrow 4a = 6$

$\Rightarrow a = \frac{6}{4} = 1.5$



10. (i) Area ABCD + Area $\triangle$ADE

$= (5)(2) + \frac{1}{2}(5)(4x - 2)$

$= 10 + 10x - 5$

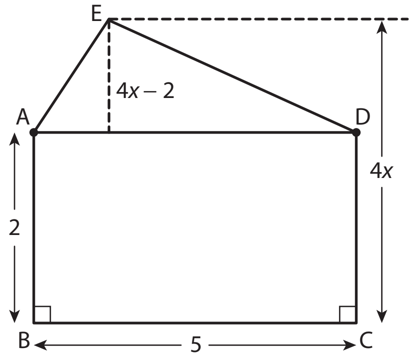
$= 10x + 5 = 5(2x + 1)$

Area EFGH + Area HGQP

$= (3)(x) + (3)(3x)$

$= 3x + 9x$

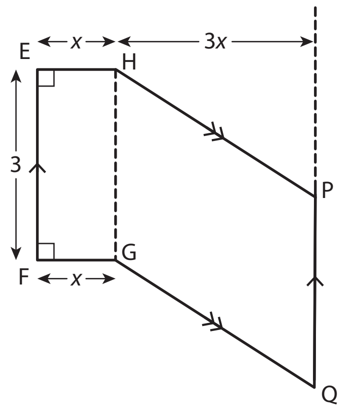
$= 12x$



(ii) Hence, $12x = 10x + 5$

$\Rightarrow 2x = 5$

$\Rightarrow x = \frac{5}{2} = 2.5$

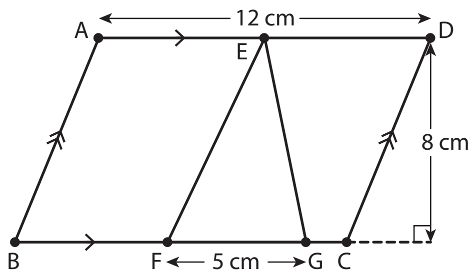


11. Shaded area = Area ABCD − Area $\triangle$EFG

$= (12)(8) - \frac{1}{2}(5)(8)$

$= 96 - 20$

$= 76\ \text{cm}^2$



12. (i) AFED is a parallelogram and diagonals [AE] and [DF] bisect each other.

In $\triangle$s AXD, FXE;

$|AX| = |XE|$

$|\angle AXD| = |\angle FXE|$

$|AX| = |XF|$

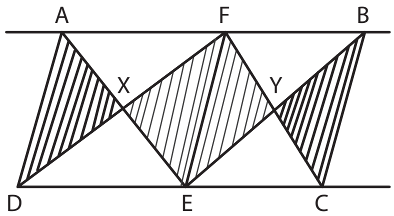
$\Rightarrow$ $\triangle$s are congruent by S.A.S.

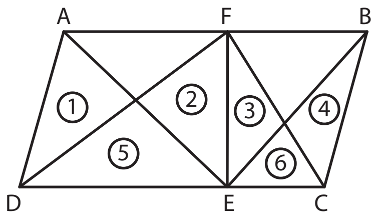
$\Rightarrow$ $\triangle$s AXD, FXE are equal in area.

Similarly, $\triangle$s FYE, BYC are equal in area.

Hence, Area $\triangle$AXD + Area $\triangle$BYC = Area $\triangle$FXE + Area $\triangle$FYE

ie, Blue shaded Area = Yellow shaded Area





(ii) AB∥DC ⇒ Area △ADE = Area △FDE

Hence, Area Triangles ① + ⑤ = Area △s ② + ⑤

⇒ Area △ ① = Area △ ②

Similarly, Area △FEC = Area △BEC

Hence, Area △s ③ + ⑥ = Area △s ④ + ⑥

⇒ Area △③ = Area △④

Hence, Area △s ① + ④ = Area △s ② + ③

ie, Blue shaded Area = Yellow shaded Area

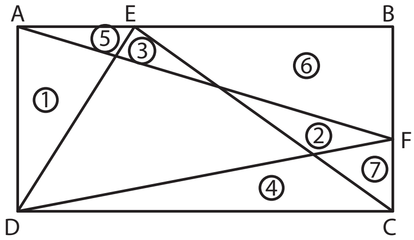
13. (i) Area △AFD = $\frac{1}{2}$ Area ABCD

Area △DEC = $\frac{1}{2}$ Area ABCD

⇒ Area △AFD = $\frac{1}{2}$ Area △DEC

ie., Area △s ① + ② + shaded green = Area △s ③ + ④ + shaded green

⇒ Area △s ① + ② = Area △s ③ + ④



(ii) Area △AFD = $\frac{1}{2}$ Area ABCD

Hence, remainder = $\frac{1}{2}$ Area ABCD

ie, Area △s ABF + FCD = $\frac{1}{2}$ Area ABCD

Hence, Area △AFD = Area △ABF + Area △FCD

⇒ Area △s ① + ② + shaded green = Area △s ⑤ + ③ + ⑥ + ④ + ⑦

⇒ shaded green = Area △s ⑤ + ⑥ + ⑦

ie, Region shaded in green = Region shaded in blue

Exercise 6.3

1. (i) $\frac{6}{3} = \frac{5}{x}$ $\Rightarrow 6x = 15$

$\Rightarrow x = \frac{15}{6} = 2.5$

(ii) $\frac{6}{4} = \frac{x}{6}$ $\Rightarrow 4x = 36$

$\Rightarrow x = \frac{36}{4} = 9$

(iii) $\frac{12}{5} = \frac{14}{x}$ $\Rightarrow 12x = 70$

$\Rightarrow x = \frac{70}{12} = 5\frac{5}{6}$

2. $\frac{5}{7} = \frac{6}{x} \Rightarrow 5x = 42$

$\Rightarrow x = \frac{42}{5} = 8.4$

$\frac{5}{2} = \frac{7}{y}$

$\Rightarrow 5y = 14$

$\Rightarrow y = 2.8$

$\frac{a}{4} = \frac{6}{4} \Rightarrow 4a = 24$

$\Rightarrow a = \frac{24}{4} = 6$

$\frac{7}{10} = \frac{b}{12}$ $\Rightarrow 10b = 84$

$\Rightarrow b = \frac{84}{10} = 8.4$

3. $\frac{|RS|}{|RX|} = \frac{|RT|}{|RY|} \Rightarrow \frac{12}{5} = \frac{|RT|}{6}$

$\Rightarrow 5|RT| = 72$

$\Rightarrow \quad |RT| = \frac{72}{5} = 14.4$

4. $\frac{|AX|}{|AB|} = \frac{|AY|}{|AC|} \Rightarrow \frac{3}{5} = \frac{|AY|}{8}$

$\Rightarrow 5|AY| = 24$

$\Rightarrow \quad |AY| = \frac{24}{5} = 4.8$

5. $\frac{|BC|}{|PQ|} = \frac{|AC|}{|AQ|} \Rightarrow \frac{|BC|}{3} = \frac{7}{5}$

$\Rightarrow 5|BC| = 21$

$\Rightarrow \quad |BC| = \frac{21}{5} = 4.2$

$\frac{|BP|}{|PA|} = \frac{|CQ|}{|QA|} \Rightarrow \frac{|BP|}{4} = \frac{2}{5}$

$\Rightarrow 5|BP| = 8$

$\Rightarrow \quad |BP| = \frac{8}{5} = 1.6$

6. (i) $\frac{|AY|}{|YC|} = \frac{|AX|}{|XB|} \Rightarrow \frac{|AY|}{10} = \frac{2}{1}$

$\Rightarrow |AY| = 20$ cm

(ii) $\frac{|XY|}{|BC|} = \frac{|AX|}{|AB|} \Rightarrow \frac{|XY|}{|BC|} = \frac{2}{3}$

$\Rightarrow |XY| : |BC| = 2 : 3$

(iii) $\frac{|XY|}{|BC|} = \frac{|AX|}{|AB|} \Rightarrow \frac{|XY|}{30} = \frac{2}{3}$

$\Rightarrow 3|XY| = 60$

$\Rightarrow \quad |XY| = \frac{60}{3} = 20$ cm

7. (i) $\frac{|DE|}{|EF|} = \frac{|GH|}{|HI|} \Rightarrow \frac{1}{1} = \frac{8}{|HI|}$

$\Rightarrow |HI| = 8$ cm

(ii) $\frac{|DE|}{|EF|} = \frac{|GJ|}{|JK|} \Rightarrow \frac{1}{1} = \frac{|GJ|}{7}$

$\Rightarrow |GJ| = 7$ cm

8. $\frac{4}{x} = \frac{|AB|}{3} \quad \Rightarrow x \cdot |AB| = 12$

$\Rightarrow \quad |AB| = \frac{12}{x}$

9. (i) $\triangle$s ABC, DEF are equiangular

because
- $|\angle BAC| = |\angle EDF|$
- $|\angle ABC| = |\angle DEF|$
- $|\angle ACB| = |\angle DFE|$.

Hence, $\triangle$s ABC, DEF are similar.

(ii) [DF]

(iii) $\frac{3.5}{8} = \frac{6}{x}$ $\Rightarrow (3.5)\,x = 48$

$\Rightarrow \quad x = \frac{48}{3.5} = \frac{96}{7}$

$\frac{3.5}{8} = \frac{4}{y}$ $\Rightarrow (3.5)\,y = 32$

$\Rightarrow \quad y = \frac{32}{3.5} = \frac{64}{7}$

10. $\frac{x}{6} = \frac{6}{8} \Rightarrow 8x = 36$

$\Rightarrow \quad x = \frac{36}{8} = 4.5$

$\frac{3}{y} = \frac{6}{8} \Rightarrow 6y = 24$

$\Rightarrow y = \frac{24}{6} = 4$

11. (i) [XY] corresponds to [AB] because $|\angle YZX| = |\angle BCA|$.

(ii) $\frac{x}{6} = \frac{15}{10} \Rightarrow 10x = 90$

$\Rightarrow \quad x = \frac{90}{10} = 9$

$\frac{y}{9} = \frac{15}{10} \Rightarrow 10y = 135$

$\Rightarrow \quad y = \frac{135}{10} = 13.5$

12. (i) $\frac{y}{6} = \frac{15}{9} \Rightarrow 9y = 90$

$\Rightarrow \quad y = \frac{90}{9} = 10$

$\frac{x}{10} = \frac{15}{9} \Rightarrow 9x = 150$

$\Rightarrow \quad x = \frac{150}{9} = 16\frac{2}{3}$

(ii) $\frac{x}{12} = \frac{4}{6}$

$\Rightarrow \quad 6x = 48 \quad \Rightarrow x = 8$

$\frac{y}{x} = \frac{4}{6}$

$\Rightarrow \frac{y}{8} = \frac{4}{6} \ldots (x = 8)$

$\Rightarrow \quad 6y = 32$

$\Rightarrow \quad y = 5\frac{2}{6} = 5\frac{1}{3}$

13. (i) $\frac{|CD|}{6} = \frac{12}{8} \Rightarrow 8|CD| = 72$

$\Rightarrow |CD| = \frac{72}{8} = 9$

(ii) $\frac{|AD|}{12} = \frac{12}{8} \Rightarrow 8|AD| = 144$

$\Rightarrow |AD| = \frac{144}{8} = 18$

14. (i) $\triangle$s DAB, DBC are equiangular because

- $|\angle DAB| = |\angle DBC|$
- $|\angle ABD| = |\angle CDB|$

Hence, $\triangle$s DAB, DBC are similar.

(ii) $\frac{4}{|DB|} = \frac{|DB|}{9} \Rightarrow |DB|^2 = 36$

$\Rightarrow |DB| = \sqrt{36} = 6$

15. $\triangle$s ABD, ACD are equiangular

because (i) $|\angle ADB| = |\angle ADC|$

(ii) $|\angle ABD| = |\angle CAD|$

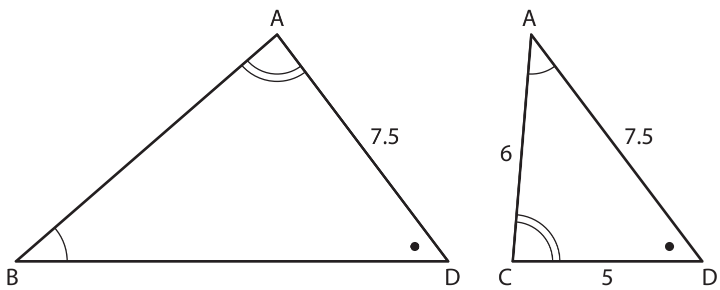
Hence, $|\angle BAD| = |\angle ACD|$

$\Rightarrow \triangle$s ABD, ACD are similar.

$\frac{|BD|}{|AD|} = \frac{|AD|}{|CD|} \Rightarrow \frac{|BD|}{7.5} = \frac{7.5}{5}$

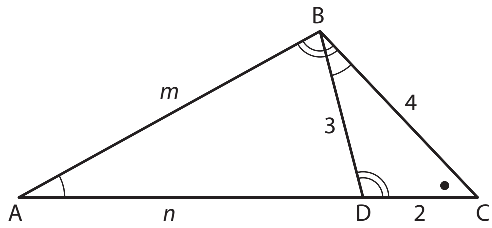
$\Rightarrow 5|BD| = 56.25$

$\Rightarrow \quad |BD| = \frac{56.25}{5} = 11.25$



$$\frac{|AB|}{|AC|} = \frac{|AD|}{|CD|} \Rightarrow \frac{|AB|}{6} = \frac{7.5}{5}$$
$$\Rightarrow 5|AB| = 45$$
$$\Rightarrow |AB| = \frac{45}{5} = 9$$

16. (i) $\triangle ABC$ and $\triangle DBC$



(ii) $\frac{|AB|}{|BD|} = \frac{|BC|}{|DC|} \Rightarrow \frac{m}{3} = \frac{4}{2}$
$$\Rightarrow 2m = 12$$
$$\Rightarrow m = 6$$
$$\frac{|AC|}{|BC|} = \frac{|BC|}{|DC|} \Rightarrow \frac{n+2}{4} = \frac{4}{2}$$
$$\Rightarrow 2n + 4 = 16$$
$$\Rightarrow 2n = 12$$
$$\Rightarrow n = 6$$

17. $\frac{3}{6} = \frac{4}{3+x}$
$$\Rightarrow 9 + 3x = 24$$
$$\Rightarrow 3x = 15$$
$$\Rightarrow x = 5$$

18. (i) $\triangle WXZ$

(ii) $\frac{|XZ|}{|WX|} = \frac{|WX|}{|XY|} \Rightarrow \frac{v+3}{5} = \frac{5}{3}$
$$\Rightarrow 3v + 9 = 25$$
$$\Rightarrow 3v = 16$$
$$\Rightarrow v = \frac{16}{3} = 5\frac{1}{3}$$
$$w^2 = \left(5\frac{1}{3}\right)^2 + (4)^2$$
$$= \frac{256}{9} + 16 = \frac{400}{9}$$
$$\Rightarrow w = \sqrt{\frac{400}{9}} = \frac{20}{3} = 6\frac{2}{3}$$

19. $\frac{x}{1} = \frac{1}{x+1}$
$$\Rightarrow x^2 + x = 1$$
$$\Rightarrow x^2 + x - 1 = 0$$
$$\Rightarrow x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$
$$= \frac{-1 + 2.236}{2}$$
$$= 0.618$$

Hence, ratio $= \frac{1}{0.618 + 1} = \frac{1}{1.618}$ or 1 : 1.618

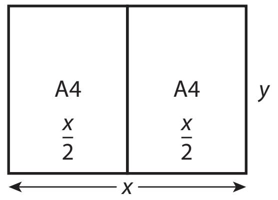
20.

$$\frac{\frac{x}{2}}{y} = \frac{y}{x}$$

$$\Rightarrow y^2 = \frac{x^2}{2}$$

$$\Rightarrow x^2 = 2y^2$$

$$\Rightarrow \quad x = \sqrt{2y^2} = \sqrt{2}y \Rightarrow \frac{x}{y} = \frac{\sqrt{2}}{1}$$



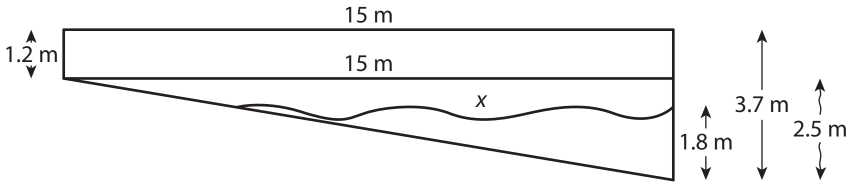
21.

$$\frac{15}{2.5} = \frac{x}{1.8}$$

$$\Rightarrow 2.5x = 27$$

$$\Rightarrow \quad x = \frac{27}{2.5}$$

$$\Rightarrow \quad x = 10.8 \text{ m}$$



Exercise 6.4

1. (i) $\angle A = 2(48°) = 96°$

$\angle B = 2(44°) = 88°$ and $\angle c = 180° - 44° = 136°$

$\angle D = 2(42°) = 84°$ and $\angle E = \frac{1}{2}(180° - 84°) = 48°$

2. $\angle A = 52°, \angle B = 44°, \angle c = 45°, \angle D = 20°$

3. $\angle A = 47°, \angle B = 2(47°) = 94°, \angle c = \frac{1}{2}(180° - 94°) = 43°$

4. $\angle f = 40°, \angle g = 180° - (85° + 40°) = 55°, \angle h = \angle g = 55°$

5. $\angle A = 42°, \angle B = 90° - 42° = 48°$

$50° + 50° = 100° \Rightarrow 180° - 100° = 80°$

$\Rightarrow \angle c = \frac{1}{2}(80°) = 40°$

$\angle D = \frac{1}{2}(180° - 70°) = 55°$

$\angle E = 90° - \angle D = 90° - 55° = 35°$

6. $\angle A = 180° - 85° = 95°$

$\angle B = 180° - 105° = 75°$

$180° - 86° = 94° \Rightarrow 2\angle c = 180° - 94° = 86°$

$\Rightarrow \angle c = \frac{1}{2}(86°) = 43°$

$\angle D = 180° - (32° + 32°) = 116°$

$\angle E = 180° - 116° = 64°$

7. $360° - (108° + 120°) = 132°$

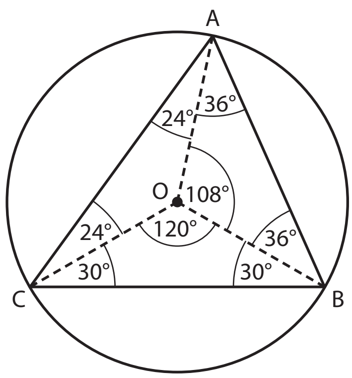
$180° - 132° = 48°$

$\Rightarrow 48° \div 2 = 24°$

$\Rightarrow \angle A = 24° + 36° = 60°$

$\angle B = 36° + 30° = 66°$

$\angle C = 24° + 30° = 54°$



8. (i) $|\angle OST| = 90°$

(ii) $|\angle OSP| = 90° - 40° = 50°$

(iii) $|\angle OPS| = |\angle OSP| = 50°$

(iv) $|\angle SOP| = 180° - (50° + 50°) = 80°$

9. $\angle a = 55°$

$\angle b = 90°, \angle c = 180° - (90° + 41°) = 49°$

$\angle d = 90° - 41° = 49°$

$\angle e = 90° - 23° = 67°$

10. (i) $|\angle BAC| = 62°$
(ii) $|\angle ABD| = 44°$

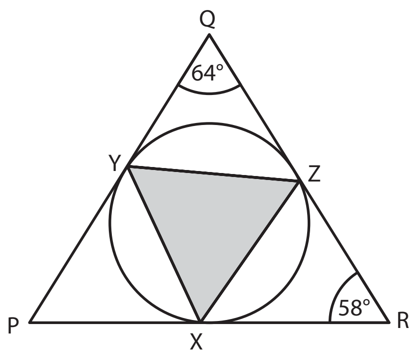
11. (i) In $\triangle$s AOP, BOP
$|OA| = |OB|$
$|\angle OAP| = |\angle OBP|$
$|OP| = |OP|$
$\Rightarrow$ $\triangle$s are congruent by R.H.S.
(ii) Since $\triangle$s AOP, BOP are congruent
$\Rightarrow$ $|PA| = |PB|$
(iii) $|\angle APB| + |\angle AOB| = 360° - (90° + 90°)$
$= 180°$

12. $|\angle AOT| = 180° - 40° = 140°$
$\Rightarrow$ $|\angle ATO| + |\angle OAT| = 2|\angle ATO| = 40°$
$= |\angle ATO| = \dfrac{40°}{2} = 20°$

13. (i) $|\angle ABC| = 180° - 34° = 146°$
(ii) $|\angle BAC| + |\angle BCA| = 180° - 146°$
$\Rightarrow 2|\angle BAC| = 34°$ .... $|\angle BAC| = |\angle BCA|$ as $\triangle ABC$ is isosceles
$\Rightarrow$ $|\angle BAC| = \dfrac{34°}{2} = 17°$

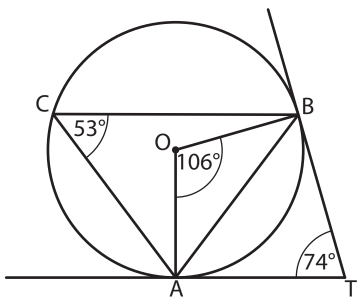
14. $\angle x + \angle x + \angle y + \angle y = 180°$
$\Rightarrow$ $2(\angle x + \angle y) = 180°$
$\Rightarrow$ $\angle x + \angle y = \dfrac{180°}{2} = 90°$

15. (i) $|PX| = |PY| \Rightarrow \triangle PXY$ is isosceles
$|QY| = |QZ| \Rightarrow \triangle QYZ$ is isosceles
$|RX| = |RZ| \Rightarrow \triangle RXZ$ is isosceles



(ii) $\angle XPY + 64° + 58° = 180°$
$\therefore\ \angle XPY = 180° - 64° - 58° = 58°$
$\triangle PXY$ is isosceles $\Rightarrow \angle PXY = \angle PYX$
$\therefore\ \angle XPY + \angle PXY + \angle PYX = 180°$
$58° + \angle PXY + \angle PXY = 180°$
$\angle PXY = \dfrac{180 - 58}{2} = 61°$
(iii) $\angle ZXR = \dfrac{180 - 58}{2} = 61° = \angle XZR$
$\angle YZQ = \dfrac{180 - 64}{2} = 58° = \angle ZYQ$
$\angle YXZ = 180° - 61° - 61° = 58°$
$\angle XZY = 180° - 58° - 61° = 61°$
$\angle XYZ = 180° - 61° - 58° = 61°$

16. (i) $|\angle AOB| = 2(53°) = 106°$
(ii) $|\angle BTA| = 180° - 106° = 74°$
(iii) $|\angle ABT| + |\angle BAT| = 180° - 74°$
$\Rightarrow 2|\angle ABT| = 106°$
$\Rightarrow$ $|\angle ABT| = \dfrac{106°}{2} = 53°$



17. (i) $|\angle DBA| + |\angle BDA| = 180° - 84°$

$\Rightarrow \quad 2|\angle DBA| = 96°$

$\Rightarrow \quad |\angle DBA| = \frac{96°}{2} = 48°$

(ii) $|\angle BCA| = |\angle BDA| = 48°$

18. In $\triangle$s ABE, ECD;

$|\angle BAE| = |\angle EDC|$

$|\angle ABE| = |\angle ECD|$

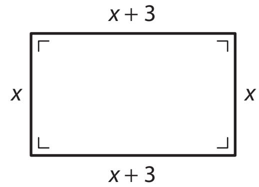
$|\angle AEB| = |\angle CED|$

$\Rightarrow \triangle$s are equiangular

$\Rightarrow \triangle$s are similar

Revision Exercise 6 (Core)

1. Let x m be the width. Then the length is $(x + 3)$ m.



The perimeter is

$2(x) + 2(x + 3)$

$= 4x + 6$

The area is

$(x)(x + 3)$

$= x^2 + 3x$

Given: $x^2 + 3x = 4x + 6$

$x^2 - x - 6 = 0$

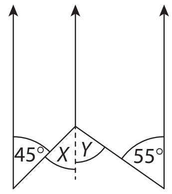
$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$ (not possible)

Thus the width is 3 m and the length is 6 m.

2. $X = 45°$ ... alternate angles

$Y = 55°$ ... alternate angles

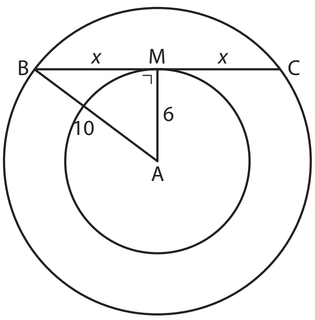


Thus

$\alpha° = 45° + 55°$

$\alpha° = 100°$

3. If $|BM| = x$ cm, then $|MC| = x$ cm.



By Pythagoras,

$x^2 + 6^2 = 10^2$

$x^2 = 64$

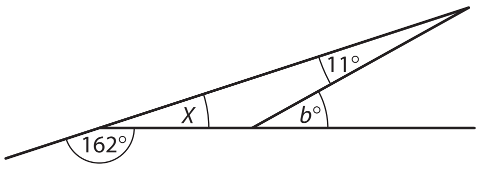
$x = 8$

Thus

$|BC| = 2(8) = 16$ cm.

4. $|\angle X| = 180° - 162°$

$= 18°$,



by a straight angle.

Then

$b° = X + 11°$

by exterior angle theorem.

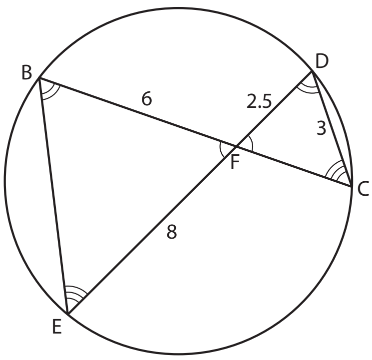
Then

$b° = 18° + 11°$

$b° = 29°$

5. $|\angle BFE| = |\angle DFC|$ ... vertically opposite angles

$|\angle EBF| = |\angle CDF|$ ... angles standing on the same arc



Hence $\triangle$BEF and $\triangle$DCF are similar, and

$|\angle BEF| = |\angle DCF|$.

Then

$$\frac{|BE|}{|DC|} = \frac{|EF|}{|FC|} = \frac{|BF|}{|FD|}.$$

(i) $\frac{|BE|}{3} = \frac{6}{2.5}$

$|BE| = 7.2$ cm

(ii) $\frac{8}{|FC|} = \frac{6}{2.5}$

$|FC| = \frac{20}{6} = \frac{10}{3}$ cm

6. (i) As the triangle is isosceles,

$3x - 8 = 2x - 2$

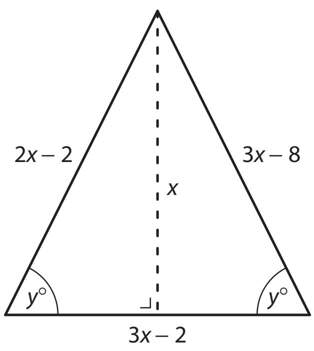
$x = 6$

(ii) The area of the triangle is

$\frac{1}{2}(3x - 2)(x)$

$= \frac{1}{2}(16)(6)$

$= 48.$



7. $r = 90° - x°$

$p = \frac{180° - x°}{2} = 90° - \frac{x°}{2}$

$q = 180° - p$

$q = 180° - \left(90° - \frac{x°}{2}\right)$

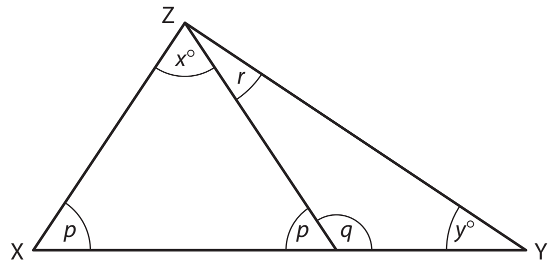
$q = 90° + \frac{x°}{2}$

Then

$y = 180° - q - r$

$y = 180° - \left(90° + \frac{x°}{2}\right) - (90° - x°)$

$y = \frac{x}{2}$



8. $\triangle$ADE and $\triangle$ACB are similar.

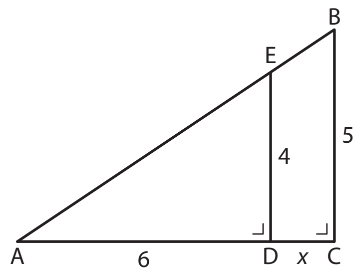
Then

$\frac{6 + x}{6} = \frac{5}{4}$

$24 + 4x = 30$

$4x = 6$

$x = 1.5$



9. By Pythagoras,

$|AC|^2 = 8^2 + 6^2 = 100$

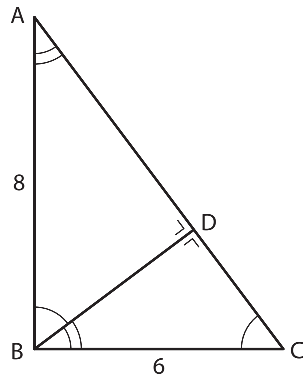
$|AC| = 10$

(i) $\triangle$ABC and $\triangle$BCD are similar.

Thus

$\frac{|BD|}{8} = \frac{6}{10}$

$|BD| = 4.8$ cm



(ii) $\triangle ABC$ and $\triangle ABD$ are similar.

Thus

$$\frac{|AD|}{8} = \frac{8}{10}$$

$$|AD| = 6.4 \text{ cm}$$

(iii) $|DC| = 10 - 6.4 = 3.6$ cm.

10. $|AC| = r + 2$, where

$r = |AB|$, the radius.

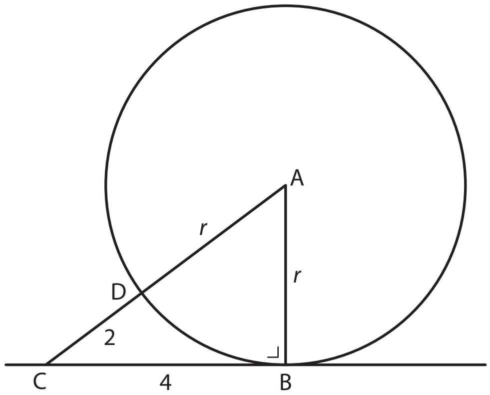
By Pythagoras,

$$(r+2)^2 = r^2 + 4^2$$

$$r^2 + 4r + 4 = r^2 + 16$$

$$4r = 12$$

$$r = 3 = |AB|.$$



Revision Exercise 6 (Advanced)

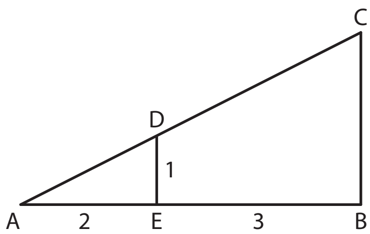
1. $\triangle ABC$ and $\triangle AED$ are similar.

Thus $$\frac{|BC|}{|DE|} = \frac{|AB|}{|AE|}$$

$$\frac{|BC|}{1} = \frac{5}{2}$$

$$|BC| = \frac{5}{2}$$

and $$|AC| = \sqrt{5^2 + \left(\frac{5}{2}\right)^2} = \frac{5\sqrt{5}}{2} \text{ cm.}$$



2. (i) $|AF| = |FE|$ ... given

$|DF| = |FC|$ ... given

$|\angle AFD| = |\angle CFE|$ ... vertically opposite angles

Thus $\triangle ADF$ and $\triangle ECF$ are congruent.

Hence $|\angle ADF| = |\angle FCE|$.

(ii) As $|DF| = |FC|$, $\triangle DFC$ is isosceles.

Hence

$$|\angle FDC| = |\angle FCD|.$$

Thus

$$|\angle ADF| + |\angle FDC| = |\angle ECF| + |\angle FCD|$$

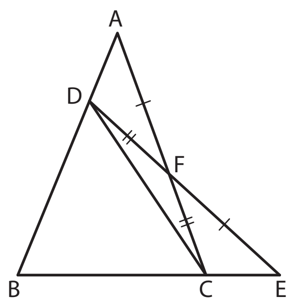
$$|\angle ADC| = |\angle ECD|$$

Hence

$$180° - |\angle ADC| = 180° - |\angle ECD|$$

$$|\angle BDC| = |\angle DCB|$$

As $\triangle BCD$ is isosceles, $|BD| = |BC|$.

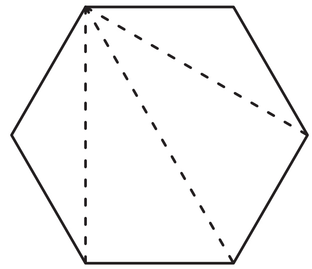


3. (i) All the angles in a regular hexagon are the angles in 4 triangles, i.e. a total of

$$4 \times 180^\circ = 720^\circ.$$

As there are 6 equal angles in a regular hexagon, each is of size

$$\frac{720^\circ}{6} = 120^\circ.$$

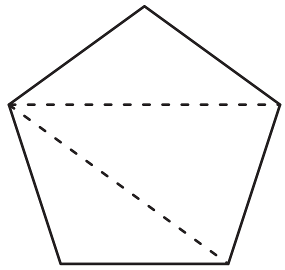


(ii) All the angles in a regular pentagon are the angles in 3 triangles, i.e. a total of

$$3 \times 180^\circ = 540^\circ.$$

As there are 5 equal angles in a regular pentagon, each is of size

$$\frac{540^\circ}{5} = 108^\circ.$$



(iii) For an n sided regular polygon, all the angles are the angles in $(n - 2)$ triangles, i.e.

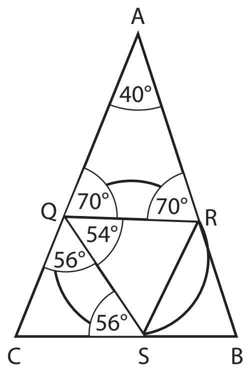
$$(n - 2)180^\circ.$$

As there are n equal angles, each is

$$\frac{(n-2)180^\circ}{n}.$$

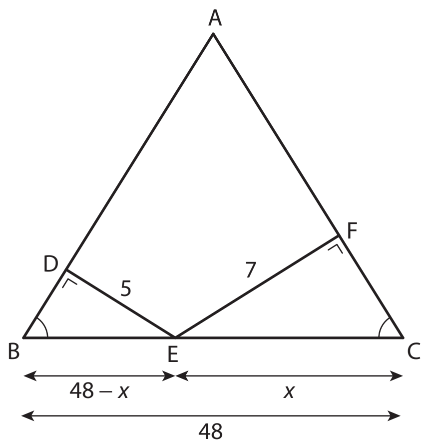
When $n = 10$, each angle is $\dfrac{8 \times 180^\circ}{10} = 144^\circ$.

4. $|\angle QAR| = 180^\circ - (70^\circ + 70^\circ) = 40^\circ$
$|\angle CQS| = 180^\circ - (70^\circ + 54^\circ) = 56^\circ$
$|\angle ACB| = 180^\circ - (56^\circ + 56^\circ) = 68^\circ$



5. $|\angle ABC| = |\angle ACB|$ … isosceles triangle
$|\angle BDE| = |\angle CFE| = 90^\circ$
Thus $\triangle BDE$ and $\triangle CFE$ are similar.
Let $|EC| = x$. Then
$|BE| = 48 - x$.
Hence

$$\frac{x}{48 - x} = \frac{7}{5}$$
$$5x = 366 - 7x$$
$$12x = 336$$
$$x = 28 = |EC|.$$



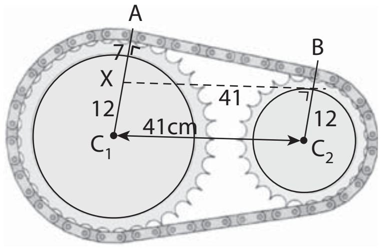
6. $x^2 + (7)^2 = (41)^2$

$\Rightarrow x^2 + 49 = 1681$

$\Rightarrow x^2 = 1681 - 49$

$\Rightarrow \quad = 1632$

$\Rightarrow x = \sqrt{1632} = 4\sqrt{102}$ (= 40.4 cm)



7. Background:

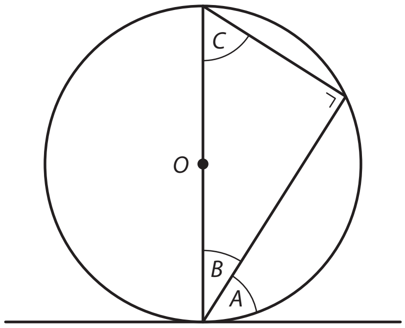
(a) In the diagram, if O is the centre of the circle, then

$|\angle A| + |\angle B| = 90°$

$|\angle B| + |\angle C| = 90°$

Thus

$|\angle A| = |\angle C|$.

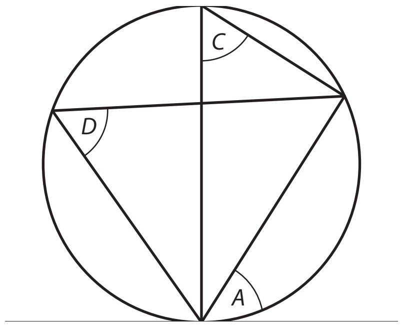


(b) By angles standing on the same arc,

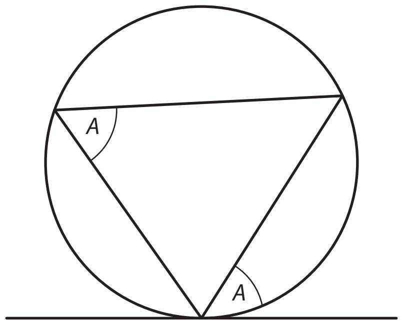
$|\angle D| = |\angle C|$

Thus

$|\angle A| = |\angle D|$.



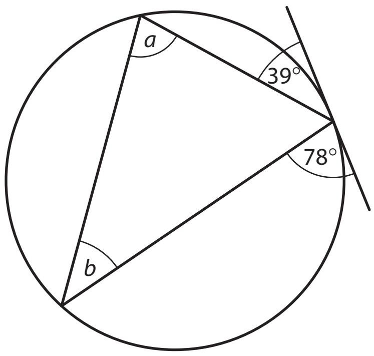
(c) Hence the angle between a tangent and a chord is equal to the angle in the opposite segment.



Solutions

(i) $a = 78°$

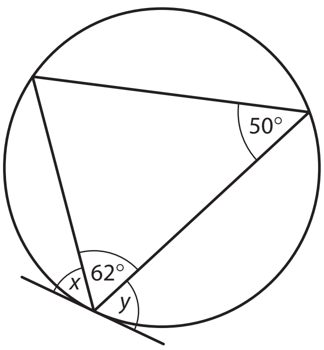
$b = 39°$



(ii) $x = 50°$

$y = 180° - 62° - 50°$

$y = 68°$



(iii) $2c + 54° = 180°$

$2c = 126°$

$c = 63°$

Then

$x = c = 63°$

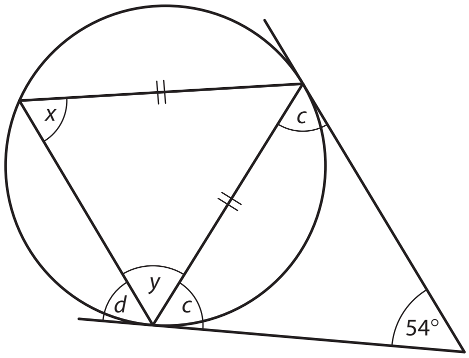
As $\triangle$ is isosceles,

$y = 63°$

Thus

$d = 180° - 63° - 63°$

$d = 54°$.



8. Radius of each circle is 3 cm.

Area of circle

$= \pi(3)^2 = 9\pi \text{ cm}^2$

Area of shaded segment

$= \frac{1}{4}(9\pi) - \frac{1}{2}(3)(3)$

$= \frac{9\pi}{4} - \frac{9}{2}$

Area of 4 segments

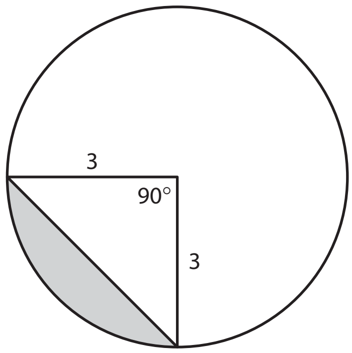
$= 4\left(\frac{9\pi}{4} - \frac{9}{2}\right)$

$= 9\pi - 18$

Area of blue shaded region

$= 9\pi - (9\pi - 18)$

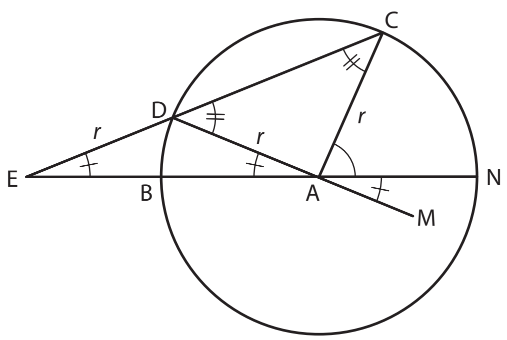
$= 18 \text{ cm}^2$



9. (i) 1. … the sum of the opposite interior angles

2. … the two base angles are equal.

(ii)



$\lvert\angle DAE\rvert = \lvert\angle DEB\rvert$	… isosceles triangle
$\lvert\angle CDA\rvert = \lvert\angle ACD\rvert$	… isosceles triangle
$\lvert\angle CAM\rvert = \lvert\angle CDA\rvert + \lvert\angle ACD\rvert$	… exterior angle
$= 2\lvert\angle CDA\rvert$	

But

$\lvert\angle CDA\rvert = \lvert\angle DEB\rvert + \lvert\angle DAE\rvert$	… exterior angle
$= 2\lvert\angle DEB\rvert$	

Thus

$\lvert\angle CAM\rvert = 2(2\lvert\angle DEB\rvert) = 4\lvert\angle DEB\rvert$

But

$|\angle NAM| = |\angle DAE|$ … vertically opposite angles

$= |\angle DEB|$

Hence

$|\angle CAN| + |\angle DEB| = 4|\angle DEB|$

$|\angle CAN| = 3|\angle DEB|$

$|\angle DEB| = \frac{1}{3}|\angle CAN|.$

Revision Exercise 6 (Extended-Response Questions)

1. In the given circle with centre O, [PA] is a tangent and [PR] is a diameter of the circle.

$|\angle APQ| = a°.$

$|\angle PQR| = 90°,$

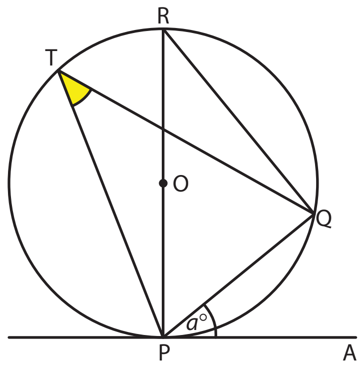
because the angle at the circle standing on a diameter is a right-angle

$|\angle PRQ| + |\angle RPQ| = 90°,$

because the sum of all angles in a triangle is 180°

$|\angle QPA| + |\angle RPQ| = 90°$

because the angle between to the point of contact with a tangent is 90°



$|\angle PRQ| + |\angle RPQ| = |\angle QPA| + |\angle RPQ|$

$|\angle PRQ| = |\angle QPA|$

But $|\angle PRQ| = |\angle PTQ|$, because angles at the circle standing on the same arc are equal.

Thus $|\angle QPA| = |\angle PTQ|$.

2. (i) $|XQ|^2 + (30)^2 = (110)^2$

$\Rightarrow |XQ|^2 + 900 = 12\,100$

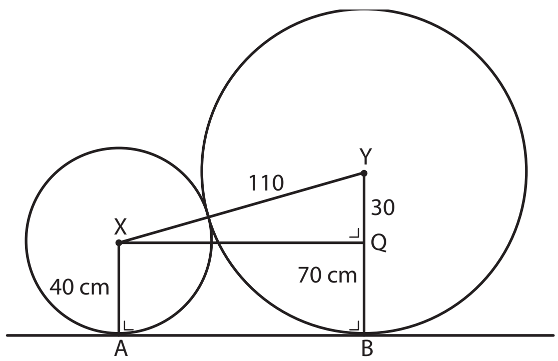
$\Rightarrow |XQ|^2 = 12\,100 - 900$

$\Rightarrow \quad = 11\,200$

$\Rightarrow \quad |XQ| = \sqrt{11\,200}$

$= 40\sqrt{7}$ (= 105.8 cm)

$\Rightarrow \quad |AB| = (40\sqrt{7})$ cm



(ii) Triangles XYQ and YZT are similar

$\Rightarrow \frac{110}{30} = \frac{70 + P}{P - 70}$

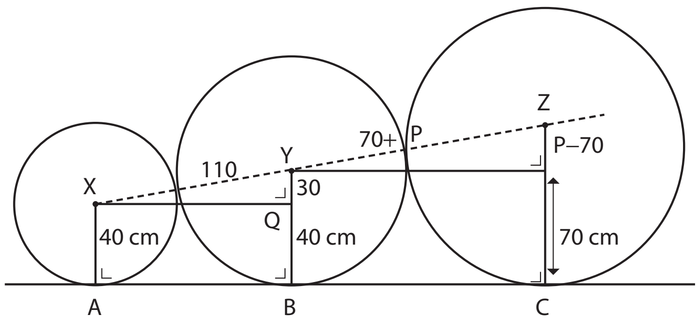
$\Rightarrow 110P - 7700 = 2100 + 30P$

$\Rightarrow \quad 110P - 30P = 2100 + 7700$

$\Rightarrow \quad 80P = 9800$

$\Rightarrow \quad P = \frac{9800}{80} = 122.5$ cm

$\Rightarrow |ZC| = 52.5$ cm



3. (i) $|\angle BAE| = |\angle ECD|$

$|\angle ABE| = |\angle ECD|$

$\Rightarrow |\angle BAE| = |\angle ABE|$

$\Rightarrow \triangle AEB$ is isosceles

(ii) (*b*) is not always true.

4. (i) $x^2 + \left(\frac{9}{2}\right)^2 = 7^2$

$x^2 + \frac{81}{4} = 49$

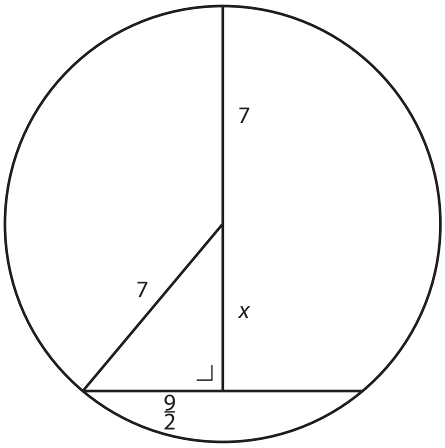
$x^2 = \frac{115}{4}$

$x = 5.36$

Then

$h = x + 7$

$h = 12.4$ cm



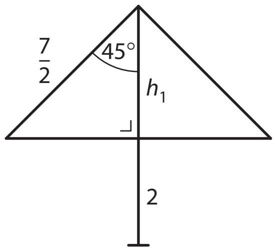
(ii) (a) $\frac{h_1}{\frac{7}{2}} = \cos 45°$

$h_1 = \frac{7}{2\sqrt{2}}$

$h = h_1 + 2 - 1.5$

$h = \frac{7}{2\sqrt{2}} + \frac{1}{2}$

$h = \frac{2 + 7\sqrt{2}}{4}$



(b) $(x + 4)^2 = \left(\frac{7}{2}\right)^2 + \left(\frac{7}{2}\right)^2$

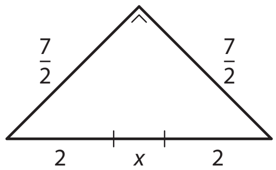
$(x + 4)^2 = \frac{49}{2}$

$x + 4 = \frac{7}{\sqrt{2}}$

$x + 4 = \frac{7\sqrt{2}}{2}$

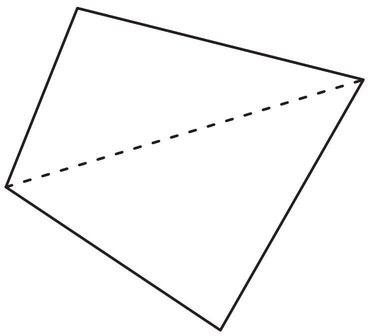
$x = \frac{7\sqrt{2}}{2} - 4$

$x = \frac{7\sqrt{2} - 8}{2}$



5. (i) Divide the quadrilateral into two triangles.
The sum of the angles in the quadrilateral is then the sum of the angles in two triangles, i.e.

$2(180°) = 360°$.



(ii) (a) $v° + 65° + 109° + 55° = 360°$

$v° = 131°$

(b) $x° = 360° - 21° - 42° - 15°$

$x° = 282°$

and

$y° = 360° - 282°$

$y° = 78°$

(iii) $|\angle BED| = \frac{1}{2}(140°) = 70°$

$|\angle BCD| = 180° - 70° = 110°$

(iv) By symmetry,

$|\angle ADC| = 87°$.

As $|\angle ABC| + |\angle ADC| = 2(87°)$

$= 184°$

$\neq 180°$,

ABCD is not a cyclic quadrilateral.

6. (i) $|\angle ABC| = |\angle ADC|$ … right angles

$|AB| = |AD|$ … radius

[AC] is common

Thus $\triangle ABC$ and $\triangle ADC$ are congruent (RHS).

(ii) Let x and y be the lengths shown.

We are given that

$x + y = 12$.

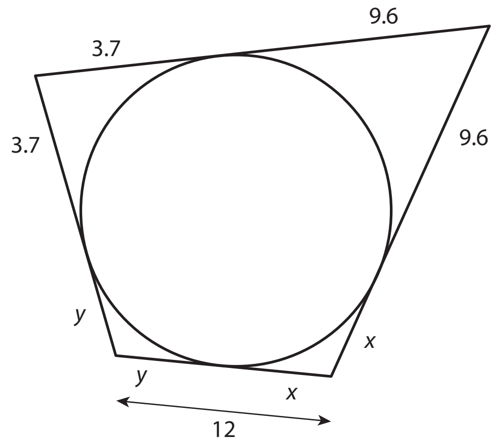
From the diagram, the perimeter is

$2(3.7) + 2(9.6) + 2x + 2y$

$= 2(3.7) + 2(9.6) + 2(x + y)$

$= 2(3.7) + 2(9.6) + 2(12)$

$= 50.6$ cm



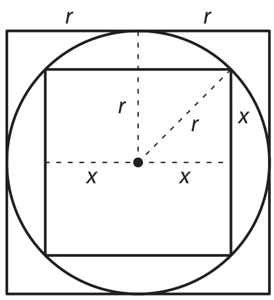
(iii) Let the length of a side of the smaller square be $2x$ and r be the radius of the circle.

Then the length of a side of the larger square is $2r$.

By Pythagoras,

$x^2 + x^2 = r^2$

$2x^2 = r^2$.



Area of small square

$= (2x)^2$

$= 4x^2$

Area of large square

$= (2r)^2$

$= 4r^2$

$= 4(2x^2)$

$= 8x^2$

Thus

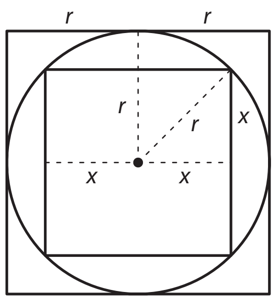
$$\frac{\text{Area of large square}}{\text{Area of small square}} = \frac{8x^2}{4x^2} = 2 = 2:1$$

(iv) Let r be the radius of the smaller circle and r_1 be the radius of the larger circle.

By Pythagoras,

$r_1^2 = r^2 + r^2$

$r_1^2 = 2r^2$



Area of smaller circle

$= \pi r^2$

Area of larger circle

$= \pi r_1^2$

$= \pi(2r^2)$

$= 2\pi r^2$

Thus

$$\frac{\text{Area of large circle}}{\text{Area of small circle}} = \frac{2\pi r^2}{\pi r^2} = 2.$$

7. (i) Circumference of rim

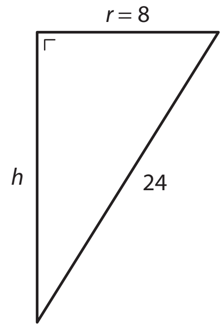
$$= 2\pi(24) \times \frac{120}{360}$$
$$= 16\pi$$

Let r be the radius of the base of the cylindrical funnel. Then

$$2\pi r = 16\pi$$
$$r = 8$$

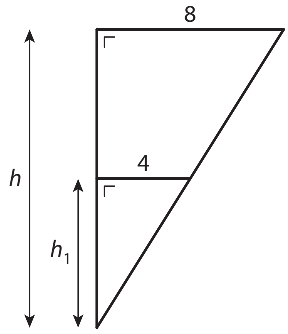
Then by Pythagoras,

$$h^2 + 8^2 = 24^2$$
$$h^2 = 512$$
$$h = 16\sqrt{2}$$



(ii) (a) By similar triangles,

$$\frac{h_1}{h} = \frac{4}{8}$$
$$h_1 = \frac{1}{2}h$$
$$h_1 = \frac{1}{2}(16\sqrt{2})$$
$$h_1 = 8\sqrt{2}$$



(b) Volume of funnel

$$= \frac{1}{3}\pi r^2 h$$

Volume of liquid

$$= \frac{1}{3}\pi\left(\frac{r}{2}\right)^2\left(\frac{h}{2}\right)$$
$$= \frac{1}{24}\pi r^2 h$$

$$\text{Fraction} = \frac{\frac{1}{24}\pi r^2 h}{\frac{1}{3}\pi r^2 h} = \frac{1}{24} \times 3 = \frac{1}{8}.$$

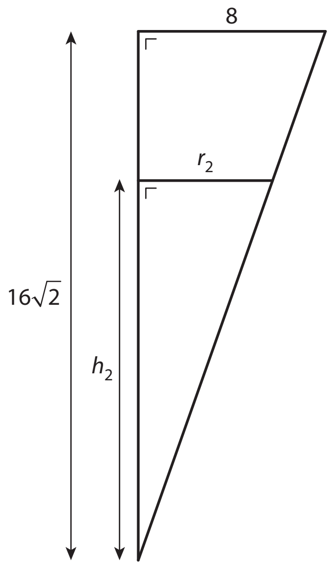
(iii) In 10 minutes the volume that flows is

$$10 \times 20 = 200 \text{ cm}^3.$$

Let r_2 and h_2 be the radius and the height, respectively, of the liquid at this time.

Then

$$\frac{r_2}{8} = \frac{h_2}{16\sqrt{2}}$$
$$r_2 = \frac{h_2}{2\sqrt{2}}$$



Then

$$\text{volume} = 200$$
$$\frac{1}{3}\pi r_2^2 h_2 = 200$$
$$\left(\frac{h_2}{2\sqrt{2}}\right)^2 h_2 = \frac{600}{\pi}$$
$$\frac{h_2^3}{8} = \frac{600}{\pi}$$
$$h_2^3 = \frac{4800}{\pi}$$

$$h_2 = \left(\frac{4800}{\pi}\right)^{\frac{1}{3}}$$
$$h_2 = 11.51 \text{ cm}$$

Then the distance of the sensor from the top is

$$16\sqrt{2} - h_2 = 16\sqrt{2} - 11.51$$
$$= 11.11 \text{ cm.}$$

8. (i) Let a and b be the lengths shown.

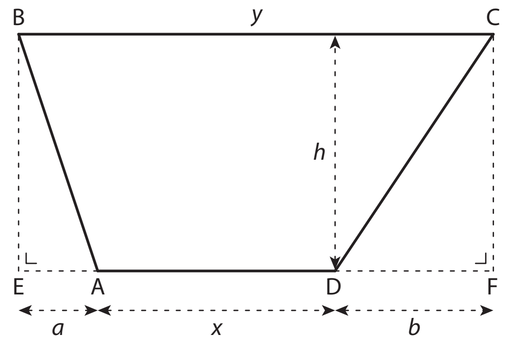
From the diagram

$a + b + x = y$

$a + b = y - x$

Area trapezium = Area BEFC − Area △BEA − Area △CDF

$$= (h)(y) - \frac{1}{2}(a)(h) - \frac{1}{2}(b)(h)$$
$$= hy - \frac{1}{2}h(a + b)$$
$$= hy - \frac{1}{2}h(y - x)$$
$$= h\left[y - \frac{1}{2}y + \frac{1}{2}x\right]$$
$$= h\left[\frac{1}{2}x + \frac{1}{2}y\right]$$
$$= \frac{1}{2}(x + y)h$$



(ii) Let h be the perpendicular distance between [AB] and [DC]. Then

$$\text{Area AEDF} = \frac{1}{2}(x + 3)h$$
$$\text{Area CBEF} = \frac{1}{2}[(30 - x) + 17]h$$
$$= \frac{1}{2}(47 - x)h$$

Given:

$$\frac{1}{2}(x + 3)h = \frac{1}{2}(47 - x)h$$
$$x + 3 = 47 - x$$
$$2x = 44$$
$$x = 22.$$

Chapter 7

Exercise 7.1

1. (i) A(2, 4) B(1, 0) $\Rightarrow$ average rate of change $= \dfrac{0-4}{1-2} = \dfrac{-4}{-1} = 4$

(ii) B(1, 0) C(-1, -2) $\Rightarrow$ average rate of change $= \dfrac{-2-0}{-1-1} = \dfrac{-2}{-2} = 1$

(iii) C(-1, -2) D(-3, 4) $\Rightarrow$ average rate of change $= \dfrac{4+2}{-3+1} = \dfrac{6}{-2} = -3$

2. Average rate of change $= \dfrac{20-4}{5+2} = \dfrac{16}{7}$

3. (i) Average rate of change $= \dfrac{5-30}{2+5} = \dfrac{-25}{7}$

(ii) Average rate of change $= \dfrac{15-3}{3-0} = \dfrac{12}{3} = 4$

4. $t = 0 \Rightarrow d(0) = \dfrac{-300}{0+6} + 50 = -50 + 50 = 0 \Rightarrow (0, 0)$

$t = 10 \Rightarrow d(10) = \dfrac{-300}{10+6} + 50 = -18.75 + 50 = 31.25 \Rightarrow (10, 31.25)$

$\Rightarrow$ average rate of change $= \dfrac{31.25-0}{10-0} = \dfrac{31.25}{10} = \dfrac{25}{8}$

5. (i) 8 years

(ii) (5, 80), (10, 140)

$\Rightarrow$ average rate of change $= \dfrac{140-80}{10-5} = \dfrac{60}{5} = 12$ (cm/year)

6. (i) $S(x) = 6x^2$

(ii) $x = 2 \Rightarrow S(2) = 6(2)^2 = 24 \Rightarrow (2, 24)$

$x = 5 \Rightarrow S(5) = 6(5)^2 = 150 \Rightarrow (5, 150)$

$\Rightarrow$ average rate of change $= \dfrac{150-24}{5-2} = \dfrac{126}{3} = 42$ cm^2/sec

7. (i) P(4, 8), Q(3, 3) $\Rightarrow$ slope PQ $= \dfrac{3-8}{3-4} = \dfrac{-5}{-1} = 5$

(ii) P(3.5, 5.25), Q(3, 3) $\Rightarrow$ slope PQ $= \dfrac{3-5.25}{3-3.5} = \dfrac{-2.25}{-0.5} = 4.5$

(iii) P(3.1, 3.41), Q(3, 3) $\Rightarrow$ slope PQ $= \dfrac{3-3.41}{3-3.1} = \dfrac{-0.41}{-0.1} = 4.1$

(iv) Slope = 4

Exercise 7.2

1. (i) $x = 0$
(ii) $x = 3$
(iii) $x = -2$
(iv) $x = 2$
(v) $x = -3$
(vi) $x = \dfrac{-\pi}{2}$ and $x = \dfrac{\pi}{2}$

2. (i) $x = 0$
(ii) $\dfrac{2}{0}$ is not defined

3. Tan $\dfrac{\pi}{2} = \dfrac{k}{0}$ which is not defined

4. (i) $x = 4$

(ii) $x = -5$ or $x = 5$

(iii) $x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0$

$\Rightarrow x = -1$ OR $x = 4$

5. (i) $\lim_{x \to 2} \frac{x + 3}{x + 2} = \frac{2 + 3}{2 + 2} = \frac{5}{4}$

(ii) $\lim_{x \to 0} (x^2 + 3x - 4) = (0)^2 + 3(0) - 4 = -4$

(iii) $\lim_{x \to 3} \frac{x^2 - x - 3}{x + 1} = \frac{(3)^2 - (3) - 3}{3 + 1} = \frac{9 - 3 - 3}{4} = \frac{3}{4}$

6. (i) $\lim_{x \to 0} \frac{x + 2}{x - 2} = \frac{0 + 2}{0 - 2} = -1$

(ii) $\lim_{x \to 0} \frac{6x - 3}{2 + x} = \frac{6(0) - 3}{2 + 0} = \frac{-3}{2}$

(iii) $\lim_{h \to 2} \frac{h^2 + 2h - 6}{h + 1} = \frac{(2)^2 + 2(2) - 6}{2 + 1} = \frac{4 + 4 - 6}{3} = \frac{2}{3}$

7. (i) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)\cancel{(x - 1)}}{\cancel{(x - 1)}} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$

(ii) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)\cancel{(x - 2)}}{\cancel{(x - 2)}} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4$

(iii) $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x + 5)\cancel{(x - 5)}}{\cancel{(x - 5)}} = \lim_{x \to 5} (x + 5) = 5 + 5 = 10$

(iv) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{(x - 2)\cancel{(x - 1)}}{\cancel{(x - 1)}} = \lim_{x \to 1} (x - 2) = 1 - 2 = -1$

(v) $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x + 2)\cancel{(x - 1)}}{\cancel{(x - 1)}} = \lim_{x \to 1} (x + 2) = 1 + 2 = 3$

(vi) $\lim_{x \to -3} \frac{x + 3}{x^2 - x - 12} = \lim_{x \to -3} \frac{\cancel{(x + 3)}}{(x - 4)\cancel{(x + 3)}} = \lim_{x \to -3} \frac{1}{x - 4} = \frac{1}{-3 - 4} = -\frac{1}{7}$

8. $f(x) = \frac{x^2 - 9}{x - 3}$

x	2.5	2.9	2.999	2.9999	3.0000	3.0001	3.001	3.1	3.5
f(x)	5.5	5.9	5.999	5.9999	Undefined	6.0001	6.001	6.1	6.5

Hence, limit = 6

9. (i) $\lim_{x \to \infty} \frac{1}{x} = 0$ (ii) $\lim_{x \to \infty} \frac{4}{3x} = 0$ (iii) $\lim_{x \to \infty} \frac{1}{x^2} = 0$ (iv) $\lim_{x \to \infty} \frac{k}{x^3} = 0$

10. (i) $\lim_{x \to \infty} \frac{3x - 2}{2x + 3} = \lim_{x \to \infty} \frac{3 - \frac{2}{x}}{2 + \frac{3}{x}} = \frac{3 - 0}{2 + 0} = \frac{3}{2}$

(ii) $\lim_{x \to \infty} \frac{4x - 3}{7x - 6} = \lim_{x \to \infty} \frac{4 - \frac{3}{x}}{7 - \frac{6}{x}} = \frac{4 - 0}{7 - 0} = \frac{4}{7}$

(iii) $\lim_{x \to \infty} \frac{1 - 3x}{4x + 2} = \lim_{x \to \infty} \frac{\frac{1}{x} - 3}{4 + \frac{2}{x}} = \frac{0 - 3}{4 + 0} = -\frac{3}{4}$

11. (i) $\lim_{n \to \infty} \frac{n^2 + 4}{3n^2 - 4n} = \lim_{n \to \infty} \frac{1 + \frac{4}{n^2}}{3 - \frac{4}{n^2}} = \frac{1 + 0}{3 - 0} = \frac{1}{3}$

(ii) $\lim_{n \to \infty} \frac{5n^2 - 3}{2n^2 - 6n + 5} = \lim_{n \to \infty} \frac{5 - \frac{3}{n^2}}{2 - \frac{6}{n} + \frac{5}{n^2}} = \frac{5 - 0}{2 - 0 + 0} = \frac{5}{2}$

(iii) $\lim_{n \to \infty} \frac{2n^2 - 3n + 2}{6n^2 + 5n - 6} = \lim_{n \to \infty} \frac{2 - \frac{3}{n} + \frac{2}{n^2}}{6 + \frac{5}{n} - \frac{6}{n^2}} = \frac{2 - 0 + 0}{6 + 0 - 0} = \frac{2}{6} = \frac{1}{3}$

12. (i) $f(x) = 2x - 3$

$f(x+h) = 2(x+h) - 3 = 2x + 2h - 3$

$\Rightarrow f(x+h) - f(x) = 2x + 2h - 3 - (2x - 3)$

$= 2x + 2h - 3 - 2x + 3 = 2h$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2h}{h} = 2$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2 = 2$$

(ii) $f(x) = x^2$

$\Rightarrow f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$

$= 2xh + h^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x$$

(iii) $f(x) = x^2 + 5$

$\Rightarrow f(x+h) = (x+h)^2 + 5 = x^2 + 2xh + h^2 + 5$

$\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 + 5 - (x^2 + 5)$

$= x^2 + 2xh + h^2 + 5 - x^2 - 5$

$= 2xh + h^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

13. $$\lim_{x \to 3} \frac{x - 3}{x^3 - 27} = \lim_{x \to 3} \frac{\cancel{(x - 3)}}{\cancel{(x - 3)}(x^2 + 3x + 9)}$$

$$= \lim_{x \to 3} \frac{1}{x^2 + 3x + 9} = \frac{1}{(3)^2 + 3(3) + 9} = \frac{1}{27}$$

14. $f(n) = \left(1 + \frac{1}{n}\right)^n$

n	1	2	5	10	100	1000	10000
***f*(*n*)**	2	2.25	2.488	2.594	2.705	2.717	2.718

Hence, approximate value for e = 2.718

Exercise 7.3

1. (i) $f(x) = 5x$

$\Rightarrow f(x+h) = 5(x+h) = 5x + 5h$

$\Rightarrow f(x+h) - f(x) = 5x + 5h - 5x$

$= 5h$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{5h}{h} = 5$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 5 = 5$$

$\therefore f'(x) = 5$

(ii) $f(x) = 3x - 4$

$\Rightarrow \ f(x+h) = 3(x+h) - 4 = 3x + 3h - 4$

$\Rightarrow \ f(x+h) - f(x) = 3x + 3h - 4 - (3x - 4)$

$= 3x + 3h - 4 - 3x + 4$

$= 3h$

$$\Rightarrow \ \frac{f(x+h) - f(x)}{h} = \frac{3h}{h} = 3$$

$$\Rightarrow \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3 = 3$$

$\therefore f'(x) = 3$

(iii) $f(x) = 6 - 4x$

$\Rightarrow \ f(x+h) = 6 - 4(x+h) = 6 - 4x - 4h$

$\Rightarrow \ f(x+h) - f(x) = 6 - 4x - 4h - (6 - 4x)$

$= 6 - 4x - 4h - 6 + 4x$

$= -4h$

$$\Rightarrow \ \frac{f(x+h) - f(x)}{h} = \frac{-4h}{h} = -4$$

$$\Rightarrow \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} -4 = -4$$

$\therefore f'(x) = -4$

2. (i) $f(x) = x^2$

$\Rightarrow \ f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$\Rightarrow \ f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$

$= 2xh + h^2$

$$\Rightarrow \ \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

$$\Rightarrow \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

$\therefore f'(x) = 2x$

(ii) $f(x) = 2x^2 + 9x$

$\Rightarrow \ f(x+h) = 2(x+h)^2 + 9(x+h) = 2x^2 + 4xh + h^2 + 9x + 9h$

$\Rightarrow \ f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 9x + 9h - (2x^2 + 9x)$

$= 2x^2 + 4xh + 2h^2 + 9x + 9h - 2x^2 - 9x$

$= 4xh + 2h^2 + 9h$

$$\Rightarrow \ \frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 + 9h}{h} = 4x + 2h + 9$$

$$\Rightarrow \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (4x + 2h + 9) = 4x + 9$$

$\therefore f'(x) = 4x + 9$

(iii) $f(x) = 3x^2 - 4x - 6$

$\Rightarrow \ f(x+h) = 3(x+h)^2 - 4(x+h) - 6 = 3x^2 + 6xh + 3h^2 - 4x - 4h - 6$

$\Rightarrow \ f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 - 4x - 4h - 6 - (3x^2 - 4x - 6)$

$= 3x^2 + 6xh + 3h^2 - 4x - 4h - 6 - 3x^2 + 4x + 6$

$= 6xh + 3h^2 - 4h$

$$\Rightarrow \ \frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 4h}{h} = 6x + 3h - 4$$

$$\Rightarrow \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h - 4) = 6x - 4$$

$\therefore f'(x) = 6x - 4$

3. (i) $f(x) = x^2 - 2x + 5$

$\Rightarrow f(x+h) = (x+h)^2 - 2(x+h) + 5 = x^2 + 2xh + h^2 - 2x - 2h + 5$

$\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 - 2x - 2h + 5 - (x^2 - 2x + 5)$

$= x^2 + 2xh + h^2 - 2x - 2h + 5 - x^2 + 2x - 5$

$= 2xh + h^2 - 2h$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 2h}{h} = 2x + h - 2$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + 2h - 2) = 2x - 2$$

$\therefore f'(x) = 2x - 2$

(ii) Point (2, 5) $\Rightarrow$ slope $= 2(2) - 2 = 4 - 2 = 2$

(iii) Equation of Tangent: $y - 5 = 2(x - 2)$

$\Rightarrow y - 5 = 2x - 4$

$\Rightarrow 2x - y + 1 = 0$

4. $f(x) = kx^2$

$\Rightarrow f(x+h) = k(x+h)^2 = kx^2 + 2kxh + kh^2$

$\Rightarrow f(x+h) - f(x) = kx^2 + 2kxh + kh^2 - kx^2$

$= 2kxh + kh^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2kxh + kh^2}{h} = 2kx + kh$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2kx + kh) = 2kx$$

$\therefore f'(x) = 2kx$

5. (i) $f(x) = -x^2$

$\Rightarrow f(x+h) = -(x+h)^2 = -(x^2 + 2xh + h^2) = -x^2 - 2xh - h^2$

$\Rightarrow f(x+h) - f(x) = -x^2 - 2xh - h^2 - (-x^2)$

$= -x^2 - 2xh - h^2 + x^2 = -2xh - h^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{-2xh - h^2}{h} = -2x - h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-2x - h) = -2x$$

$\therefore f'(x) = -2x$

(ii) $f(x) = 4x - x^2$

$\Rightarrow f(x+h) = 4(x+h) - (x+h)^2 = 4x + 4h - x^2 - 2xh - h^2$

$\Rightarrow f(x+h) - f(x) = 4x + 4h - x^2 - 2xh - h^2 - (4x - x^2)$

$= 4x + 4h - x^2 - 2xh - h^2 - 4x + x^2$

$= 4h - 2xh - h^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{4h - 2xh - h^2}{h} = 4 - 2x - h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (4 - 2x - h) = 4 - 2x$$

$\therefore f'(x) = 4 - 2x$

(iii) $f(x) = 2 - x - 3x^2$

$\Rightarrow f(x+h) = 2 - (x+h) - 3(x+h)^2 = 2 - x - h - 3x^2 - 6xh - 3h^2$

$\Rightarrow f(x+h) - f(x) = 2 - x - h - 3x^2 - 6xh - 3h^2 - (2 - x - 3x^2)$

$= 2 - x - h - 3x^2 - 6xh - 3h^2 - 2 + x + 3x^2$

$= -h - 6xh - 3h^2$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{-h - 6xh - 3h^2}{h} = -1 - 6x - 3h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-1 - 6x - 3h) = -1 - 6x$$

$\therefore f'(x) = -1 - 6x$

6. $f(x) = 2x^2 - 3x - 2$

$\Rightarrow f(x+h) = 2(x+h)^2 - 3(x+h) - 2 = 2x^2 + 4xh + 2h^2 - 3x - 3h - 2$

$\Rightarrow f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 - 3x - 3h - 2 - (2x^2 - 3x - 2)$

$= 2x^2 + 4xh + 2h^2 - 3x - 3h - 2 - 2x^2 + 3x + 2$

$= 4xh + 2h^2 - 3h$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 - 3h}{h} = 4x + 2h - 3$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (4x + 2h - 3) = 4x - 3$$

$\therefore f'(x) = 4x - 3$

(i) Point $(3, 7) \Rightarrow$ slope $= 4(3) - 3 = 12 - 3 = 9$

(ii) Equation of Tangent: $y - 7 = 9(x - 3)$

$\Rightarrow y - 7 = 9x - 27$

$\Rightarrow 9x - y - 20 = 0$

7. $A = f(r) = \pi r^2$

$\Rightarrow f(r+h) = \pi(r+h)^2 = \pi(r^2 + 2rh + h^2) = \pi r^2 + 2\pi rh + \pi h^2$

$\Rightarrow f(r+h) - f(r) = \pi r^2 + 2\pi rh + \pi h^2 - \pi r^2$

$= 2\pi rh + \pi h^2$

$$\Rightarrow \frac{f(r+h) - f(r)}{h} = \frac{2\pi rh + \pi h^2}{h} = 2\pi r + \pi h$$

$$\Rightarrow \lim_{h \to 0} \frac{f(r+h) - f(r)}{h} = \lim_{h \to 0} (2\pi r + \pi h) = 2\pi r$$

$\therefore f'(r) = \frac{dA}{dr} = 2\pi r$

8. $f(x) = x^2 - 3x + 1$

$\Rightarrow f(x+h) = (x+h)^2 - 3(x+h) + 1 = x^2 + 2xh + h^2 - 3x - 3h + 1$

$\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 - 3x - 3h + 1 - (x^2 - 3x + 1)$

$= x^2 + 2xh + h^2 - 3x - 3h + 1 - x^2 + 3x - 1$

$= 2xh + h^2 - 3h$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 3h}{h} = 2x + h - 3$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h - 3) = 2x - 3$$

$\therefore f'(x) = 2x - 3$

Slope of tangent is zero $\Rightarrow 2x - 3 = 0$

$\Rightarrow 2x = 3 \quad \Rightarrow x = \frac{3}{2} = 1\frac{1}{2}$

$$\Rightarrow f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1$$

$$= \frac{9}{4} - \frac{9}{2} + 1 = -1\frac{1}{4} \Rightarrow \text{Point} = \left(1\frac{1}{2}, -1\frac{1}{4}\right)$$

Exercise 7.4

1. (i) $y = 5x^5 \Rightarrow \frac{dy}{dx} = 25x^4$

(ii) $y = 5x^2 - 4x \Rightarrow \frac{dy}{dx} = 10x - 4$

(iii) $y = 6x^2 + 5x - 4 \Rightarrow \frac{dy}{dx} = 12x + 5$

(iv) $y = x^3 - 8x + 2 \Rightarrow \frac{dy}{dx} = 3x^2 - 8$

(v) $y = x^2 + 2x + \frac{1}{x} = x^2 + 2x + x^{-1}$

$\frac{dy}{dx} = 2x + 2 - 1x^{-2} = 2x + 2 - \frac{1}{x^2}$

(vi) $y = 2x^3 + x^2 + \frac{1}{x^2} = 2x^3 + x^2 + x^{-2}$

$\frac{dy}{dx} = 6x^2 + 2x - 2x^{-3} = 6x^2 + 2x - \frac{2}{x^3}$

2. (i) $f(x) = 7x^2 - \frac{3}{x} = 7x^2 - 3x^{-1}$

$\Rightarrow f'(x) = 14x + 3x^{-2} = 14x + \frac{3}{x^2}$

(ii) $f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$

$\Rightarrow f'(x) = 3 \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$

(iii) $f(x) = 2\sqrt{x} + \frac{2}{x^2} = 2x^{\frac{1}{2}} + 2x^{-2}$

$\Rightarrow f'(x) = 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 4x^{-3} = \frac{1}{\sqrt{x}} - \frac{4}{x^3}$

(iv) $f(x) = x^2 - 5\sqrt{x} = x^2 - 5x^{\frac{1}{2}}$

$\Rightarrow f'(x) = 2x - \frac{5}{2}x^{-\frac{1}{2}} = 2x - \frac{5}{2\sqrt{x}}$

(v) $f(x) = \frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$

$\Rightarrow f'(x) = -\frac{3}{2}x^{-\frac{3}{2}} = \frac{-3}{2x^{\frac{3}{2}}} = \frac{-3}{2\sqrt{x^3}}$

(vi) $f(x) = 3x^{-2} + \frac{1}{2\sqrt{x}} = 3x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow f'(x) = -6x^{-3} - \frac{1}{4}x^{\frac{-3}{2}} = \frac{-6}{x^3} - \frac{1}{4\sqrt{x^3}}$

3. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$

$\frac{dy}{dx} = \frac{1}{3} \cdot 3x^2 + \frac{1}{2} \cdot 2x - 6 = x^2 + x - 6$

4. (i) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$

(ii) $f(x) = 3\sqrt{x} - \frac{1}{x^2} = 3x^{\frac{1}{2}} - x^{-2}$

$\Rightarrow f'(x) = \frac{3}{2}x^{-\frac{1}{2}} + 2x^{-3} = \frac{3}{2\sqrt{x}} + \frac{2}{x^3}$

(iii) $f(x) = \frac{4}{x} + \frac{3}{\sqrt{x}} = 4x^{-1} + 3x^{-\frac{1}{2}}$

$\Rightarrow f'(x) = -4x^{-2} - \frac{3}{2}x^{-\frac{3}{2}} = -\frac{4}{x^2} - \frac{3}{2\sqrt{x^3}}$

(iv) $f(x) = 6 - \frac{3}{x} = 6 - 3x^{-1}$

$\Rightarrow f'(x) = 0 + 3x^{-2} = \frac{3}{x^2}$

(v) $f(x) = 2\sqrt{x} + \sqrt[3]{x} = 2x^{\frac{1}{2}} + x^{\frac{1}{3}}$

$\Rightarrow f'(x) = 1x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$

(vi) $f(x) = x^2 + 3 - \frac{4}{x^{-2}} = x^2 + 3 - 4x^2 = 3 - 3x^2$

$\Rightarrow f'(x) = 0 - 6x = -6x$

5. $y = \sqrt{x}(1 + \sqrt{x}) = \sqrt{x} + x = x^{\frac{1}{2}} + x$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 1 = \frac{1}{2\sqrt{x}} + 1$

When $x = 4 \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4}} + 1 = \frac{1}{4} + 1 = 1\frac{1}{4}$

6. $f(x) = x^3 + 2\sqrt{x} = x^3 + 2x^{\frac{1}{2}}$

$\Rightarrow f'(x) = 3x^2 + 1 \,.\, x^{-\frac{1}{2}} = 3x^2 + \frac{1}{\sqrt{x}}$

$\Rightarrow f'(4) = 3(4)^2 + \frac{1}{\sqrt{4}} = 48 + \frac{1}{2} = 48\frac{1}{2}$

7. $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\Rightarrow f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$

$\Rightarrow f'(4) = -\frac{1}{2\sqrt{(4)^3}} = -\frac{1}{16}$

8. $y = x^{\frac{5}{2}} \Rightarrow \frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}\sqrt{x^3}$

When $x = 2 \Rightarrow \frac{dy}{dx} = \frac{5}{2}\sqrt{(2)^3}$

$= \frac{5}{2}\sqrt{8} = \frac{5}{2} \cdot 2\sqrt{2} = 5\sqrt{2}$

$\Rightarrow p = 5$

9. $f(x) = x^2 + kx \Rightarrow f'(x) = 2x + k$

$\Rightarrow f'(-1) = 2(-1) + k = 3$

$\Rightarrow -2 + k = 3 \quad \Rightarrow \quad k = 5$

10. $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}}$

$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

$= \frac{x - 1}{2x\sqrt{x}}$

11. $y = x^2 - 2x - 3$

$\Rightarrow \frac{dy}{dx} = 2x - 2$

Point $(2, 3) \Rightarrow$ slope $= \frac{dy}{dx} = 2(2) - 2 = 2$

12. $y = 2x^2 - 3x + 4$

$\Rightarrow \dfrac{dy}{dx} = 4x - 3$

Point (1, 3) $\Rightarrow$ slope $= \dfrac{dy}{dx} = 4(1) - 3 = 1$

Equation of tangent: $y - 3 = 1(x - 1)$

$\Rightarrow y - 3 = x - 1$

$\Rightarrow x - y + 2 = 0$

13. $y = 6 + x - x^2$

$\Rightarrow \dfrac{dy}{dx} = 1 - 2x$

Point (2, 4) $\Rightarrow$ slope $= \dfrac{dy}{dx} = 1 - 2(2) = -3$

Equation of tangent: $y - 4 = -3(x - 2)$

$\Rightarrow y - 4 = -3x + 6$

$\Rightarrow 3x + y - 10 = 0$

14. $y = 8 + 2x - x^2$

$\Rightarrow \dfrac{dy}{dx} = 2 - 2x$

Slope $= 6 \Rightarrow 2 - 2x = 6$

$\Rightarrow -2x = 6 - 2 = 4$

$\Rightarrow 2x = -4 \Rightarrow x = -2$

15. $y = x^2 - x \Rightarrow \dfrac{dy}{dx} = 2x - 1$

Slope $= 1 \Rightarrow 2x - 1 = 1$

$\Rightarrow 2x = 2 \Rightarrow x = 1$

$\Rightarrow y = (1)^2 - 1 = 0 \Rightarrow$ Point (1, 0)

16. $y = 2x^2 - x - 4 \Rightarrow \dfrac{dy}{dx} = 4x - 1$

Slope $= 3 \Rightarrow 4x - 1 = 3$

$\Rightarrow 4x = 4 \Rightarrow x = 1$

$\Rightarrow y = 2(1)^2 - 1 - 4 = 2 - 1 - 4 = -3 \Rightarrow$ Point (1, −3)

17. $y = x^2 + ax \Rightarrow \dfrac{dy}{dx} = 2x + a$

When $x = -1 \Rightarrow$ slope $= \dfrac{dy}{dx} = 2(-1) + a = 3$

$\Rightarrow -2 + a = 3 \Rightarrow a = 5$

18. $y = x^2 - 3x + 4 \Rightarrow \dfrac{dy}{dx} = 2x - 3$

When $x = 1\frac{1}{2} \Rightarrow \dfrac{dy}{dx} = 2\left(1\frac{1}{2}\right) - 3 = 3 - 3 = 0$

Hence, tangent is parallel to the x-axis.

19. $y = 2x^2 - 8x + 3 \Rightarrow \dfrac{dy}{dx} = 4x - 8$

Line: $4x - y + 2 = 0 \Rightarrow$ slope $= -\dfrac{a}{b} = -\dfrac{4}{-1} = 4$

Hence, $4x - 8 = 4$

$\Rightarrow 4x = 12 \Rightarrow x = 3$

$\Rightarrow y = 2(3)^2 - 8(3) + 3 = 18 - 24 + 3 = -3 \Rightarrow$ Point (3, −3)

20. $y = 2x^2 + 3x \Rightarrow \dfrac{dy}{dx} = 4x + 3$

Tangent is parallel to the x-axis $\Rightarrow$ slope $= 0$

$\Rightarrow 4x + 3 = 0$

$\Rightarrow 4x = -3 \Rightarrow x = -\dfrac{3}{4}$

$\Rightarrow y = 2\left(\dfrac{-3}{4}\right)^2 + 3\left(\dfrac{-3}{4}\right) = \dfrac{9}{8} - \dfrac{9}{4} = \dfrac{-9}{8} \Rightarrow P = \left(\dfrac{-3}{4}, \dfrac{-9}{8}\right)$

21. $y = a\sqrt{x} + b = ax^{\frac{1}{2}} + b$

$\Rightarrow \dfrac{dy}{dx} = a \cdot \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{a}{2\sqrt{x}}$

$x = 4 \Rightarrow \dfrac{dy}{dx} = \dfrac{a}{2\sqrt{4}} = \dfrac{a}{4} = 3 \Rightarrow a = 12$

Point $(4, 6) \Rightarrow 6 = a\sqrt{4} + b$

$\Rightarrow 6 = 12.2 + b$

$\Rightarrow -b = -6 + 24 \Rightarrow b = -18$

22. $y = \dfrac{3}{x} = 3x^{-1} \Rightarrow \dfrac{dy}{dx} = -3x^{-2} = \dfrac{-3}{x^2}$

Point $\left(2, \dfrac{3}{2}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{-3}{(2)^2} = \dfrac{-3}{4}$

Equation of Tangent: $y - \dfrac{3}{2} = -\dfrac{3}{4}(x - 2)$

$\Rightarrow 4y - 6 = -3x + 6$

$\Rightarrow 3x + 4y - 12 = 0$

Point A on x-axis $\Rightarrow y = 0 \quad \Rightarrow 3x = 12$

$\Rightarrow x = 4 \Rightarrow A(4, 0)$

Point B on y-axis $\Rightarrow x = 0 \quad \Rightarrow 4y = 12$

$\Rightarrow y = 3 \Rightarrow B(0, 3)$

Area Triangle AOB $= \dfrac{1}{2}|x_1y_2 - x_2y_1|$

$= \dfrac{1}{2}|(4)(3) - 0.0|$

$\dfrac{1}{2}|12| = 6$ sq.units

Exercise 7.5

1. (i) $y = (3x + 4)(x - 2)$

Product Rule $\Rightarrow u = 3x + 4$ and $v = x - 2$

$\Rightarrow \dfrac{du}{dx} = 3 \qquad \Rightarrow \dfrac{dv}{dx} = 1$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = (3x + 4).(1) + (x - 2)(3)$

$= 3x + 4 + 3x - 6 = 6x - 2$

(ii) $y = (3x - 4)(4x + 5)$

Product Rule $\Rightarrow u = 3x - 4$ and $v = 4x + 5$

$\Rightarrow \dfrac{du}{dx} = 3 \qquad \Rightarrow \dfrac{dv}{dx} = 4$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = (3x - 4).(4) + (4x + 5).(3)$

$= 12x - 16 + 12x + 15 = 24x - 1$

(iii) $y = (x^2 + 2)(x - 1)$

Product Rule $\Rightarrow u = x^2 + 2$ and $v = x - 1$

$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^2 + 2)\,.\,(1) + (x - 1)\,.\,(2x)$$
$$= x^2 + 2 + 2x^2 - 2x = 3x^2 - 2x + 2$$

(iv) $y = (2x - 1)(x^2 - 2)$

Product Rule $\Rightarrow u = 2x - 1$ and $v = x^2 - 2$

$\Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (2x - 1)\,.\,(2x) + (x^2 - 2)\,.\,(2)$$
$$= 4x^2 - 2x + 2x^2 - 4$$
$$= 6x^2 - 2x - 4$$

(v) $y = (1 - x)(2 - x^2)$

Product Rule $\Rightarrow u = 1 - x$ and $v = 2 - x^2$

$\Rightarrow \frac{du}{dx} = -1 \qquad \Rightarrow \frac{dv}{dx} = -2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (1 - x)(-2x) + (2 - x^2)\,.\,(-1)$$
$$= -2x + 2x^2 - 2 + x^2$$
$$= 3x^2 - 2x - 2$$

(vi) $y = (x^3 - 1)(2x + 1)$

Product Rule $\Rightarrow u = x^3 - 1$ and $v = 2x + 1$

$\Rightarrow \frac{du}{dx} = 3x^2 \qquad \Rightarrow \frac{dv}{dx} = 2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^3 - 1)\,.\,(2) + (2x + 1)\,.\,(3x^2)$$
$$= 2x^3 - 2 + 6x^3 + 3x^2$$
$$= 8x^3 + 3x^2 - 2$$

2. (i) $f(x) = \frac{3x}{2x + 6}$

Quotient Rule $\Rightarrow u = 3x$ and $v = 2x + 6$

$\Rightarrow \frac{du}{dx} = 3 \qquad \Rightarrow \frac{dv}{dx} = 2$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x + 6)(3) - (3x)\,.\,(2)}{(2x + 6)^2}$$
$$= \frac{6x + 18 - 6x}{(2x + 6)^2} = \frac{18}{(2x + 6)^2}$$

(ii) $f(x) = \frac{2x + 3}{x - 1}$

Quotient Rule $\Rightarrow u = 2x + 3$ and $v = x - 1$

$\Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow \frac{dv}{dx} = 1$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x - 1)(2) - (2x + 3)\cdot(1)}{(x - 1)^2}$$
$$= \frac{2x - 2 - 2x - 3}{(x - 1)^2} = \frac{-5}{(x - 1)^2}$$

(iii) $f(x) = \dfrac{x^2}{2x+3}$

Quotient Rule $\Rightarrow u = x^2$ and $v = 2x + 3$

$\Rightarrow \dfrac{du}{dx} = 2x \qquad \Rightarrow \dfrac{dv}{dx} = 2$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x+3).(2x) - (x^2).(2)}{(2x+3)^2}$$

$$= \frac{4x^2 + 6x - 2x^2}{(2x+3)^2} = \frac{2x^2 + 6x}{(2x+3)^2}$$

(iv) $f(x) = \dfrac{2x^2 - 1}{2x - 3}$

Quotient Rule $\Rightarrow u = 2x^2 - 1$ and $v = 2x - 3$

$\Rightarrow \dfrac{du}{dx} = 4x \qquad \Rightarrow \dfrac{dv}{dx} = 2$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x-3).(4x) - (2x^2-1).(2)}{(2x-3)^2}$$

$$= \frac{8x^2 - 12x - 4x^2 + 2}{(2x-3)^2} = \frac{4x^2 - 12x + 2}{(2x-3)^2}$$

(v) $f(x) = \dfrac{2x^3}{1 - 2x}$

Quotient Rule $\Rightarrow u = 2x^3$ and $v = 1 - 2x$

$\Rightarrow \dfrac{du}{dx} = 6x^2 \qquad \Rightarrow \dfrac{dv}{dx} = -2$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1-2x).(6x^2) - (2x^3).(-2)}{(1-2x)^2}$$

$$= \frac{6x^2 - 12x^3 + 4x^3}{(1-2x)^2}$$

$$= \frac{-8x^3 + 6x^2}{(1-2x)^2}$$

(vi) $f(x) = \dfrac{3x + 2}{x^2 - 3}$

Quotient Rule $\Rightarrow u = 3x + 2$ and $v = x^2 - 3$

$\Rightarrow \dfrac{du}{dx} = 3 \qquad \Rightarrow \dfrac{dv}{dx} = 2x$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2-3).(3) - (3x+2).(2x)}{(x^2-3)^2}$$

$$= \frac{3x^2 - 9 - 6x^2 - 4x}{(x^2-3)^2}$$

$$= \frac{-3x^2 - 4x - 9}{(x^2-3)^2}$$

3. $y = \dfrac{x^2 + 1}{3x - 1}$

Quotient Rule $\Rightarrow u = x^2 + 1$ and $v = 3x - 1$

$\Rightarrow \dfrac{du}{dx} = 2x \qquad \Rightarrow \dfrac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(3x-1).(2x) - (x^2+1)\cdot(3)}{(3x-1)^2}$$

$$= \frac{6x^2 - 2x - 3x^2 - 3}{(3x-1)^2}$$

$$= \frac{3x^2 - 2x - 3}{(3x-1)^2}$$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{3(0)^2 - 2(0) - 3}{[3(0) - 1]^2} = \dfrac{-3}{(-1)^2} = \dfrac{-3}{1} = -3$

4. $y = \sqrt{x}\,.\,(2x - 1)$

Product Rule $\Rightarrow u = \sqrt{x} = x^{\frac{1}{2}}$ and $v = 2x - 1$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ $\quad \Rightarrow \dfrac{dv}{dx} = 2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (\sqrt{x})\,.\,2 + (2x - 1)\,.\,\frac{1}{2\sqrt{x}}$$
$$= \frac{(2\sqrt{x})\,.\,(2\sqrt{x}) + 2x - 1}{2\sqrt{x}}$$
$$= \frac{4x + 2x - 1}{2\sqrt{x}} = \frac{6x - 1}{2\sqrt{x}}$$

5. $y = (\sqrt{x} + 4)(\sqrt{x} - 4)$

Product Rule $\Rightarrow u = \sqrt{x} + 4 = x^{\frac{1}{2}} + 4$ and $v = \sqrt{x} - 4 = x^{\frac{1}{2}} - 4$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ $\quad \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (\sqrt{x} + 4)\,.\left(\frac{1}{2\sqrt{x}}\right) + (\sqrt{x} - 4)\,.\left(\frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{2} + \frac{4}{2\sqrt{x}} + \frac{1}{2} - \frac{4}{2\sqrt{x}} = 1$$

6. $y = \dfrac{x}{1 - x^2}$

Quotient Rule $\Rightarrow u = x$ and $v = 1 - x^2$

$\Rightarrow \dfrac{du}{dx} = 1$ $\quad \Rightarrow \dfrac{dv}{dx} = -2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1 - x^2)\,.\,(1) - (x)(-2x)}{(1 - x^2)^2}$$
$$= \frac{1 - x^2 + 2x^2}{(1 - x^2)^2} = \frac{1 + x^2}{(1 - x^2)^2} > 0$$

7. (i) $y = (x + 4)^2$

Chain Rule $\Rightarrow$ let $u = x + 4$ and $y = u^2$

$\Rightarrow \dfrac{du}{dx} = 1$ $\quad \Rightarrow \dfrac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 2u\,.\,1 = 2u = 2(x + 4)$$

(ii) $y = (2x - 1)^3$

Chain Rule $\Rightarrow$ let $u = 2x - 1$ and $y = u^3$

$\Rightarrow \dfrac{du}{dx} = 2$ $\quad \Rightarrow \dfrac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2\,.\,2 = 6u^2 = 6(2x - 1)^2$$

(iii) $y = (3x + 5)^3$

Chain Rule $\Rightarrow$ let $u = 3x + 5$ and $y = u^3$

$\Rightarrow \dfrac{du}{dx} = 3$ $\quad \Rightarrow \dfrac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2\,.\,3 = 9u^2 = 9(3x + 5)^2$$

(iv) $y = (x^2 - 1)^2$

Chain Rule $\Rightarrow$ let $u = x^2 - 1$ and $y = u^2$

$\Rightarrow \dfrac{du}{dx} = 2x$ $\quad \Rightarrow \dfrac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 2u\,.\,2x = 2(x^2 - 1)\,.\,2x = 4x\,.\,(x^2 - 1)$$

(v) $y = (2x^2 + 3)^4$

Chain Rule $\Rightarrow$ let $u = 2x^2 + 3$ and $y = u^4$

$\Rightarrow \dfrac{du}{dx} = 4x$ $\qquad \Rightarrow \dfrac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 . 4x = 4(2x^2 + 3)^3 . 4x = 16x(2x^2 + 3)^3$$

(vi) $y = (1 - 3x)^5$

Chain Rule $\Rightarrow$ let $u = 1 - 3x$ and $y = u^5$

$\Rightarrow \dfrac{du}{dx} = -3$ $\qquad \Rightarrow \dfrac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 . -3 = -15u^4 = -15(1 - 3x)^4$$

8. (i) $f(x) = \sqrt{4x + 1} = (4x + 1)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(4x + 1)^{-\frac{1}{2}} . 4 = \dfrac{2}{\sqrt{4x + 1}}$

(ii) $f(x) = \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} . 2x = \dfrac{x}{\sqrt{x^2 - 4}}$

(iii) $f(x) = \sqrt{x^3 - 2x} = (x^3 - 2x)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(x^3 - 2x)^{-\frac{1}{2}} . (3x^2 - 2)$

$$= \frac{3x^2 - 2}{2\sqrt{x^3 - 2x}}$$

9. (i) $y = 2x . (2x + 5)^3$

Product Rule $\Rightarrow$ let $u = 2x$ and $v = (2x + 5)^3$

$\Rightarrow \dfrac{du}{dx} = 2$ $\qquad \Rightarrow \dfrac{dv}{dx} = 3(2x + 5)^2 . 2$

$= 6(2x + 5)^2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 2x . 6(2x + 5)^2 + (2x + 5)^3 . 2$$

$$= 2(2x + 5)^3 + 12x(2x + 5)^2$$

(ii) $y = (x^2 - 1)(3x + 2)^2$

Product Rule $\Rightarrow$ let $u = x^2 - 1$ and $v = (3x + 2)^2$

$\Rightarrow \dfrac{du}{dx} = 2x$ $\qquad \Rightarrow \dfrac{dv}{dx} = 2(3x + 2)^1 . 3 = 6(3x + 2)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^2 - 1) . 6(3x + 2) + (3x + 2)^2 . 2x$$

$$= 6(3x + 2)(x^2 - 1) + 2x(3x + 2)^2$$

(iii) $y = (x + 4)^2 . (x - 2)$

Product Rule $\Rightarrow$ let $u = (x + 4)^2$ and $v = x - 2$

$\Rightarrow \dfrac{du}{dx} = 2(x + 4)^1 . 1$ $\qquad \Rightarrow \dfrac{dv}{dx} = 1$

$= 2(x + 4)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x + 4)^2 . 1 + (x - 2) . 2(x + 4)$$

$$= (x + 4)^2 + 2(x - 2)(x + 4)$$

10. $y = (x^2 - 3)^3 \Rightarrow \dfrac{dy}{dx} = 3(x^2 - 3)^2 . 2x = 6x(x^2 - 3)^2$

When $x = 1 \Rightarrow \dfrac{dy}{dx} = 6(1)[(1)^2 - 3]^2 = 6(-2)^2 = 24$

11. $y = \dfrac{(2x-1)^2}{3x+4}$

Quotient Rule $\Rightarrow u = (2x-1)^2$ and $v = 3x+4$

$$\frac{du}{dx} = 2(2x-1)^1 \,.\, 2 \qquad \Rightarrow \frac{dv}{dx} = 3$$
$$= 4(2x-1)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(3x+4)\,.\,4(2x-1) - (2x-1)^2\,.\,3}{(3x+4)^2}$$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{(3(0)+4)\,.\,4(2(0)-1) - (2(0)-1)^2\,.\,3}{(3(0)+4)^2}$

$$= \frac{(4)(4)(-1) - (1)(3)}{(4)^2} = \frac{-16-3}{16} = \frac{-19}{16}$$

12. $y = (2x^2-3)^7 \Rightarrow \dfrac{dy}{dx} = 7(2x^2-3)^6\,.\,4x$

At $x = -1 \Rightarrow \dfrac{dy}{dx} = 7(2(-1)^2-3)^6\,.\,4(-1)$

$$= 7(2-3)^6\,.\,(-4)$$
$$= 7(-1)^6\,.\,(-4) = 7\,.\,(1)\,.\,(-4) = -28$$

13. $f(x) = x\sqrt{x+1}$

Product Rule $\Rightarrow u = x$ and $v = \sqrt{x+1} = (x+1)^{-\frac{1}{2}}$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}$$

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx} = x \cdot \frac{1}{2\sqrt{x+1}} + \sqrt{x+1}\,.\,1$$
$$= \frac{x + 2\sqrt{x+1}\cdot\sqrt{x+1}}{2\sqrt{x+1}}$$
$$= \frac{x + 2(x+1)}{2\sqrt{x+1}}$$
$$= \frac{x+2x+2}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

14. $4x^2 + 2xy = 5$

$$\Rightarrow 2xy = 5 - 4x^2$$
$$\Rightarrow y = \frac{5-4x^2}{2x}$$

Quotient Rule $\Rightarrow u = 5 - 4x^2$ and $v = 2x$

$$\Rightarrow \frac{du}{dx} = -8x \qquad \Rightarrow \frac{dv}{dx} = 2$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x)(-8x) - (5-4x^2)(2)}{(2x)^2}$$
$$= \frac{-16x^2 - 10 + 8x^2}{4x^2}$$
$$= \frac{-8x^2-10}{4x^2} = \frac{-4x^2-5}{2x^2}$$

15. $y = \dfrac{x}{\sqrt{x}+1}$

Quotient Rule $\Rightarrow u = x$ and $v = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(\sqrt{x}+1)\,.\,(1) - x\,.\,\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{2\sqrt{x}(\sqrt{x}+1)-x}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{2x+2\sqrt{x}-x}{2\sqrt{x}(\sqrt{x}+1)^2}$$

$$= \frac{x+2\sqrt{x}}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{\sqrt{x}(\sqrt{x}+2)}{2\sqrt{x}(\sqrt{x}+1)^2}$$

$$= \frac{\sqrt{x}+2}{2(\sqrt{x}+1)^2}$$

At $x = 1 \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1}+2}{2(\sqrt{1}+1)^2} = \frac{1+2}{2(2)^2} = \frac{3}{8}$

16. $y = \frac{3x+1}{1-2x}$

Quotient Rule $\Rightarrow u = 3x + 1$ and $v = 1 - 2x$

$\Rightarrow \frac{du}{dx} = 3$ $\quad\Rightarrow \frac{dv}{dx} = -2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1-2x).3 - (3x+1).(-2)}{(1-2x)^2}$$

$$= \frac{3-6x+6x+2}{(1-2x)^2} = \frac{5}{(1-2x)^2}$$

17. $f(x) = \sqrt{3x^2-2} = (3x^2-2)^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2}(3x^2-2)^{-\frac{1}{2}} \cdot 6x = \frac{3x}{\sqrt{3x^2-2}}$

When $x = 1 \Rightarrow f'(1) = \frac{3(1)}{\sqrt{3(1)^2-2}} = \frac{3}{\sqrt{1}} = \frac{3}{1} = 3$

18. $y = (x-1)^{\frac{3}{2}} - 3(x-1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}(x-1)^{\frac{1}{2}} - 3 \cdot \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$= \frac{3\sqrt{x-1}}{2} - \frac{3}{2\sqrt{x-1}} = \frac{3(\sqrt{x-1})(\sqrt{x-1})-3}{2\sqrt{x-1}}$$

$$= \frac{3(x-1)-3}{2\sqrt{x-1}}$$

$$= \frac{3x-3-3}{2\sqrt{x-1}}$$

$$= \frac{3x-6}{2\sqrt{x-1}}$$

$$= \frac{3(x-2)}{2\sqrt{x-1}}$$

19. $y = ax^3 + 2bx^2 + 3cx$

$\frac{dy}{dx} = 3ax^2 + 4bx + 3c = 6x^2 + 6x - 6$

$\Rightarrow 3a = 6$ and $4b = 6$ and $3c = -6$

$\Rightarrow a = 2$ $\quad\Rightarrow b = \frac{6}{4} = 1\frac{1}{2}$ $\quad\Rightarrow c = -2$

20. $y = \frac{4x}{x+3}$

Quotient Rule $\Rightarrow u = 4x$ and $v = x + 3$

$\Rightarrow \frac{du}{dx} = 4$ $\quad\Rightarrow \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+3).(4) - (4x).1}{(x+3)^2}$$

$$= \frac{4x + 12 - 4x}{(x+3)^2} = \frac{12}{(x+3)^2}$$

$$f(x) = \sqrt{\frac{4x}{x+3}} = \left(\frac{4x}{x+3}\right)^{\frac{1}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{2}\left(\frac{4x}{x+3}\right)^{-\frac{1}{2}} \cdot \frac{12}{(x+3)^2}$$

$$= \frac{6}{\sqrt{\frac{4x}{x+3}} \cdot (x+3)^2}$$

$$f'(1) = \frac{6}{\sqrt{\frac{4(1)}{1+3}} \cdot (1+3)^2}$$

$$= \frac{6}{\sqrt{\frac{4}{4}} \cdot 16}$$

$$= \frac{6}{1.16} = \frac{3}{8}$$

21. $y = (8 - 2x^2)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}(8 - 2x^2)^{-\frac{1}{3}} \cdot (-4x)$$

$$= \frac{-8x}{3}(8 - 2x^2)^{-\frac{1}{3}} \quad \Rightarrow \text{Answer is A}$$

22. $f(x) = 3x + 1$ and $g(x) = x^2 - 2$

(a) (i) $p(x) = f(g(x))$

$= f(x^2 - 2)$

$= 3(x^2 - 2) + 1$

$= 3x^2 - 6 + 1 = 3x^2 - 5$

(ii) $q(x) = g(f(x))$

$= g(3x + 1)$

$= (3x + 1)^2 - 2$

$= 9x^2 + 6x + 1 - 2 = 9x^2 + 6x - 1$

(b) $p(x) = 3x^2 - 5 \Rightarrow p'(x) = 6x$

$q(x) = 9x^2 + 6x - 1 \Rightarrow q'(x) = 18x + 6$

Solve $p'(x) = q'(x)$

$\Rightarrow 6x = 18x + 6$

$\Rightarrow -12x = 6$

$\Rightarrow -2x = 1$

$\Rightarrow x = -\frac{1}{2}$

Exercise 7.6

1. $y = x^3 + 2x^2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 4x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x + 4$$

2. $y = x^4 - 3x^2 + 6$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - 6x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 12x^2 - 6$$

3. $y = \dfrac{1}{x} = x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = -1x^{-2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2x^{-3} = \dfrac{2}{x^3}$

4. $y = \dfrac{1}{x^2} + 3x^2 = x^{-2} + 3x^2$

$\Rightarrow \dfrac{dy}{dx} = -2x^{-3} + 6x$

$\Rightarrow \dfrac{d^2y}{dx^2} = 6x^{-4} + 6 = \dfrac{6}{x^4} + 6$

5. $y = 3x + \dfrac{1}{x} + 4 = 3x + x^{-1} + 4$

$\Rightarrow \dfrac{dy}{dx} = 3 - x^{-2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2x^{-3} = \dfrac{2}{x^3}$

6. $y = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{1}{4}x^{-\frac{3}{2}} = -\dfrac{1}{4\sqrt{x^3}}$

7. $y = \sqrt{2x+3} = (2x+3)^{\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}(2x+3)^{-\frac{1}{2}} \cdot 2 = (2x+3)^{-\frac{1}{2}}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{1}{2}(2x+3)^{-\frac{3}{2}} \cdot 2 = \dfrac{-1}{\sqrt{(2x+3)^3}}$

8. $y = (3x-2)^3$

$\Rightarrow \dfrac{dy}{dx} = 3(3x-2)^2 \,.\, 3 = 9(3x-2)^2$

$\Rightarrow \dfrac{d^2y}{dx^2} = 18(3x-2)^1 \,.\, 3 = 54(3x-2)$

9. $y = \dfrac{1}{x+4} = (x+4)^{-1}$

$\Rightarrow \dfrac{dy}{dx} = -1(x+4)^{-2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2(x+4)^{-3} = \dfrac{2}{(x+4)^3}$

10. $y = x^4 - x^3 + 4x - 1$

$\Rightarrow \dfrac{dy}{dx} = 4x^3 - 3x^2 + 4$

$\Rightarrow \dfrac{d^2y}{dx^2} = 12x^2 - 6x$

Solve $\Rightarrow \dfrac{d^2y}{dx^2} = 0 \quad \Rightarrow 12x^2 - 6x = 0$

$\Rightarrow 6x(2x-1) = 0$

$\Rightarrow x = 0, \dfrac{1}{2}$

11. $y = 3x + \dfrac{4}{x} = 3x + 4x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = 3 - 4x^{-2} = 3 - \dfrac{4}{x^2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 8x^{-3} = \dfrac{8}{x^3}$

Hence, $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y$

$= x^2 \cdot \left(\dfrac{8}{x^3}\right) + x \cdot \left(3 - \dfrac{4}{x^2}\right) - \left(3x + \dfrac{4}{x}\right)$

$= \dfrac{8}{x} + 3x - \dfrac{4}{x} - 3x - \dfrac{4}{x}$

$= \dfrac{8}{x} - \dfrac{8}{x} = 0$

12. $f(x) = \dfrac{2}{x} + 4\sqrt{x} = 2x^{-1} + 4x^{\frac{1}{2}}$

$\Rightarrow f'(x) = -2x^{-2} + 2x^{-\frac{1}{2}}$

$\Rightarrow f''(x) = 4x^{-3} - x^{-\frac{3}{2}} = \dfrac{4}{x^3} - \dfrac{1}{\sqrt{x^3}}$

Hence, $f''(4) = \dfrac{4}{(4)^3} - \dfrac{1}{\sqrt{(4)^3}}$

$= \dfrac{4}{64} - \dfrac{1}{\sqrt{64}} = \dfrac{1}{16} - \dfrac{1}{8} = -\dfrac{1}{16}$

13. $y = x^4 \quad \Rightarrow \dfrac{dy}{dx} = 4x^3 \quad \Rightarrow \dfrac{d^2y}{dx^2} = 12x^2$

Hence, $\dfrac{4x^4}{3}\left(\dfrac{d^2y}{dx^2}\right) - \left(\dfrac{dy}{dx}\right)^2$

$= \dfrac{4x^4}{3}(12x^2) - (4x^3)^2$

$= \dfrac{48}{3}x^6 - 16x^6$

$= 16x^6 - 16x^6 = 0$

14. $y = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{2}x^{-\frac{3}{2}} = -\dfrac{1}{2\sqrt{x^3}}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{3}{4}x^{-\frac{5}{2}} = \dfrac{3}{4\sqrt{x^5}}$

Hence, $2x\left(\dfrac{d^2y}{dx^2}\right) + 3\dfrac{dy}{dx}$

$= 2x \cdot \left(\dfrac{3}{4}x^{-\frac{5}{2}}\right) + 3\left(-\dfrac{1}{2}x^{-\frac{3}{2}}\right)$

$= \dfrac{3}{2}x^{-\frac{3}{2}} - \dfrac{3}{2}x^{-\frac{3}{2}} = 0$

Exercise 7.7

1. (i) $y = \sin 2x \quad \Rightarrow \dfrac{dy}{dx} = \cos 2x \,.\, 2 = 2\cos 2x$

(ii) $y = \cos 6x \quad \Rightarrow \dfrac{dy}{dx} = -\sin 6x \,.\, 6 = -6\sin 6x$

(iii) $y = \tan 4x \quad \Rightarrow \dfrac{dy}{dx} = \sec^2 4x \,.\, 4 = 4\sec^2 4x$

(iv) $y = \sin(2x + 3) \quad \Rightarrow \dfrac{dy}{dx} = \cos(2x + 3) \,.\, 2 = 2\cos(2x + 3)$

(v) $y = \cos(3x - 1) \quad \Rightarrow \dfrac{dy}{dx} = -\sin(3x - 1) \,.\, 3 = -3\sin(3x - 1)$

(vi) $y = \tan(x^2) \quad \Rightarrow \dfrac{dy}{dx} = \sec^2(x^2) \,.\, 2x = 2x\sec^2(x^2)$

(vii) $y = \sin\left(\dfrac{1}{2}x\right) \quad \Rightarrow \dfrac{dy}{dx} = \cos\left(\dfrac{1}{2}x\right) \cdot \dfrac{1}{2} = \dfrac{1}{2}\cos\dfrac{1}{2}x$

(viii) $y = \cos(x^2 - 1) \quad \Rightarrow \dfrac{dy}{dx} = -\sin(x^2 - 1) \,.\, 2x = -2x\sin(x^2 - 1)$

(ix) $y = \sin 2x + \cos 4x \quad \Rightarrow \dfrac{dy}{dx} = \cos 2x \,.\, 2 - \sin 4x \,.\, 4$

$= 2\cos 2x - 4\sin 4x$

2. (i) $y = \sin^2 x \quad \Rightarrow \dfrac{dy}{dx} = 2\sin x\cos x$

(ii) $y = \cos^3 x \quad \Rightarrow \dfrac{dy}{dx} = 3\cos^2 x \,.\, -\sin x = -3\cos^2 x\sin x$

(iii) $y = \tan^4 x \quad \Rightarrow \dfrac{dy}{dx} = 4\tan^3 x\sec^2 x$

(iv) $y = \sin^3(4x) \quad \Rightarrow \dfrac{dy}{dx} = 3\sin^2(4x) \,.\, \cos(4x) \,.\, 4$

$= 12\sin^2(4x)\cos(4x)$

(v) $y = \cos^2(2x + 1) \quad \Rightarrow \dfrac{dy}{dx} = 2\cos(2x + 1) \,.\, -\sin(2x + 1) \,.\, 2$

$= -4\cos(2x + 1)\sin(2x + 1)$

(vi) $y = \tan^3(4x + 3) \quad \Rightarrow \dfrac{dy}{dx} = 3\tan^2(4x + 3) \,.\, \sec^2(4x + 3) \,.\, 4$

$= 12\tan^2(4x + 3)\sec^2(4x + 3)$

3. (i) $y = 2\sin 3\theta + \cos 2\theta \quad \Rightarrow \dfrac{dy}{d\theta} = 2\cos 3\theta \,.\, 3 - \sin 2\theta \,.\, 2$

$= 6\cos 3\theta - 2\sin 2\theta$

(ii) $y = \tan^2\theta + \tan 2\theta$

$\Rightarrow \dfrac{dy}{d\theta} = 2\tan\theta \,.\, \sec^2\theta + \sec^2 2\theta \,.\, 2$

$= 2\tan\theta\sec^2\theta + 2\sec^2 2\theta$

(iii) $y = \cos 4\theta - \cos\dfrac{\theta}{4}$

$\dfrac{dy}{d\theta} = -\sin 4\theta \,.\, 4 + \sin\dfrac{\theta}{4} \cdot \dfrac{1}{4}$

$= -4\sin 4\theta + \dfrac{1}{4}\sin\dfrac{\theta}{4}$

(iv) $y = \tan^3\theta + 5$

$\Rightarrow \dfrac{dy}{d\theta} = 3\tan^2\theta \,.\, \sec^2\theta$

4. (i) $y = x \sin 2x$

Product Rule $\Rightarrow$ let $u = x$ and $v = \sin 2x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \cos 2x \,.\, 2$$
$$= 2\cos 2x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x \,.\, 2\cos 2x + \sin 2x \,.\, 1$$
$$= 2x\cos 2x + \sin 2x$$

(ii) $y = x^2 \cos x$

Product Rule $\Rightarrow$ let $u = x^2$ and $v = \cos x$

$$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^2 \,.\, -\sin x + \cos x \,.\, 2x$$
$$= 2x\cos x - x^2 \sin x$$

(iii) $y = (x + 3)\sin x$

Product Rule $\Rightarrow$ let $u = x + 3$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x + 3) \,.\, \cos x + \sin x \,.\, 1$$
$$= (x + 3)\cos x + \sin x$$

5. $y = \sin x \cos x$

Product Rule $\Rightarrow$ let $u = \sin x$ and $v = \cos x$

$$\Rightarrow \frac{du}{dx} = \cos x \qquad \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \sin x \,.\, -\sin x + \cos x \,.\, \cos x$$
$$= \cos^2 x - \sin^2 x$$
$$= \cos 2x$$

6. $f(x) = \cos x \,.\, \tan x$

$$= \cos x \cdot \frac{\sin x}{\cos x}$$
$$= \sin x$$

$\Rightarrow f'(x) = \cos x$

7. (i) $y = \sin 2x \qquad \Rightarrow \dfrac{dy}{dx} = \cos 2x \,.\, 2 = 2\cos 2x$

At $x = \pi \qquad \Rightarrow \dfrac{dy}{dx} = 2\cos 2\pi = 2 \,.\, 1 = 2$

(ii) $y = x \cos x$

Product Rule $\Rightarrow u = x$ and $v = \cos x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x \,.\, -\sin x + \cos x \,.\, 1$$
$$= -x\sin x + \cos x$$

At $x = \pi \qquad \Rightarrow \dfrac{dy}{dx} = -\pi \sin \pi + \cos \pi$

$$= -\pi(0) - 1 = -1$$

(iii) $y = \sin^2 x \quad \Rightarrow \dfrac{dy}{dx} = 2\sin x \cos x$

At $x = \pi \quad \Rightarrow \dfrac{dy}{dx} = 2\sin \pi \,.\, \cos \pi = 2(0)(-1) = 0$

8. $y = \tan x = \dfrac{\sin x}{\cos x}$

Quotient Rule $\Rightarrow u = \sin x$ and $v = \cos x$

$\Rightarrow \dfrac{du}{dx} = \cos x \qquad \Rightarrow \dfrac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\cos x\,.\,(\cos x) - \sin x\,.\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

9. $f(x) = (\sin x + 1)^2$

$\Rightarrow f'(x) = 2(\sin x + 1)\,.\,\cos x$

$\Rightarrow f'\left(\dfrac{\pi}{6}\right) = 2\left(\sin\dfrac{\pi}{6} + 1\right)\cdot\cos\dfrac{\pi}{6}$

$= 2\left(\dfrac{1}{2} + 1\right)\cdot\dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2}$

10. $y = \sin x + 3\cos x$

$\Rightarrow \dfrac{dy}{dx} = \cos x + 3(-\sin x) = \cos x - 3\sin x$

Hence, $\cos x\dfrac{dy}{dx} + y\sin x$

$= \cos x(\cos x - 3\sin x) + (\sin x + 3\cos x)\sin x$

$= \cos^2 x - 3\sin x\cos x + \sin^2 x + 3\sin x\cos x$

$= 1$

11. $y = \sin 2x - 2x$

$\Rightarrow \dfrac{dy}{dx} = \cos 2x\,.\,2 - 2 = 2\cos 2x - 2$

$= 2(1 - 2\sin 2x) - 2$

$= 2 - 4\sin 2x - 2 = -4\sin 2x$

12. $y = \cos\left(\dfrac{1}{4}\pi x\right)$

$\Rightarrow \dfrac{dy}{dx} = -\sin\left(\dfrac{1}{4}\pi x\right)\cdot\dfrac{1}{4}\pi$

At $x = 4 \quad \Rightarrow \dfrac{dy}{dx} = -\sin\left(\dfrac{1}{4}\cdot\pi\,.\,4\right)\cdot\dfrac{1}{4}\pi$

$= -\sin\pi\cdot\dfrac{1}{4}\pi = 0\cdot\dfrac{1}{4}\pi = 0$

13. $f(x) = \cos^3(2x) \quad \Rightarrow f'(x) = 3\cos^2(2x)\,.\,-\sin(2x)\,.\,2$

$= -6\cos^2(2x)\,.\,\sin(2x)$

At $x = \dfrac{\pi}{6} \quad \Rightarrow f'\left(\dfrac{\pi}{6}\right) = -6\cos^2\left(2\cdot\dfrac{\pi}{6}\right)\sin\left(2\cdot\dfrac{\pi}{6}\right)$

$= -6\cos^2\dfrac{\pi}{3}\sin\dfrac{\pi}{3} = -6\left(\dfrac{1}{2}\right)^2\cdot\dfrac{\sqrt{3}}{2} = -\dfrac{3\sqrt{3}}{4}$

14. (i) $f(x) = \cos 2x \quad \Rightarrow f'(x) = -\sin 2x\,.\,2 = -2\sin 2x$

(ii) $g(x) = 2\sin^2 x \quad \Rightarrow g'(x) = 4\sin x\,.\,\cos x$

Hence, $f'(x) + g'(x) = -2\sin 2x + 4\sin x\cos x$

$= -2\,.\,2\sin x\cos x + 4\sin x\cos x$

$= -4\sin x\cos x + 4\sin x\cos x = 0$

15. $y = \sin 3x \Rightarrow \dfrac{dy}{dx} = \cos 3x \,.\, 3 = 3\cos 3x$

$$\Rightarrow \frac{d^2y}{dx^2} = 3(-\sin 3x \,.\, 3)$$
$$= -9\sin 3x = -9y$$

16. $y = \tan x + \dfrac{1}{3}\tan^3 x$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x + \frac{1}{3}\cdot 3\tan^2 x \,.\, \sec^2 x$$
$$= \sec^2 x + \tan^2 x \,.\, \sec^2 x$$
$$= \sec^2 x(1 + \tan^2 x) = \sec^2 x \,.\, \sec^2 x = \sec^4 x$$

17. $y = 3\sin x + k\sin 3x$

$$\Rightarrow \frac{dy}{dx} = 3\cos x + k\cos 3x \,.\, 3$$
$$= 3\cos x + 3k\cos 3x$$

When $x = \dfrac{\pi}{3}$ $\Rightarrow \dfrac{dy}{dx} = 3\cos\dfrac{\pi}{3} + 3k\cos 3\cdot\left(\dfrac{\pi}{3}\right) = 0$

$$\Rightarrow 3\cos\frac{\pi}{3} + 3k\cos\pi = 0$$
$$\Rightarrow 3\cdot\frac{1}{2} + 3k(-1) = 0$$
$$\Rightarrow \frac{3}{2} - 3k = 0$$
$$\Rightarrow 3 - 6k = 0$$
$$\Rightarrow \quad -6k = -3$$
$$\Rightarrow \quad k = \frac{1}{2}$$

18. $y = \dfrac{\sin x}{2 + \cos x}$

Quotient Rule $\Rightarrow u = \sin x$ and $v = 2 + \cos x$

$$\Rightarrow \frac{du}{dx} = \cos x \qquad \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(2 + \cos x)\cos x - \sin x(-\sin x)}{(2 + \cos x)^2}$$
$$= \frac{2\cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2}$$
$$= \frac{1 + 2\cos x}{(2 + \cos x)^2} = \frac{a + b\cos x}{(2 + \cos x)^2}$$
$$\Rightarrow a = 1, b = 2$$

Exercise 7.8

1. (i) $y = \sin^{-1} 6x$

Chain Rule $\Rightarrow u = 6x$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 6 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1 - u^2}}\cdot 6 = \frac{6}{\sqrt{1 - 36x^2}}$$

(ii) $y = \tan^{-1} 3x$

Chain Rule $\Rightarrow u = 3x$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 3 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1 + u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1 + u^2}\cdot 3 = \frac{3}{1 + 9x^2}$$

(iii) $y = \sin^{-1}(2x + 1)$

Chain Rule $\Rightarrow u = 2x + 1$ and $y = \sin^{-1} u$

$\Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2 = \frac{2}{\sqrt{1-(2x+1)^2}}$$

$$= \frac{2}{\sqrt{1-(4x^2+4x+1)}}$$

$$= \frac{2}{\sqrt{1-4x^2-4x-1}} = \frac{2}{\sqrt{-4x^2-4x}}$$

(iv) $y = \tan^{-1}(x^2)$

Chain Rule $\Rightarrow u = x^2$ and $y = \tan^{-1} u$

$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1 + u^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 2x = \frac{2}{1+(x^2)^2} = \frac{2x}{1+x^4}$$

2. $y = \sin^{-1}(3x - 1)$

Chain Rule $\Rightarrow u = 3x - 1$ and $y = \sin^{-1} u$

$\Rightarrow \frac{du}{dx} = 3 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 3 = \frac{3}{\sqrt{1-(3x-1)^2}}$$

$$= \frac{3}{\sqrt{1-(9x^2-6x+1)}}$$

$$= \frac{3}{\sqrt{1-9x^2+6x-1}}$$

$$= \frac{3}{\sqrt{6x-9x^2}}$$

3. (i) $y = \sin^{-1} 2x$

Chain Rule $\Rightarrow u = 2x$ and $y = \sin^{-1} u$

$\Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

At $x = 0$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4(0)^2}} = \frac{2}{\sqrt{1-0}} = \frac{2}{1} = 2$

(ii) $y = \tan^{-1} 4x$

Chain Rule $\Rightarrow u = 4x$ and $y = \tan^{-1} u$

$\Rightarrow \frac{du}{dx} = 4 \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1 + u^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 4 = \frac{4}{1+16x^2}$$

At $x = \frac{1}{4}$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{4}{1+16\left(\frac{1}{4}\right)^2} = \frac{4}{1+16 \cdot \frac{1}{16}} = \frac{4}{2} = 2$

4. (i) $f(x) = \sin^{-1}\frac{3}{x}$

Chain Rule $\Rightarrow u = \frac{3}{x} = -3x^{-2}$ and $y = \sin^{-1} u$

$\Rightarrow \frac{du}{dx} = -3x^{-2} = \frac{-3}{x^2} \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{-3}{x^2} = \frac{-3}{x^2\sqrt{1-\frac{9}{x^2}}}$$

$$= \frac{-3}{x^2\sqrt{\frac{x^2-9}{x^2}}}$$

$$= \frac{-3}{x^2\frac{\sqrt{x^2-9}}{\sqrt{x^2}}}$$

$$= \frac{-3}{x^2\frac{\sqrt{x^2-9}}{x}}$$

$$= \frac{-3}{x\sqrt{x^2-9}}$$

(ii) $f(x) = \tan^{-1}\frac{x}{4}$

Chain Rule $\Rightarrow u = \frac{x}{4}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = \frac{1}{4} \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot \frac{1}{4}$$

$$= \frac{1}{1+\left(\frac{x}{4}\right)^2} \cdot \frac{1}{4}$$

$$= \frac{1}{1+\frac{x^2}{16}} \cdot \frac{1}{4}$$

$$= \frac{1}{\frac{16+x^2}{16}} \cdot \frac{1}{4} = \frac{1}{\frac{x^2+16}{4}} = \frac{4}{x^2+16}$$

5. (i) $y = x\sin^{-1} x$

Product Rule $\Rightarrow u = x$ and $v = \sin^{-1} x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \,.\, 1$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$$

(ii) $y = 2x \,.\, \tan^{-1} x$

Product Rule $\Rightarrow u = 2x$ and $v = \tan^{-1} x$

$$\Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 2x \cdot \frac{1}{1+x^2} + \tan^{-1} x \,.\, 2$$

$$= 2\tan^{-1} x + \frac{2x}{1+x^2}$$

6. $y = (\sin^{-1} x)^2$

Chain Rule $\Rightarrow u = \sin^{-1} x$ and $y = u^2$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \frac{1}{\sqrt{1-x^2}} = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

7. $y = f(x) = \sin^{-1}(\cos x)$

Chain Rule $\Rightarrow u = \cos x$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = -\sin x \qquad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\cdot -\sin x$$

$$= \frac{1}{\sqrt{1-\cos^2 x}}\cdot -\sin x$$

$$= \frac{1}{\sqrt{\sin^2 x}}\cdot -\sin x$$

$$= \frac{-\sin x}{\sin x} = -1 \Rightarrow k = -1$$

8. $y = f(x) = \tan^{-1}(\cos x)$

Chain Rule $\Rightarrow u = \cos x$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -\sin x \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+u^2}\cdot -\sin x$$

$$= \frac{-\sin x}{1+\cos^2 x}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{-\sin\frac{\pi}{6}}{1+\cos^2\frac{\pi}{6}} = \frac{-\frac{1}{2}}{1+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{-\frac{1}{2}}{1+\frac{3}{4}}$$

$$= \frac{-\frac{1}{2}}{\frac{7}{4}} = -\frac{2}{7}$$

9. $y = \tan^{-1}\frac{1}{x}$

Chain Rule $\Rightarrow u = \frac{1}{x} = x^{-1}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -x^{-2} = \frac{-1}{x^2} \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+u^2}\cdot\frac{-1}{x^2}$$

$$= \frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2} = \frac{-1}{x^2+1}$$

At $x = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{-1}{(1)^2+1} = -\frac{1}{2}$

10. $y = \tan^{-1}(3x^2)$

Chain Rule $\Rightarrow u = 3x^2$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 6x \qquad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+u^2}\cdot 6x = \frac{1}{1+(3x^2)^2}\cdot 6x$$

$$= \frac{6x}{1+9x^4}$$

At $x = \frac{1}{3} \quad \Rightarrow \frac{dy}{dx} = \frac{6\left(\frac{1}{3}\right)}{1+9\left(\frac{1}{3}\right)^4} = \frac{2}{1+9\cdot\frac{1}{81}} = \frac{2}{1\frac{1}{9}} = \frac{9}{5}$

11. $y = \tan^{-1} x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1 + x^2} = (1 + x^2)^{-1}$

$$\Rightarrow \frac{d^2y}{dx^2} = -1(1 + x^2)^{-2} . 2x = \frac{-2x}{(1 + x^2)^2}$$

Hence, $\dfrac{d^2y}{dx^2}(1 + x^2) + 2x\dfrac{dy}{dx}$

$$= \frac{-2x}{(1 + x^2)^2} \cdot (1 + x^2) + 2x \cdot \frac{1}{1 + x^2}$$

$$= \frac{-2x}{1 + x^2} + \frac{2x}{1 + x^2} = 0$$

Exercise 7.9

1. (i) $y = e^{4x} \Rightarrow \dfrac{dy}{dx} = e^{4x} . 4 = 4e^{4x}$

(ii) $y = e^{-3x} \Rightarrow \dfrac{dy}{dx} = e^{-3x} . -3 = -3e^{-3x}$

(iii) $y = e^{x^2} \Rightarrow \dfrac{dy}{dx} = e^{x^2} . 2x = 2xe^{x^2}$

(iv) $y = e^{2x+4} \Rightarrow \dfrac{dy}{dx} = e^{2x+4} . 2 = 2e^{2x+4}$

(v) $y = e^{x^2+3x} \Rightarrow \dfrac{dy}{dx} = e^{x^2+3x} . (2x + 3) = (2x + 3)e^{x^2+3x}$

(vi) $y = e^{\sin x} \Rightarrow \dfrac{dy}{dx} = e^{\sin x} . \cos x = \cos xe^{\sin x}$

2. (i) $y = e^{\frac{x}{2}} \Rightarrow \dfrac{dy}{dx} = e^{\frac{x}{2}} \cdot \dfrac{1}{2} = \dfrac{1}{2}e^{\frac{x}{2}}$

(ii) $y = e^{\sin^2 x} \Rightarrow \dfrac{dy}{dx} = \left(e^{\sin^2 x}\right)2 \sin x . \cos x$

$$= 2 \sin x \cos x\left(e^{\sin^2 x}\right)$$

(iii) $y = xe^{2x}$

Product Rule $\Rightarrow u = x$ and $v = e^{2x}$

$\Rightarrow \dfrac{du}{dx} = 1$ $\quad \Rightarrow \dfrac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x . 2e^{2x} + e^{2x} . 1$$

$$= e^{2x}(1 + 2x)$$

3. (i) $y = e^{2x} \sin x$

Product Rule $\Rightarrow u = e^{2x}$ and $v = \sin x$

$\Rightarrow \dfrac{du}{dx} = e^{2x} . 2 = 2e^{2x}$ $\quad \Rightarrow \dfrac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{2x} . \cos x + \sin x . 2e^{2x}$$

$$= e^{2x}(2 \sin x + \cos x)$$

(ii) $y = (e^x - 1)^2$

Chain Rule $\Rightarrow u = e^x - 1$ and $y = u^2$

$\Rightarrow \dfrac{du}{dx} = e^x$ $\quad \Rightarrow \dfrac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u . e^x$$

$$= 2(e^x - 1)e^x = 2e^x(e^x - 1)$$

(iii) $y = \dfrac{e^{2x+1}}{e^x} = e^{2x+1-x} = e^{x+1}$

$\Rightarrow \dfrac{dy}{dx} = e^{x+1}$

4. (i) $y = e^{2x}(1 + e^x) = e^{2x} + e^{3x}$

$\Rightarrow \dfrac{dy}{dx} = 2e^{2x} + 3e^{3x}$

(ii) $t = \dfrac{e^{2x}}{x}$

Quotient Rule $\Rightarrow u = e^{2x}$ and $v = x$

$\Rightarrow \dfrac{du}{dx} = 2e^{2x}$ $\qquad \Rightarrow \dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x\,.\,2e^{2x} - e^{2x}\,.\,1}{x^2}$$

$$= \frac{e^{2x}(2x-1)}{x^2}$$

(iii) $y = x^2e^{\cos x}$

Product Rule $\Rightarrow u = x^2$ and $v = e^{\cos x}$

$\Rightarrow \dfrac{du}{dx} = 2x$ $\qquad \Rightarrow \dfrac{dv}{dx} = e^{\cos x} - \sin x$

$= -\sin xe^{\cos x}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$= x^2(-\sin x\,.\,e^{\cos x}) + e^{\cos x}(2x)$

$= -x^2 \sin x\,.\,e^{\cos x} + 2xe^{\cos x}$

$= xe^{\cos x}(-x\sin x + 2)$

5. (i) $y = e^{3x} \sin(\pi x)$

Product Rule $\Rightarrow u = e^{3x}$ and $v = \sin(\pi x)$

$\Rightarrow \dfrac{du}{dx} = 3e^{3x}$ $\qquad \Rightarrow \dfrac{dv}{dx} = \cos(\pi x)\,.\,\pi$

$= \pi\cos(\pi x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{3x}\,.\,\pi\cos(\pi x) + \sin(\pi x)\,.\,3e^{3x}$$

At $x = 1 \Rightarrow \dfrac{dy}{dx} = e^{3(1)}\,.\,\pi\cos(\pi\,.\,1) + \sin(\pi\,.\,1)\,.\,3e^{3(1)}$

$= e^3\,.\,\pi(-1) + 0.3e^3$

$= -\pi e^3$

6. $y = e^{2x} \Rightarrow \dfrac{dy}{dx} = 2e^{2x}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\,.\,2e^{2x} = 4e^{2x}$

Hence, $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y$

$= 4e^{2x} - 3(2e^{2x}) + 2(e^{2x})$

$= 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$

7. $y = e^x(\cos x - \sin x)$

Product Rule $\Rightarrow u = e^x$ and $v = \cos x - \sin x$

$\Rightarrow \dfrac{du}{dx} = e^x$ $\qquad \Rightarrow \dfrac{dv}{dx} = -\sin x - \cos x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$$
$$= -e^x \sin x - e^x \cos x + e^x \cos x - e^x \sin x$$
$$= -2e^x \sin x$$

8. $y = xe^x$

Product Rule $\Rightarrow u = x$ and $v = e^x$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow \frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x \,.\, e^x + e^x \,.\, 1 = e^x(x + 1)$$
$$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx} = e^x(1) + (x + 1)e^x = e^x(1 + x + 1) = e^x(x + 2)$$
$$\frac{d^2y}{dx^2} + y = e^x(x + 2) + xe^x = e^x(x + 2 + x) = e^x(2x + 2) = 2e^x(x + 1)$$
$$= 2\frac{dy}{dx}$$

9. $f(x) = e^{2x} - ae^x$

$\Rightarrow f'(x) = 2e^{2x} - ae^x$ OR $2(e^x)^2 - ae^x$

When $e^x = \frac{a}{2} \Rightarrow f'(x) = 2\left(\frac{a}{2}\right)^2 - a\left(\frac{a}{2}\right)$

$$= 2\frac{a^2}{4} - \frac{a^2}{2} = \frac{a^2}{2} - \frac{a^2}{2} = 0$$

10. $y = e^{mx} \Rightarrow \frac{dy}{dx} = e^{mx} \,.\, m = me^{mx}$

$\Rightarrow \frac{d^2y}{dx^2} = me^{mx} \,.\, m = m^2e^{mx}$

Hence, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

$\Rightarrow m^2e^{mx} - 3me^{mx} - 4e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 - 3m - 4) = 0$

$\Rightarrow (m + 1)(m - 4) = 0$

$\Rightarrow m = -1, m = 4$

11. $f(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$

$\Rightarrow f'(x) = \frac{1}{2}(e^x + e^{-x} \,.\, -1) = \frac{1}{2}(e^x - e^{-x})$

$\Rightarrow f''(x) = \frac{1}{2}(e^x - e^{-x} \,.\, -1)$

$= \frac{1}{2}(e^x + e^{-x}) = f(x)$

12. $y = 3e^x - \sin x + 5$

$\Rightarrow \frac{dy}{dx} = 3e^x - \cos x$

At $x = 0 \Rightarrow \frac{dy}{dx} = 3e^0 - \cos 0$

$= 3.(1) - 1 = 2$

At $x = 0 \Rightarrow$ $y = 3e^0 - \sin 0 + 5$

$= 3(1) - 0 + 5 = 8$

Slope $= 2$ Point $= (0, 8)$

$\Rightarrow$ equation of tangent: $y - 8 = 2(x - 0)$

$\Rightarrow y - 8 = 2x$

$\Rightarrow y = 2x + 8$

13. $\ell_1 : y = 2e^x - x$

$\Rightarrow \dfrac{dy}{dx} = 2e^x - 1$

Point $(0, 2) \Rightarrow \dfrac{dy}{dx} = 2e^o - 1 = 2(1) - 1 = 1$

$\Rightarrow$ equation of tangent: $y - 2 = 1(x - 0)$

$\Rightarrow y = x + 2$

$\ell_2 : y = \sin 2x - x^2$

$\Rightarrow \dfrac{dy}{dx} = \cos 2x \,.\, 2 - 2x = 2\cos 2x - 2x$

Point $(0, 0) \Rightarrow \dfrac{dy}{dx} = 2\cos 2(0) - 2(0)$

$= 2\cos 0 - 0 = 2.1 = 2$

$\Rightarrow$ equation of tangent: $y - 0 = 2(x - 0)$

$\Rightarrow y = 2x$

Point of intersection: $y = 2x \cap y = x + 2$

$\Rightarrow 2x = x + 2$

$\Rightarrow x = 2$

$\Rightarrow y = 2(2) = 4 \Rightarrow$ Point $= (2, 4)$

Exercise 7.10

1. $y = \log_e 5x \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{5x} \cdot 5 = \dfrac{1}{x}$

2. $y = \log_e(2x + 3) \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2x + 3} \cdot 2 = \dfrac{2}{2x + 3}$

3. $y = \log_e(3x^2) \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3x^2} \cdot 6x = \dfrac{2}{x}$

4. $y = \log_e(\sin x) \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sin x} \cdot \cos x = \dfrac{\cos x}{\sin x} = \cot\text{an}\, x$

5. $y = \log_e(x^2 - 6x) \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x^2 - 6x} \cdot 2x - 6 = \dfrac{2(x - 3)}{x^2 - 6x}$

6. $y = \log_e(\cos 3x) \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos 3x} \cdot -\sin 3x \,.\, 3$

$= \dfrac{-3\sin 3x}{\cos 3x} = -3\tan 3x$

7. $y = x \log_e x$

Product Rule $\Rightarrow \quad u = x \quad$ and $\quad v = \log_e x$

$\Rightarrow \dfrac{du}{dx} = 1 \qquad \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = x \cdot \dfrac{1}{x} + \log_e x \,.\, 1 = \log_e x + 1$

8. $y = x^2 \ell n(3x)$

Product Rule $\Rightarrow$ $u = x^2$ and $v = \ell n(3x)$

$$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^2 \cdot \frac{1}{x} + \ell n\, 3x \,.\, 2x$$

$$= x + 2x\,\ell n\, 3x$$

9. $y = \dfrac{\ell n\, x}{x}$

Quotient Rule $\Rightarrow$ $u = \ell n\, x$ and $v = x$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \qquad \Rightarrow \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x \cdot \frac{1}{x} - \ell n\, x\,.\,1}{x^2}$$

$$= \frac{1 - \ell n\, x}{x^2}$$

10. (i) $y = \log_e(3x+1)^3 = 3\log_e(3x+1)$

$$\Rightarrow \frac{dy}{dx} = 3 \cdot \frac{1}{3x+1} \cdot 3 = \frac{9}{3x+1}$$

(ii) $y = \log_e\left(\dfrac{2x+1}{1-3x}\right) = \log_e(2x+1) - \log_e(1-3x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2x+1} \cdot 2 - \frac{1}{1-3x} \cdot -3$$

$$= \frac{2}{2x+1} + \frac{3}{1-3x}$$

$$= \frac{2(1-3x) + 3(2x+1)}{(2x+1)(1-3x)}$$

$$= \frac{2 - 6x + 6x + 3}{(2x+1)(1-3x)} = \frac{5}{(2x+1)(1-3x)}$$

(iii) $y = \log_e\sqrt{1+x^2} = \log_e(1+x^2)^{\frac{1}{2}} = \dfrac{1}{2}\log_e(1+x^2)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x = \frac{x}{1+x^2}$$

(iv) $y = \log_e\sqrt{\sin x} = \log_e(\sin x)^{\frac{1}{2}} = \dfrac{1}{2}\log_e(\sin x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x = \frac{1}{2}\frac{\cos x}{\sin x} = \frac{1}{2}\cot x$$

(v) $y = \log_e(x^2+4)^2 = 2\log_e(x^2+4)$

$$\Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{x^2+4} \cdot 2x = \frac{4x}{x^2+4}$$

(vi) $y = \log_e\sqrt{\dfrac{x}{1+x}} = \log_e\left(\dfrac{x}{1+x}\right)^{\frac{1}{2}}$

$$= \frac{1}{2}\log_e\left(\frac{x}{1+x}\right) = \frac{1}{2}[\log x - \log(1+x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{1+x}\right)$$

$$= \frac{1}{2}\left[\frac{1(1+x) - 1(x)}{x(1+x)}\right]$$

$$= \frac{1}{2}\left[\frac{1+x-x}{x(1+x)}\right] = \frac{1}{2x(x+1)}$$

11. $y = \ell n\, 3x^4 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3x^4} \cdot 12x^3 = \dfrac{4}{x} = 4x^{-1}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -4x^{-2} = \dfrac{-4}{x^2}$

12. $y = [\log_e(x + 4)]^2$

$\Rightarrow \dfrac{dy}{dx} = 2[\log_e(x + 4)] \cdot \dfrac{1}{x + 4} \cdot 1 = \dfrac{2\log_e(x + 4)}{x + 4}$

13. $y = x \log_e x$

Product Rule $\Rightarrow u = x$ and $v = \log_e x$

$\Rightarrow \dfrac{du}{dx} = 1 \qquad \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = x \cdot \dfrac{1}{x} + \log_e x\,.\,1 = 1 + \log_e x$

$\dfrac{d^2y}{dx^2} = 0 + \dfrac{1}{x} = \dfrac{1}{x}$

14. $y = \log_e x - 2x + x^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - 2 + 2x$

When $x = 2 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2} - 2 + 2(2) = \dfrac{1}{2} - 2 + 4 = \dfrac{5}{2}$

15. $y = (\ell n\, x)^2$

$\Rightarrow \dfrac{dy}{dx} = 2(\ell n\, x) \cdot \dfrac{1}{x} = \dfrac{2\,\ell n\, x}{x}$

At $x = e \Rightarrow \dfrac{dy}{dx} = \dfrac{2\ell n\, e}{e} = \dfrac{2.1}{e} = \dfrac{2}{e}$

16. $y = \ell n(1 + \sin t) \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{1 + \sin t} \cdot \cos t = \dfrac{\cos t}{1 + \sin t}$

$\dfrac{dy}{dt} = \dfrac{\cos t}{1 + \sin t}$

Quotient Rule $\Rightarrow u = \cos t$ and $v = 1 + \sin t$

$\dfrac{du}{dt} = -\sin t \qquad \Rightarrow \dfrac{dv}{dt} = \cos t$

$\dfrac{d^2y}{dt^2} = \dfrac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} = \dfrac{(1 + \sin t)(-\sin t) - (\cos t)\,.\,(\cos t)}{(1 + \sin t)^2}$

$= \dfrac{-\sin t - \sin^2 t - \cos^2 t}{(1 + \sin t)^2}$

$= \dfrac{-(\sin t + \sin^2 t + \cos^2 t)}{(1 + \sin t)^2} = \dfrac{-(1 + \sin t)}{(1 + \sin t)^2} = \dfrac{-1}{1 + \sin t}$

If $(1 + \sin t)\dfrac{d^2y}{dt^2} + k = 0$

$\Rightarrow (1 + \sin t) \cdot \dfrac{-1}{(1 + \sin t)} + k = 0$

$\Rightarrow -1 + k = 0$

$\Rightarrow k = 1$

17. $y = \ell n(e^x \cos x) = \ell n\, e^x + \ell n \cos x$

$= x + \ell n \cos x$

$\Rightarrow \dfrac{dy}{dx} = 1 + \dfrac{1}{\cos x} \cdot (-\sin x) = 1 - \dfrac{\sin x}{\cos x} = 1 - \tan x$

Revision Exercise 7 (Core)

1. (i) $y = x^2 + \frac{1}{x} = x^2 + x^{-1}$

$\Rightarrow \frac{dy}{dx} = 2x - x^{-2} = 2x - \frac{1}{x^2}$

(ii) $y = (2x + 3)^3$

$\Rightarrow \frac{dy}{dx} = 3(2x + 3)^2 . 2 = 6(2x + 3)^2$

(iii) $y = \sqrt{1 + 3x} = (1 + 3x)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(1 + 3x)^{-\frac{1}{2}} . 3 = \frac{3}{2\sqrt{1 + 3x}}$

2. $f(x) = x^2 + 3x - 4$

$\Rightarrow f(x + h) = (x + h)^2 + 3(x + h) - 4$

$= x^2 + 2xh + h^2 + 3x + 3h - 4$

$\Rightarrow f(x + h) - f(x) = x^2 + 2xh + h^2 + 3x + 3h - 4 - (x^2 + 3x - 4)$

$= x^2 + 2xh + h^2 + 3x + 3h - 4 - x^2 - 3x + 4$

$= 2xh + h^2 + 3h$

$\Rightarrow \frac{f(x + h) - f(x)}{h} = \frac{2xh + h^2 + 3h}{h} = 2x + h + 3$

$\Rightarrow \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (2x + h + 3) = 2x + 3$

$\therefore f'(x) = 2x + 3$

3. (i) $y = \frac{1}{3}(x + 2)^3$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \cdot 3(x + 2)^2 . 1 = (x + 2)^2$

(ii) $y = \frac{2x}{x + 1}$

Quotient Rule: $u = 2x$ and $v = x + 1$

$\Rightarrow \frac{du}{dx} = 2$ $\quad \Rightarrow \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x + 1)(2) - (2x)(1)}{(x + 1)^2}$

$= \frac{2x + 2 - 2x}{(x + 1)^2} = \frac{2}{(x + 1)^2}$

4. (i) $f(x) = 2x^2 - \frac{3}{x^2} = 2x^2 - 3x^{-2}$

$\Rightarrow f'(x) = 4x + 6x^{-3} = 4x + \frac{6}{x^3}$

(ii) $y = 4 \sin 6x \Rightarrow \frac{dy}{dx} = 4 \cos 6x . 6 = 24 \cos 6x$

(iii) $y = 3e^{x^2} \Rightarrow \frac{dy}{dx} = 3e^{x^2} . 2x = 6xe^{x^2}$

5. $y = \frac{2x + 3}{x - 4}$

Quotient Rule $\Rightarrow u = 2x + 3$ and $v = x - 4$

$\Rightarrow \frac{du}{dx} = 2$ $\quad \Rightarrow \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x - 4) . 2 - (2x + 3) . 1}{(x - 4)^2}$

$= \frac{2x - 8 - 2x - 3}{(x - 4)^2} = \frac{-11}{(x - 4)^2}$

$\Rightarrow k = -11$

6. (i) $y = 6x^2 - x^3$

$\Rightarrow \dfrac{dy}{dx} = 12x - 3x^2 = 12$ (Gradient)

$\Rightarrow 3x^2 - 12x + 12 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)(x-2) = 0 \Rightarrow x = 2$

(ii) $x = 2 \Rightarrow y = 6(2)^2 - (2)^3 = 24 - 8 = 16$

Point $= (2, 16)$ slope $= 12$

Equation of Tangent: $y - 16 = 12(x - 2)$

$\Rightarrow y - 16 = 12x - 24$

$\Rightarrow 12x - y - 8 = 0$

7. (i) $y = 3x^2 - x + \dfrac{3}{x} = 3x^2 - x + 3x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = 6x - 1 - 3x^{-2} = 6x - 1 - \dfrac{3}{x^2}$

(ii) $y = \dfrac{3x^2}{x-1}$

Quotient Rule: $u = 3x^2$ and $v = x - 1$

$\Rightarrow \dfrac{du}{dx} = 6x$ $\Rightarrow \dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x-1).(6x) - (3x)^2.(1)}{(x-1)^2}$$

$$= \frac{6x^2 - 6x - 3x^2}{(x-1)^2} = \frac{3x^2 - 6x}{(x-1)^2}$$

(iii) $y = \cos^2 4x$

$\Rightarrow \dfrac{dy}{dx} = 2\cos 4x . -\sin 4x . 4$

$= -8\cos 4x \sin 4x$

8. $y = \dfrac{4x^2 + 6}{x} = \dfrac{4x^2}{x} + \dfrac{6}{x} = 4x + 6x^{-1}$

$\dfrac{dy}{dx} = 4 - 6x^{-2} = 4 - \dfrac{6}{x^2}$

9. $f(x) = a\sin 3x$

$\Rightarrow f'(x) = a\cos 3x . 3 = 3a\cos 3x$

$\Rightarrow f'(\pi) = 3a\cos 3(\pi) = 2$

$\Rightarrow 3a\cos 3\pi = 2$

$\Rightarrow 3a(-1) = 2$

$\Rightarrow -3a = 2 \quad \Rightarrow a = -\dfrac{2}{3}$

10. $y = x\sin 2x$

Product Rule $\Rightarrow u = x$ and $v = \sin 2x$

$\Rightarrow \dfrac{du}{dx} = 1$ $\Rightarrow \dfrac{dv}{dx} = \cos 2x . 2 = 2\cos 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x(2\cos 2x) + (\sin 2x).(1)$$

$$= 2x\cos 2x + \sin 2x$$

$$x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = 2\left(\frac{\pi}{3}\right)\cos 2\left(\frac{\pi}{3}\right) + \sin 2\left(\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3}\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3}\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = \frac{-\pi}{3} + \frac{\sqrt{3}}{2}$$

11. $\frac{dy}{dx} = (x+1)(x-2)$

$P(1, 2) \Rightarrow \frac{dy}{dx} = (1+1)(1-2) = (2)(-1) = -2$

$\Rightarrow$ Equation of Tangent $\Rightarrow y - 2 = -2(x-1)$

$\Rightarrow y - 2 = -2x + 2$

$\Rightarrow 2x + y - 4 = 0$

12. $f(x) = \sqrt{x} + \frac{1}{x^2} = x^{\frac{1}{2}} + x^{-2}$

$\Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^3}$

$\Rightarrow f'(4) = \frac{1}{2\sqrt{4}} - \frac{2}{(4)^3} = \frac{1}{4} - \frac{2}{64} = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$

13. $y = 2x^2 - 1$

(i) $x = 1 \quad \Rightarrow \quad y = 2(1)^2 - 1 = 1$ Point (1, 1)

$x = 4 \quad \Rightarrow \quad y = 2(4)^2 - 1 = 31$ Point (4, 31)

$\Rightarrow$ average rate of change $= \frac{31-1}{4-1} = \frac{30}{3} = 10$

(ii) $\frac{dy}{dx} = 4x$

When $x = 4 \Rightarrow \frac{dy}{dx} = 4(4) = 16$

14. $y = \tan^{-1}(5x)$

Chain Rule $\Rightarrow u = 5x$ and $y = \tan^{-1} u$

$\Rightarrow \frac{du}{dx} = 5 \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 5 = \frac{5}{1+(5x)^2} = \frac{5}{1+25x^2}$

15. $y = 2x^2 - 2x + 3$

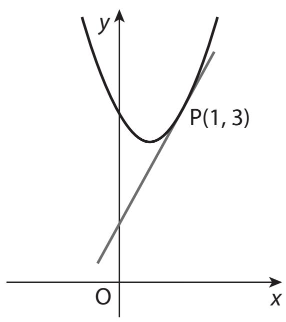
$\frac{dy}{dx} = 4x - 2$

$P(1, 3) \Rightarrow \frac{dy}{dx} = 4(1) - 2 = 2$

$\Rightarrow$ Equation of Tangent: $y - 3 = 2(x-1)$

$\Rightarrow y - 3 = 2x - 2$

$\Rightarrow 2x - y + 1 = 0$



16. $f(x) = 2x^{-3} + \frac{k}{2}x^{-2} - x$

$\Rightarrow f'(x) = -6x^{-4} + \frac{k}{2}(-2x^{-3}) - 1$

$= \frac{-6}{x^4} - \frac{k}{x^3} - 1$

$\Rightarrow f'(-2) = \frac{-6}{(-2)^4} - \frac{k}{(-2)^3} - 1 = 0$

$\Rightarrow \frac{-6}{16} - \frac{k}{-8} - 1 = 0$

$\Rightarrow \frac{-3}{8} + \frac{k}{8} - 1 = 0$

$\Rightarrow -3 + k - 8 = 0$

$\Rightarrow k = 11$

Revision Exercise 7 (Advanced)

1. $f(x) = \sin x - \cos x$

$\Rightarrow f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

At $x = \frac{\pi}{2} \Rightarrow f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)$

$= 0 + 1 = 1$

2. $y = x^2 \sin x$

Product Rule $\Rightarrow u = x^2$ and $v = \sin x$

$\Rightarrow \frac{du}{dx} = 2x$ $\Rightarrow \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^2(\cos x) + \sin x(2x)$

$= x^2 \cos x + 2x \sin x$

At $x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \left(\frac{\pi}{2}\right)^2\left(\cos\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}$

$= \frac{\pi^2}{4}\cdot 0 + \pi . 1 = \pi$

3. $y = x^2 + \ln x$

$\Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} = 3$

$\Rightarrow 2x^2 + 1 = 3x \Rightarrow 2x^2 - 3x + 1 = 0$

$\Rightarrow (x-1)(2x-1) = 0$

$\Rightarrow x = 1$ or $x = \frac{1}{2}$

$\Rightarrow y = (1)^2 + \ln(1) = 1$ OR $y = \left(\frac{1}{2}\right)^2 + \ln\frac{1}{2}$

$= \frac{1}{4} + \ln 1 - \ln 2$

$= \frac{1}{4} - \ln 2$

Points are $(1, 1), \left(\frac{1}{2}, \frac{1}{4} - \ln 2\right)$

4. $y = x - 1 + \frac{1}{x-1} = x - 1 + (x-1)^{-1}$

$\frac{dy}{dx} = 1 - 1(x-1)^{-2} = 1 - \frac{1}{(x-1)^2} = 0$

$\Rightarrow (x-1)^2 - 1 = 0$

$\Rightarrow x^2 - 2x + 1 - 1 = 0$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0$ $\Rightarrow x = 0, 2$

5. (i) $y = \ln(3x^4)$ $\Rightarrow \frac{dy}{dx} = \frac{1}{3x^4}\cdot 12x^3 = \frac{4}{x}$

(ii) $y = \ln\left(\frac{3}{\sqrt{x}}\right) = \ln 3 - \ln\sqrt{x} = \ln 3 - \ln x^{\frac{1}{2}}$

$= \ln 3 - \frac{1}{2}\ln x$

$\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{2}\left(\frac{1}{x}\right) = -\frac{1}{2x}$

6. $y = e^{nx} \Rightarrow \dfrac{dy}{dx} = e^{nx}\,.\,n = ne^{nx}$

$$\Rightarrow \frac{d^2y}{dx^2} ne^{nx}\,.\,n = n^2e^{nx}$$

Hence, $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$

$\Rightarrow n^2e^{nx} - 5(ne^{nx}) + 6(e^{nx}) = 0$

$\Rightarrow n^2e^{nx} - 5ne^{nx} + 6e^{nx} = 0$

$\Rightarrow e^{nx}(n^2 - 5n + 6) = 0$

$\Rightarrow (n - 2)(n - 3) = 0 \quad \Rightarrow n = 2, 3$

7. $y = x^3 - 3x^2 - 5x + 10$

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 6x - 5$

Line $\quad y = 4x - 7 \quad$ has slope $= 4$

Hence, $3x^2 - 6x - 5 = 4$

$\Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = -1, \quad x = 3$

$\Rightarrow y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$ and $y = (3)^3 - 3(3)^2 - 5(3) + 10$

$= -1 - 3 + 5 + 10 = 11 \qquad = 27 - 27 - 15 + 10 = -5$

$\Rightarrow$ Points $= (-1, 11), (3, -5)$

8. $y = a\sqrt{x} - 5 = ax^{\frac{1}{2}} - 5$

$\Rightarrow \dfrac{dy}{dx} = a \cdot \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{a}{2\sqrt{x}}$

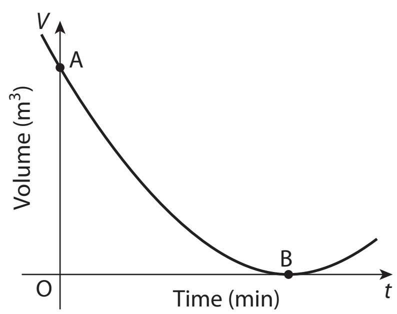
At point $(4, b) \Rightarrow \dfrac{dy}{dx} = \dfrac{a}{2\sqrt{4}} = 2$

$\Rightarrow \dfrac{a}{4} = 2 \quad \Rightarrow a = 8$

Curve: $y = 8\sqrt{x} - 5$

Point $(4, b) \Rightarrow b = 8\sqrt{4} - 5 = 16 - 5 = 11$

9. $V = 80(30 - t)^3$



(i) Point A $\Rightarrow t = 0 \quad \Rightarrow V = 80(30 - 0)^3 = 2,160,000$

$\Rightarrow$ A $= (0, 2160\,000)$

Point B $\Rightarrow V = 0 \Rightarrow 80(30 - t)^3 = 0$

$\Rightarrow t = 30$

$\Rightarrow$ B $= (30, 0)$

$\Rightarrow$ Full tank $= 2,160,000$ m^3

and empty tank occurs after 30 mins

(ii) $t = 10 \quad \Rightarrow V = 80(30 - 10)^3 = 640,000$ m^3

(iii) $(0, 2, 160,000)$ and $(10, 640,000)$

$\Rightarrow$ average rate $= \dfrac{640,000 - 2,160,000}{10 - 0} = \dfrac{1,520,000}{10}$

$= 152,000$ m^3/min

(iv) $V = 80(30 - t)^3 \Rightarrow \dfrac{dV}{dt} = 240(30 - t)^2\,.\,-1 = -240(30 - t)^2$

$t = 10 \quad \Rightarrow \dfrac{dV}{dt} = -240(30 - 10)^2 = -96,000$

$\Rightarrow$ water is draining at rate $= 96,000$ m^3/min

10. $y = \dfrac{x}{\sqrt{1-x^2}}$

Quotient Rule: $u = x$ and $v = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{1-x^2}\cdot 1 - x \cdot \frac{-x}{\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2} = \frac{\frac{\sqrt{1-x^2}\cdot\sqrt{1-x^2}+x^2}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1-x^2+x^2}{(1-x^2)\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow k = \frac{3}{2}$$

11. $y = \tan^{-1}\left(\dfrac{1}{x}\right)$

Chain Rule: $u = \dfrac{1}{x} = x^{-1}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -1x^{-2} = \frac{-1}{x^2} \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+u^2}\cdot\frac{-1}{x^2} = \frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2} = \frac{-1}{x^2+1}$$

At $x = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{(1)^2+1} = -\dfrac{1}{2}$

12. $y = x^3e^x$

Product Rule: $u = x^3$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 3x^2 \qquad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^3 . e^x + e^x . 3x^2 = x^3e^x + 3x^2e^x$$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = (0)^3e^0 + 3(0)^2 . e^0 = 0 + 0 = 0 \Rightarrow$ slope $= 0$

and $y = 0^2 . e^0 = 0 \qquad \Rightarrow$ Point $(0, 0)$

$\Rightarrow$ Equation of Tangent: $y - 0 = 0(x-0) \Rightarrow y = 0$

13. $y = kx^2 \Rightarrow \dfrac{dy}{dx} = k . 2x = 2kx$

$$x\frac{dy}{dx} + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y = 0$$

$$\Rightarrow x . 2kx + \frac{1}{2}(2kx)^2 + kx^2 = 0$$

$$\Rightarrow 2kx^2 + \frac{1}{2}\cdot 4k^2x^2 + kx^2 = 0$$

$$\Rightarrow \quad 2k^2x^2 + 3kx^2 = 0$$

$$\Rightarrow \quad 2k^2 + 3k = 0$$

$$\Rightarrow \quad k(2k+3) = 0$$

$$\Rightarrow k = 0 \quad \text{or} \quad k = \frac{-3}{2} \qquad \Rightarrow \text{Ans: } k = \frac{-3}{2}$$

14. $f(x) = x^3 + x^2 - 1 \qquad \Rightarrow \qquad f'(x) = 3x^2 + 2x$

$x_1 = 1 \Rightarrow f(1) = (1)^3 + (1)^2 - 1 = 1$ and $f'(1) = 3(1)^2 + 2(1) = 5$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{5} = \frac{4}{5}$$

15. $y = \ell n(1 + e^x) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1 + e^x} \cdot e^x = \dfrac{e^x}{1 + e^x}$

Quotient Rule: $u = e^x$ and $v = 1 + e^x$

$$\Rightarrow \frac{du}{dx} = e^x \qquad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1 + e^x) \,.\, e^x - e^x \,.\, e^x}{(1 + e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1 + e^x)^2}$$

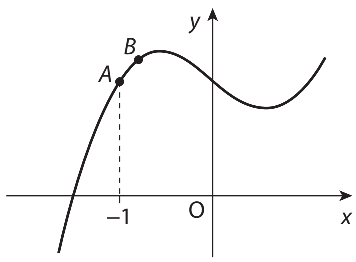
$$= \frac{e^x}{(1 + e^x)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{e^x}{(1 + e^x)^2} + \left(\frac{e^x}{1 + e^x}\right)^2$$

$$= \frac{e^x}{(1 + e^x)^2} + \frac{(e^x)^2}{(1 + e^x)^2}$$

$$= \frac{e^x(1 + e^x)}{(1 + e^x)^2} = \frac{e^x}{1 + e^x} = \frac{dy}{dx}$$

16. $y = x^3 - x + 1$



(i) At B: $x = -1 + h \Rightarrow y = (-1 + h)^3 - (-1 + h) + 1$

$$= -1 + 3h - 3h^2 + h^3 + 1 - h + 1$$
$$= 1 + 2h - 3h^2 + h^3$$

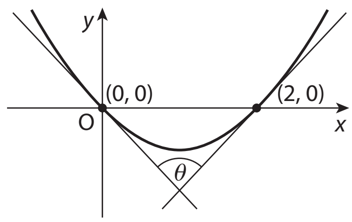
(ii) At A: $x = -1 \Rightarrow y = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$

$A(-1, 1) \quad B(-1 + h, 1 + 2h - 3h^2 + h^3)$

$$\text{Gradient } AB = \frac{1 + 2h - 3h^2 + h^3 - 1}{-1 + h + 1} = \frac{2h - 3h^2 + h^3}{h}$$
$$= 2 - 3h + h^2$$

(iii) As h becomes smaller $\Rightarrow$ gradient $= 2 - 3(0) + (0)^2 = 2$

Revision Exercise 7 (Extended-Response)



1. (i) $y = x(x - 2) \quad = x^2 - 2x$

On x-axis, $y = 0 \Rightarrow x(x - 2) = 0$

$\Rightarrow x = 0, x = 2$

$\Rightarrow$ points are (0, 0) and (2, 0)

(ii) Slope $= \dfrac{dy}{dx} = 2x - 2$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = 2(0) - 2 = -2$

At $x = 2 \Rightarrow \dfrac{dy}{dx} = 2(2) - 2 = +2$

Curve is symmetrical; hence, the slopes of the tangents at any two points on the curve which have the same y-value will differ only in sign.

(iii) Point (0, 0), slope $= -2$

$\Rightarrow$ equation of the tangent: $y - 0 = -2(x - 0)$

$\Rightarrow \qquad y = -2x$

$\Rightarrow 2x + y = 0$

Point (2, 0), slope $= +2$

$\Rightarrow$ equation of the tangent: $y - 0 = -2(x - 2)$

$\Rightarrow y = 2x - 4$

$\Rightarrow 2x - y - 4 = 0$

(iv) $\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1m_2} = \pm\dfrac{2-(-2)}{1+2(-2)} = \pm\dfrac{4}{-3} = \pm\dfrac{4}{3}$

acute angle $\Rightarrow \tan\theta = +\dfrac{4}{3}$

$\Rightarrow \theta = \tan^{-1}\dfrac{4}{3} = 53.13° = 53°$

(v) $y = x(x-2)(x-5) = x^3 - 7x^2 + 10x$

On x-axis, $y = 0 \Rightarrow x(x-2)(x-5) = 0$

$\Rightarrow x = 0, x = 2$ or $x = 5$

$\dfrac{dy}{dx} = 3x^2 - 14x + 10$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = 3(0)^2 - 14(0) + 10 = 10 = p$

At $x = 2 \Rightarrow \dfrac{dy}{dx} = 3(2)^2 - 14(2) + 10 = -6 = q$

At $x = 5 \Rightarrow \dfrac{dy}{dx} = 3(5)^2 - 14(5) + 10 = 15 = r$

Hence, $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = \dfrac{1}{10} + \dfrac{1}{-6} + \dfrac{1}{15}$

$= \dfrac{1}{10} - \dfrac{1}{6} + \dfrac{1}{15} = 0$

2. (i) $y = 2\ell n\left(x\sqrt{x^2+1}\right)$

$= 2\ell n\left[x\,.\,(x^2+1)^{\frac{1}{2}}\right]$

$= 2\left[\ell n\, x + \ell n(x^2+1)^{\frac{1}{2}}\right]$

$= 2\left[\ell n\, x + \dfrac{1}{2}\ell n(x^2+1)\right]$

$= 2\ell n\, x + \ell n(x^2+1)$

$\dfrac{dy}{dx} = 2\cdot\dfrac{1}{x} + \dfrac{1}{x^2+1}\cdot 2x = \dfrac{2}{x} + \dfrac{2x}{x^2+1}$

$= \dfrac{2(x^2+1) + 2x(x)}{x(x^2+1)} = \dfrac{2x^2 + 2 + 2x^2}{x(x^2+1)}$

$= \dfrac{4x^2+2}{x(x^2+1)}$

$= \dfrac{2(2x^2+1)}{x(x^2+1)} \Rightarrow k = 2$

(ii) $y = 2\,\ell n\left(x\sqrt{x^2+1}\right)$

When $x = 1 \Rightarrow y = 2\ell n\left(1\sqrt{1^2+1}\right) = 2\ell n\sqrt{2} = \ell n(\sqrt{2})^2 = \ell n\, 2 \Rightarrow$ point $= (1, \ell n\, 2)$

$\dfrac{dy}{dx} = \dfrac{2(2x^2+1)}{x(x^2+1)}$

When $x = 1 \quad\Rightarrow$ slope $= \dfrac{dy}{dx} = \dfrac{2(2(1)^2+1)}{1((1)^2+1)} = 3$

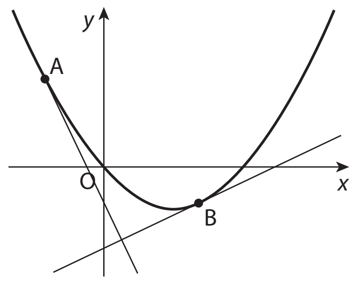
$\Rightarrow$ equation of the tangent: $y - y_1 = m(x - x_1)$

$\Rightarrow y - \ell n\, 2 = 3(x-1)$

$\Rightarrow y - \ell n\, 2 = 3x - 3$

$\Rightarrow y = 3x + \ell n\, 2 - 3$

3. $y = \frac{x^2}{4} - x = \frac{1}{4}x^2 - x$



(i) $\frac{dy}{dx} = \frac{1}{4}(2x) - 1 = \frac{x}{2} - 1$

When $x = 3 \quad \Rightarrow$ slope $= \frac{dy}{dx} = \frac{3}{2} - 1 = \frac{1}{2}$ and point $B\left(3, -\frac{3}{4}\right)$

$\Rightarrow$ equation of the tangent: $y + \frac{3}{4} = \frac{1}{2}(x - 3)$

$$\Rightarrow 2y + \frac{3}{2} = x - 3$$
$$\Rightarrow 4y + 3 = 2x - 6$$
$$\Rightarrow 2x - 4y - 9 = 0$$

(ii) Slope $= \frac{1}{2} \Rightarrow$ perpendicular slope $= -2$

$$\Rightarrow \frac{x}{2} - 1 = -2 \Rightarrow x - 2 = -4 \Rightarrow x = -2$$

When $x = -2 \Rightarrow y = \frac{(-2)^2}{4} - (-2) = 3 \Rightarrow A = (-2, 3)$

(iii) Tangent at $A(-2, 3)$ with slope $= -2$

$\Rightarrow y - 3 = -2(x + 2)$

$\Rightarrow y - 3 = -2x - 4 \qquad \Rightarrow 2x + y = -1$

and $\underline{-2x + 4y = -9}$

add $\Rightarrow \quad 5y = -10 \qquad \Rightarrow y = -2$

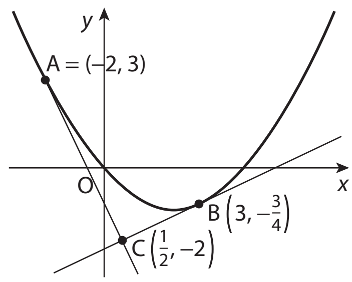
$\Rightarrow \qquad 2x - 2 = -1$

$\Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow$ point $C = \left(\frac{1}{2}, -2\right)$

(iv) $A(-2, 3)$, $\quad B\left(3, -\frac{3}{4}\right)$, $\quad C\left(\frac{1}{2}, -2\right)$

$+2 \downarrow -3 \qquad +2 \downarrow -3 \qquad +2 \downarrow -3$

$(0, 0) \qquad \left(\underset{x_1}{5}, \underset{y_1}{-3\frac{3}{4}}\right), \qquad \left(\underset{x_1}{2\frac{1}{2}}, \underset{y_2}{-5}\right)$



Area Triangle ABC $= \frac{1}{2}|x_1y_2 - x_2y_1|$

$$= \frac{1}{2}\left|(5)(-5) - \left(2\frac{1}{2}\right)\left(-3\frac{3}{4}\right)\right| = \frac{1}{2}\left|\frac{-125}{8}\right| = \frac{125}{16}$$

4. (a) $f(x) = x^3 - x$

at Q, $x = (2 + h) \Rightarrow f(2 + h) = (2 + h)^3 - (2 + h) = h^3 + 6h^2 + 12h + 8 - 2 - h$

$$= h^3 + 6h^2 + 11h + 6$$

$\Rightarrow Q = (2 + h, h^3 + 6h^2 + 11h + 6)$ and $P = (2, 6)$

slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{h^3 + 6h^2 + 11h + 6 - 6}{2 + h - 2} = \frac{h^3 + 6h^2 + 11h}{h}$

(b) (i) $h = 0.5 \Rightarrow$ slope $= \frac{(0.5)^3 + 6(0.5)^2 + 11(0.5)}{0.5} = \frac{7.125}{0.5} = 14.25$

(ii) $h = 0.1 \Rightarrow$ slope $= \frac{(0.1)^3 + 6(0.1)^2 + 11(0.1)}{0.1} = \frac{1.161}{0.1} = 11.61$

(iii) $h = 0.01 \Rightarrow$ slope $= \frac{(0.01)^3 + 6(0.01)^2 + 11(0.01)}{0.01} = \frac{0.110601}{0.01} = 11.0601$

(iv) $h = 0.001 \Rightarrow$ slope $= \frac{(0.001)^3 + 6(0.001)^2 + 11(0.001)}{0.001} = \frac{0.011006001}{0.001} = 11.006001$

(c) As $h \to 0$, slope of PQ $= 11$

(d) Hence, slope of the curve at $P(2, 6) = 11$

(e) At Q, $x = (a + h) \Rightarrow f(a + h) = (a + h)^3 - (a + h)$

$$= a^3 + 3a^2h + 3ah^2 + h^3 - a - h$$

$\Rightarrow Q = [a + h, a^3 + 3a^2h + 3ah^2 + h^3 - a - h]$ $\quad P = [a, a^3 - a]$

$$\text{Slope PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{a + h - a}$$

$$= \frac{3a^2h + 3ah^2 + h^3 - h}{h}$$

$$= 3a^2 + 3ah + h^2 - 1$$

(f) As $h \rightarrow 0$, slope $= 3a^2 + 3a(0) + (0)^2 - 1 = 3a^2 - 1$

5. (a) $f(x) = e^{-\frac{1}{2}x^2}$

$$\Rightarrow f'(x) = e^{-\frac{1}{2}x^2} \cdot -\frac{1}{2}(2x) = -x \,.\, e^{-\frac{1}{2}x^2}$$

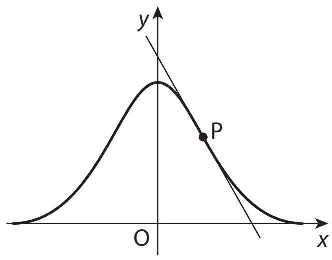
Find $f''(x) \Rightarrow$ Product Rule: $u = -x$ and $v = e^{-\frac{1}{2}x^2}$

$$\Rightarrow \frac{du}{dx} = -1 \qquad \Rightarrow \frac{dv}{dx} = -xe^{-\frac{1}{2}x^2}$$

$$\Rightarrow f''(x) = u\frac{dv}{dx} + v\frac{du}{dx} = (-x)\,.\left(-xe^{-\frac{1}{2}x^2}\right) + \left(e^{-\frac{1}{2}x^2}\right)(-1)$$

$$= (x^2 - 1)e^{-\frac{1}{2}x^2}$$

(b) Point of inflection at P $\Rightarrow f''(x) = 0$



$$\Rightarrow (x^2 - 1)\,.\,e^{-\frac{1}{2}x^2} = 0$$

$$\Rightarrow (x - 1)(x + 1) = 0$$

$$\Rightarrow x = 1 \text{ or } x = -1 \text{ (Not in first quadrant)}$$

When $x = 1 \Rightarrow$ slope $= f'(1) = (-1)e^{-\frac{1}{2}(1)^2} = (-1)e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}$

When $x = 1 \Rightarrow f(1) = e^{-\frac{1}{2}(1)^2} = e^{-\frac{1}{2}} \Rightarrow$ Point $P\left(1, e^{-\frac{1}{2}}\right)$

$\Rightarrow$ Equation of Tangent: $y - y_1 = m(x - x_1)$ $\qquad \Rightarrow y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1)$

$$\Rightarrow y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}x + e^{-\frac{1}{2}}$$

$$\Rightarrow e^{-\frac{1}{2}}x + y = 2e^{-\frac{1}{2}}$$

On x-axis, $y = 0 \Rightarrow e^{-\frac{1}{2}}x + 0 = 2e^{-\frac{1}{2}}$

$$\Rightarrow x = \frac{2e^{-\frac{1}{2}}}{e^{-\frac{1}{2}}} = 2 \Rightarrow \text{Point} = (2, 0)$$

6. $D = 50e^{kt}$

(a) $\dfrac{dD}{dt} = 50e^{kt}\,.\,k = k\,.\,50e^{kt} = kD =$ a constant times D

(b) $k = 0.2$ and $D = 100 \Rightarrow 50e^{0.2t} = 100$

$$\Rightarrow \quad e^{0\cdot 2t} = \frac{100}{50} = 2$$

$$\Rightarrow \ell n\, e^{0.2t} = \ell n\, 2$$

$$\Rightarrow 0.2t(\ell n\, e) = \ell n\, 2$$

$$\Rightarrow t = \frac{\ell n\, 2}{0.2} = 3.4657$$

$$\Rightarrow \frac{dD}{dt} = (0.2)50e^{(0.2)(3.4657)}$$

$$= 10e^{0.69314} = 19.99 = 20 \text{ cm/year}$$

7. (i) $y = \ell n\sqrt{1 + \sin 2x} = \ell n(1 + \sin 2x)^{\frac{1}{2}}$

$$= \frac{1}{2}\ell n(1 + \sin 2x)$$

$$\text{Slope} = \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + \sin 2x} \cdot \cos 2x \,.\, 2 = \frac{\cos 2x}{1 + \sin 2x}$$

$$\text{When } x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{\cos 2\left(\frac{\pi}{2}\right)}{1 + \sin 2\left(\frac{\pi}{2}\right)} = \frac{\cos \pi}{1 + \sin \pi} = \frac{-1}{1 + 0} = -1$$

(ii) $y = (x^2 - 1)^n \Rightarrow \dfrac{dy}{dx} = n(x^2 - 1)^{n-1} \,.\, (2x) = 2nx(x^2 - 1)^{n-1}$

Hence, $(x^2 - 1)\dfrac{dy}{dx} - 2nxy$

$= (x^2 - 1) \,.\, 2nx(x^2 - 1)^{n-1} - 2nx(x^2 - 1)^n$

$= 2nx \,.\, (x^2 - 1) \,.\, (x^2 - 1)^{n-1} - 2nx(x^2 - 1)^n$

$= 2nx \,.\, (x^2 - 1)^n - 2nx(x^2 - 1)^n$

$= 0$

(iii) $f(x) = x^3 - 6x^2 + 12x + 5$

$\Rightarrow f'(x) = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x - 2)(x - 2)$

$= 3(x - 2)^2$, which is always positive

8. $f(x) = x(x - k)^2$

$= x(x^2 - 2kx + k^2) = x^3 - 2kx^2 + k^2x$

(i) $f'(x) = 3x^2 - 4kx + k^2$

$= (x - k)(3x - k)$

(ii) Tangents are parallel to x-axis $\Rightarrow f'(x) = 0$

$\Rightarrow (x - k)(3x - k) = 0$

$\Rightarrow x = k \quad \text{or} \quad 3x = k$

$\Rightarrow x = \dfrac{k}{3}$

When $x = k \Rightarrow f(k) = k(k - k)^2 = k \,.\, 0 = 0 \Rightarrow$ Point $(k, 0)$

$$\text{When } x = \frac{k}{3} \Rightarrow f\left(\frac{k}{3}\right) = \frac{k}{3}\left(\frac{k}{3} - k\right)^2 = \frac{k}{3}\left(\frac{-2k}{3}\right)^2 = \frac{k}{3} \cdot \frac{4k^2}{9}$$

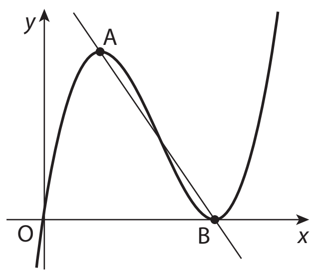
$$= \frac{4k^3}{27} \Rightarrow \text{Point}\left(\frac{k}{3}, \frac{4k^3}{27}\right)$$

(iii) $(k, 0), \left(\dfrac{k}{3}, \dfrac{4k^3}{27}\right) \Rightarrow \text{slope} = \dfrac{\frac{4k^3}{27} - 0}{\frac{k}{3} - k} = \dfrac{\frac{4k^3}{27}}{\frac{-2k}{3}} = -\dfrac{2k^2}{9}$

Equation of the line: $y - 0 = \dfrac{-2k^2}{9}(x - k)$

$\Rightarrow y = \dfrac{-2k^2}{9}(x - k)$

(iv) $A\left(\dfrac{k}{3}, \dfrac{4k^3}{27}\right)$, $B(k, 0)$



$$\text{Midpoint} = \left[\frac{\frac{k}{3} + k}{2}, \frac{\frac{4k^3}{27} + 0}{2}\right] = \left(\frac{2k}{3}, \frac{2k^3}{27}\right)$$

$f(x) = x(x - k)^2$

$\Rightarrow f\left(\dfrac{2k}{3}\right) = \dfrac{2k}{3}\left(\dfrac{2k}{3} - k\right)^2$

$= \dfrac{2k}{3}\left(\dfrac{-k}{3}\right)^2$

$= \dfrac{2k}{3} \cdot \dfrac{k^2}{9} = \dfrac{2k^3}{27}$

Hence, AB intersects the curve at the midpoint of $[AB]$.

Chapter 8

Exercise 8.1

1. Prove: $\cos A \tan A = \sin A$

Proof: $\cos A \tan A = \cos A \cdot \dfrac{\sin A}{\cos A}$

$= \sin A$

2. Prove: $\sin\theta \sec\theta = \tan\theta$

Proof: $\sin\theta \sec\theta = \sin\theta \cdot \dfrac{1}{\cos\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$

3. Prove: $\sin\theta \tan\theta + \cos\theta = \sec\theta$

Proof: $\sin\theta \tan\theta + \cos\theta = \sin\theta \cdot \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1}$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta}$

$= \sec\theta$

4. Prove: $\dfrac{\sin\theta}{\sqrt{1 - \sin^2\theta}} = \tan\theta$

Proof: $\dfrac{\sin\theta}{\sqrt{1 - \sin^2\theta}} = \dfrac{\sin\theta}{\sqrt{\cos^2\theta}}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$

5. Prove: $\sec A - \sin A \tan A = \cos A$

Proof: $\sec A - \sin A \tan A = \dfrac{1}{\cos A} - \sin A \cdot \dfrac{\sin A}{\cos A}$

$= \dfrac{1 - \sin^2 A}{\cos A}$

$= \dfrac{\cos^2 A}{\cos A}$

$= \cos A$

6. Prove: $1 - \tan^2\theta \cos^2\theta = \cos^2\theta$

Proof: $1 - \tan^2\theta \cos^2\theta = 1 - \dfrac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta$

$= 1 - \sin^2\theta$

$= \cos^2\theta$

7. Prove: $\dfrac{(1 + \cos\theta)(1 - \cos\theta)}{\cos^2\theta} = \tan^2\theta$

Proof: $\dfrac{(1 + \cos\theta)(1 - \cos\theta)}{\cos^2\theta} = \dfrac{1 - \cos\theta + \cos\theta - \cos^2\theta}{\cos^2\theta}$

$= \dfrac{1 - \cos^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

8. Prove: $\sec^2 A - \tan^2 A = 1$

Proof: $\sec^2 A - \tan^2 A = \dfrac{1}{\cos^2 A} - \dfrac{\sin^2 A}{\cos^2 A}$

$= \dfrac{1 - \sin^2 A}{\cos^2 A}$

$= \dfrac{\cos^2 A}{\cos^2 A}$

$= 1$

9. Prove: $\dfrac{\sqrt{1 - \cos^2 \theta}}{\tan \theta} = \cos \theta$

Proof: $\dfrac{\sqrt{1 - \cos^2 \theta}}{\tan \theta} = \dfrac{\sqrt{\sin^2 \theta}}{\frac{\sin \theta}{\cos \theta}}$

$= \sin \theta \cdot \dfrac{\cos \theta}{\sin \theta}$

$= \cos \theta$

10. Prove: $(1 + \tan^2 \theta)\cos^2 \theta = 1$

Proof: $(1 + \tan^2 \theta) \cos^2 \theta = \sec^2 \theta \cos^2 \theta$

$= \dfrac{1}{\cos^2 \theta} \cdot \cos^2 \theta$

$= 1$

11. Prove: $(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 = 2$

Proof: $(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2$

$= \cos^2 \theta + 2 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta$

$= 2(\cos^2 \theta + \sin^2 \theta)$

$= 2 . 1 = 2$

12. Prove: $(1 + \tan^2 A)(1 - \sin^2 A) = 1$

Proof: $(1 + \tan^2 A)(1 - \sin^2 A) = \sec^2 A \cos^2 A$

$= \dfrac{1}{\cos^2 A} \cdot \cos^2 A$

$= 1$

13. Prove: $(\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta = 1$

Proof: $(\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta$

$= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta - 2 \sin \theta \cos \theta$

$= \sin^2 \theta + \cos^2 \theta$

$= 1$

14. Prove: $\dfrac{1 - \cos^2 A}{\sin A \cos A} = \tan A$

Proof: $\dfrac{1 - \cos^2 A}{\sin A \cos A} = \dfrac{\sin^2 A}{\sin A \cos A}$

$= \dfrac{\sin A}{\cos A}$

$= \tan A$

15. Prove: $\dfrac{1}{1-\sin A}+\dfrac{1}{1+\sin A}=2\sec^2 A$

Proof: $$\begin{aligned}\frac{1}{1-\sin A}+\frac{1}{1+\sin A}&=\frac{1(1+\sin A)+1(1-\sin A)}{(1-\sin A)(1+\sin A)}\\&=\frac{1+\sin A+1-\sin A}{1+\sin A-\sin A-\sin^2 A}\\&=\frac{2}{1-\sin^2 A}\\&=\frac{2}{\cos^2 A}\\&=2\sec^2 A\end{aligned}$$

16. Prove: $(1-\sin^2 A)\tan^2 A+\cos^2 A=1$

Proof: $$\begin{aligned}(1-\sin^2 A)\tan^2 A+\cos^2 A&=\cos^2 A\cdot\frac{\sin^2 A}{\cos^2 A}+\cos^2 A\\&=\sin^2 A+\cos^2 A=1\end{aligned}$$

17. Prove: $\operatorname{cosec}^2\theta(\tan^2\theta-\sin^2\theta)=\tan^2\theta$

Proof: $\operatorname{cosec}^2\theta(\tan^2\theta-\sin^2\theta)$
$$\begin{aligned}&=\frac{1}{\sin^2\theta}\left(\frac{\sin^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{1}\right)\\&=\frac{1}{\sin^2\theta}\left(\frac{\sin^2\theta-\sin^2\theta\cos^2\theta}{\cos^2\theta}\right)\\&=\frac{1}{\sin^2\theta}\cdot\sin^2\theta\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)=\frac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta\end{aligned}$$

18. Prove: $(1-\sin A)(\sec A+\tan A)=\cos A$

Proof: $$\begin{aligned}(1-\sin A)(\sec A+\tan A)&=(1-\sin A)\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}\right)\\&=(1-\sin A)\cdot\frac{(1+\sin A)}{\cos A}\\&=\frac{1+\sin A-\sin A-\sin^2 A}{\cos A}\\&=\frac{1-\sin^2 A}{\cos A}=\frac{\cos^2 A}{\cos A}=\cos A\end{aligned}$$

19. Prove: $b\cos C+c\cos B=a$

Proof: cosine rule: $\cos B=\dfrac{a^2+c^2-b^2}{2ac}$, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

Thus, $$\begin{aligned}b\cos C+c\cos B&=b\cdot\frac{(a^2+b^2-c^2)}{2ab}+c\cdot\frac{(a^2+c^2-b^2)}{2ac}\\&=\frac{a^2+b^2-c^2}{2a}+\frac{a^2+c^2-b^2}{2a}\\&=\frac{a^2+b^2-c^2+a^2+c^2-b^2}{2a}=\frac{2a^2}{2a}=a\end{aligned}$$

20. Prove: $bc\cos A+ca\cos B=c^2$

Proof: cosine rule: $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, $\cos B=\dfrac{a^2+c^2-b^2}{2ac}$

Hence, $$\begin{aligned}bc\cos A+ca\cos B&=\frac{bc(b^2+c^2-a^2)}{2bc}+\frac{ca(a^2+c^2-b^2)}{2ac}\\&=\frac{b^2+c^2-a^2}{2}+\frac{a^2+c^2-b^2}{2}\\&=\frac{b^2+c^2-a^2+a^2+c^2-b^2}{2}\\&=\frac{2c^2}{2}=c^2\end{aligned}$$

21. Prove: $c = b\cos A + a\cos B$

Proof: cosine rule: $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

Hence, $b\cos A + a\cos B = \dfrac{b(b^2+c^2-a^2)}{2bc} + \dfrac{a(a^2+c^2-b^2)}{2ac}$

$$= \frac{b^2+c^2-a^2}{2c} + \frac{a^2+c^2-b^2}{2c}$$

$$= \frac{b^2+c^2-a^2+a^2+c^2-b^2}{2c}$$

$$= \frac{2c^2}{2c} = c$$

22. Prove: $a\cos B - b\cos A = \dfrac{a^2-b^2}{c}$

Proof: cosine rule: $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

Hence, $a\cos B - b\cos A = \dfrac{a(a^2+c^2-b^2)}{2ac} - \dfrac{b(b^2+c^2-a^2)}{2bc}$

$$= \frac{a^2+c^2-b^2}{2c} - \frac{(b^2+c^2-a^2)}{2c}$$

$$= \frac{a^2+c^2-b^2-b^2-c^2+a^2}{2c}$$

$$= \frac{2(a^2-b^2)}{2c} = \frac{a^2-b^2}{c}$$

23. Prove: $ab\cos C - ac\cos B = b^2 - c^2$

Proof: cosine rule: $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Hence, $ab\cos C - ac\cos B = \dfrac{ab(a^2+b^2-c^2)}{2ab} - \dfrac{ac(a^2+c^2-b^2)}{2ac}$

$$= \frac{a^2+b^2-c^2}{2} - \frac{(a^2+c^2-b^2)}{2}$$

$$= \frac{a^2+b^2-c^2-a^2-c^2+b^2}{2}$$

$$= \frac{2(b^2-c^2)}{2} = b^2-c^2$$

24. Prove: $c\cos B - b\cos C = \dfrac{c^2-b^2}{a}$

Proof: cosine rule: $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Hence, $c\cos B - b\cos C = \dfrac{c(a^2+c^2-b^2)}{2ac} - \dfrac{b(a^2+b^2-c^2)}{2ab}$

$$= \frac{a^2+c^2-b^2}{2a} - \frac{(a^2+b^2-c^2)}{2a}$$

$$= \frac{a^2+c^2-b^2-a^2-b^2+c^2}{2a}$$

$$= \frac{2(c^2-b^2)}{2a} = \frac{c^2-b^2}{a}$$

25. Prove: $\dfrac{\sin A - \sin B}{\sin B} = \dfrac{a-b}{b}$

Proof: sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow b \sin A = a \sin B$

$$\Rightarrow \sin A = \frac{a \sin B}{b}$$

Hence, $\dfrac{\sin A - \sin B}{\sin B} = \dfrac{\frac{a \sin B}{b} - \frac{\sin B}{1}}{\sin B}$

$$= \frac{\frac{a \sin B - b \sin B}{b}}{\sin B}$$

$$= \frac{\sin B\left[\frac{a-b}{b}\right]}{\sin B}$$

$$= \frac{a-b}{b}$$

26. Prove: $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$

Proof: cosine rule: $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$; $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$; $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Hence, $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{\frac{b^2+c^2-a^2}{2bc}}{a} + \dfrac{\frac{a^2+c^2-b^2}{2ac}}{b} + \dfrac{\frac{a^2+b^2-c^2}{2ab}}{c}$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

$$= \frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

Exercise 8.2

1. (i) $\cos 15° = \cos(60° - 45°)$

$$= \cos 60° \cos 45° + \sin 60° \sin 45°$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\sin 75° = \sin(45° + 30°)$

$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) $\cos 105° = \cos(60° + 45°)$

$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

2. (i) $\tan 15° = \tan(45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(ii) $\sin 135° = \sin(90° + 45°)$

$$= \sin 90° \cos 45° + \cos 90° \sin 45°$$

$$= 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(iii) $\tan 75° = \tan(45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

3. First Triangle: $\sin A = \frac{3}{5}, \cos A = \frac{4}{5}, \tan A = \frac{3}{4}$

Second Triangle: $\sin B = \frac{5}{13}, \cos B = \frac{12}{13}, \tan B = \frac{5}{12}$

(i) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

(ii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \cdot \frac{5}{12}} = \frac{\frac{1}{3}}{\frac{21}{16}} = \frac{16}{63}$$

4. (i) $\sin 45° \cos 15° + \cos 45° \sin 15°$

$$= \sin(45° + 15°) = \sin 60° = \frac{\sqrt{3}}{2}$$

(ii) $\cos 40° \cos 50° - \sin 40° \sin 50°$

$$= \cos(40° + 50°) = \cos 90° = 0$$

(iii) $\cos 80° \cos 20° + \sin 80° \sin 20°$

$$= \cos(80° - 20°) = \cos 60° = \frac{1}{2}$$

(iv) $\dfrac{\tan 25° + \tan 20°}{1 - \tan 25° \tan 20°} = \tan(25° + 20°) = \tan 45° = 1$

5. (i) $\dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \tan(2A + A) = \tan 3A$

(ii) $\sin 2\theta \cos \theta + \cos 2\theta \sin \theta$

$$= \sin(2\theta + \theta) = \sin 3\theta$$

6. (i) $\sin(90° - A) = \sin 90° \cos A - \cos 90° \sin A$
$= 1 \cos A - 0 \sin A = \cos A$

(ii) $\cos(90° + A) = \cos 90° \cos A - \sin 90° \sin A$
$= 0 \cos A - 1 \sin A = -\sin A$

7. $\tan(A - B) = 2$ and $\tan B = \frac{1}{4}$

$$\Rightarrow \frac{\tan A - \tan B}{1 + \tan A \tan B} = 2$$

$$\Rightarrow \frac{\tan A - \frac{1}{4}}{1 + \tan A \cdot \frac{1}{4}} = \frac{2}{1}$$

$$\Rightarrow \tan A - \frac{1}{4} = 2 + \tan A \cdot \frac{1}{2}$$

$$\Rightarrow 4 \tan A - 1 = 8 + 2 \tan A$$

$$\Rightarrow 2 \tan A = 9$$

$$\Rightarrow \tan A = \frac{9}{2} = 4\frac{1}{2}$$

8. $\tan A = \frac{1}{2}, \tan B = \frac{1}{3}$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$$\Rightarrow (A + B) = \tan^{-1}(1) = \frac{\pi}{4} (\text{or } 45°)$$

9. $\tan(A + B) = 1$ and $\tan A = \frac{1}{3}$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \frac{\frac{1}{3} + \tan B}{1 - \frac{1}{3} \cdot \tan B} = \frac{1}{1}$$

$$\Rightarrow \frac{1}{3} + \tan B = 1 - \frac{1}{3} \tan B$$

$$\Rightarrow 1 + 3 \tan B = 3 - \tan B$$

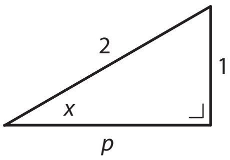
$$\Rightarrow 4 \tan B = 2$$

$$\Rightarrow \tan B = \frac{2}{4} = \frac{1}{2}$$

10. If $\sin x = \frac{1}{2}$ $\Rightarrow$ Pythagoras: $p^2 + 1^2 = 2^2$

$$\Rightarrow p^2 = 4 - 1 = 3$$

$$\Rightarrow p = \sqrt{3}$$



Hence, $\cos x = \frac{\sqrt{3}}{2}$

$$\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

11. $\tan 15° = \tan(45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Hence, $\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1}$

$$= \frac{\sqrt{3}(\sqrt{3}-1) - 1(\sqrt{3}-1)}{\sqrt{3}(\sqrt{3}-1) + 1(\sqrt{3}-1)}$$

$$= \frac{3 - \sqrt{3} - \sqrt{3} + 1}{3 - \sqrt{3} + \sqrt{3} - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Hence, $\tan^2 15° = (\tan 15°)(\tan 15°)$

$= (2 - \sqrt{3})(2 - \sqrt{3})$

$= 2(2 - \sqrt{3}) - \sqrt{3}(2 - \sqrt{3})$

$= 4 - 2\sqrt{3} - 2\sqrt{3} + 3 = 7 - 4\sqrt{3}$

12. Prove: $\tan\left(\frac{\pi}{4} + A\right) = \frac{\cos A + \sin A}{\cos A - \sin A}$

Proof: $\tan\left(\frac{\pi}{4} + A\right) = \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A}$

$$= \frac{1 + \tan A}{1 - 1 \,.\, \tan A}$$

$$= \frac{1 + \frac{\sin A}{\cos A}}{1 - \frac{\sin A}{\cos A}}$$

$$= \frac{\frac{\cos A + \sin A}{\cos A}}{\frac{\cos A - \sin A}{\cos A}}$$

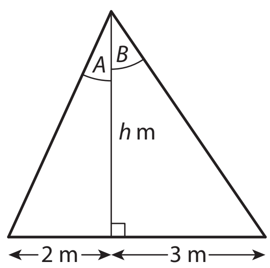
$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

13. $\cos(A + B)\cos B + \sin(A + B)\sin B$

$= \cos[(A + B) - B] = \cos(A + B - B) = \cos A$

14. First Triangle: $\tan A = \frac{2}{h}$

Second Triangle: $\tan B = \frac{3}{h}$



If $A + B = 45°$

$\Rightarrow \tan(A + B) = \tan 45°$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \frac{\frac{2}{h} + \frac{3}{h}}{1 - \frac{2}{h} \cdot \frac{3}{h}} = \frac{\frac{2+3}{h}}{\frac{h^2-6}{h^2}}$

$\Rightarrow \frac{5}{h} \cdot \frac{h^2}{h^2 - 6} = 1$

$\Rightarrow \dfrac{5h}{h^2 - 6} = \dfrac{1}{1}$

$\Rightarrow h^2 - 6 = 5h$

$\Rightarrow h^2 - 5h - 6 = 0$

$\Rightarrow (h - 6)(h + 1) = 0$

$\Rightarrow h = 6 \quad \text{or} \quad h = -1$ (not valid)

15. $\sin A = \sin (A + 30°)$

$\Rightarrow \sin A = \sin A \cos 30° + \cos A \sin 30°$

$\Rightarrow \sin A = \sin A \cdot \dfrac{\sqrt{3}}{2} + \cos A \cdot \dfrac{1}{2}$

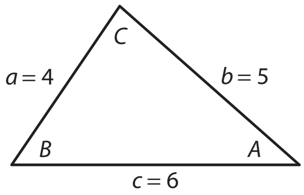
$\Rightarrow 2 \sin A = \sqrt{3} \sin A + \cos A \,.\, 1$

$\Rightarrow \sin A(2 - \sqrt{3}) = \cos A \,.\, 1$

$\Rightarrow \dfrac{\sin A}{\cos A} = \dfrac{1}{2 - \sqrt{3}}$

$\Rightarrow \tan A = \dfrac{1}{2 - \sqrt{3}} \cdot \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{2 + \sqrt{3}}{4 - 3} = \dfrac{2 + \sqrt{3}}{1} = 2 + \sqrt{3}$

16. (i) cos rule: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



$= \dfrac{(5)^2 + (6)^2 - (4)^2}{2(5)(6)}$

$= \dfrac{45}{60} = \dfrac{3}{4}$

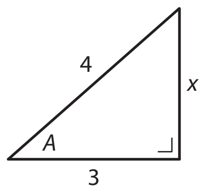
$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{(4)^2 + (5)^2 - (6)^2}{2(4)(5)}$

$= \dfrac{5}{40} = \dfrac{1}{8}$

Hence, $\cos A + \cos C = \dfrac{3}{4} + \dfrac{1}{8} = \dfrac{7}{8}$

(ii) $\cos A = \dfrac{3}{4} \quad \Rightarrow \quad$ Pythagoras: $x^2 + (3)^2 = (4)^2$



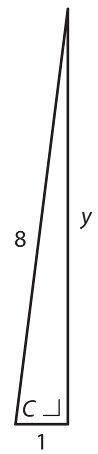
$\Rightarrow \quad x^2 + 9 = 16$

$\Rightarrow \quad x^2 = 16 - 9 = 7$

$\Rightarrow \quad x = \sqrt{7}$

Hence, $\sin A = \dfrac{\sqrt{7}}{4}$

$\cos C = \dfrac{1}{8} \quad \Rightarrow \quad$ Pythagoras: $y^2 + (1)^2 = (8)^2$



$\Rightarrow \quad y^2 + 1 = 64$

$\Rightarrow \quad y^2 = 64 - 1 = 63$

$\Rightarrow \quad y = \sqrt{63} = 3\sqrt{7}$

Hence, $\sin C = \dfrac{3\sqrt{7}}{8}$

$\cos (A + C) = \cos A \cos C - \sin A \sin C$

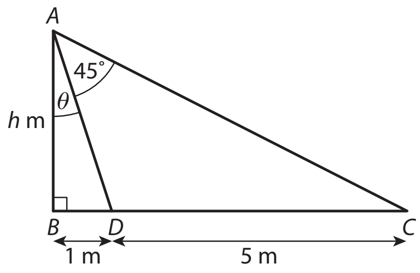
$= \dfrac{3}{4} \cdot \dfrac{1}{8} - \dfrac{\sqrt{7}}{4} \cdot \dfrac{3\sqrt{7}}{8}$

$= \dfrac{3}{32} - \dfrac{3\,.\,7}{32}$

$= \dfrac{3 - 21}{32} = \dfrac{-18}{32} = \dfrac{-9}{16}$

17. Triangle $ABD \Rightarrow \tan\theta = \frac{1}{h}$

Triangle $ABC \Rightarrow \tan(\theta + 45°) = \frac{6}{h}$



$$\Rightarrow \frac{\tan\theta + \tan 45°}{1 - \tan\theta\tan 45°} = \frac{6}{h}$$

$$\Rightarrow \frac{\frac{1}{h} + 1}{1 - \frac{1}{h}\cdot 1} = \frac{6}{h}$$

$$\Rightarrow h\left(\frac{1}{h} + 1\right) = 6\left(1 - \frac{1}{h}\right)$$

$$\Rightarrow 1 + h = 6 - \frac{6}{h}$$

$$\Rightarrow h + h^2 = 6h - 6$$

$$\Rightarrow h^2 - 5h + 6 = 0$$

$$\Rightarrow (h-2)(h-3) = 0$$

$$\Rightarrow h = 2\text{ m} \quad \text{or} \quad 3\text{ m}$$

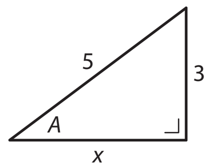
Exercise 8.3

1. $\sin A = \frac{3}{5} \Rightarrow$ Pythagoras: $x^2 + (3)^2 = (5)^2$

$\Rightarrow x^2 + 9 = 25$

$\Rightarrow x^2 = 25 - 9 = 16$

$\Rightarrow x = \sqrt{16} = 4$



Hence, $\cos A = \frac{4}{5}$ and $\tan A = \frac{3}{4}$

(i) $\sin 2A = 2\sin A\cos A = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

(iii) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{\frac{24}{25}}{\frac{7}{25}} = \frac{24}{7}$

2. $\tan A = \frac{1}{2}$

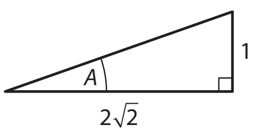
(i) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

(ii) $\sin 2A = \frac{2\tan A}{1 + \tan^2 A} = \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)^2} = \frac{1}{1 + \frac{1}{4}} = \frac{1}{1\frac{1}{4}} = \frac{4}{5}$

3. Using the diagram, $\tan A = \frac{1}{2\sqrt{2}}$.

$\Rightarrow \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$$= \frac{1 - \left(\frac{1}{2\sqrt{2}}\right)^2}{1 + \left(\frac{1}{2\sqrt{2}}\right)^2} = \frac{1 - \frac{1}{8}}{1 + \frac{1}{8}} = \frac{\frac{7}{8}}{\frac{9}{8}} = \frac{7}{9}$$



4. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$$\cos 2A = \frac{3}{8}$$

$$\Rightarrow 1 - 2\sin^2 A = \frac{3}{8}$$

$$\Rightarrow \quad -2\sin^2 A = \frac{3}{8} - 1 = \frac{-5}{8}$$

$$\Rightarrow \quad \sin^2 A = \frac{5}{16}$$

$$\Rightarrow \quad \sin A = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4} \text{ where } 0° < A < 90°$$

$$\cos 2A = \frac{3}{8}$$

$$\Rightarrow 2\cos^2 A - 1 = \frac{3}{8}$$

$$\Rightarrow \quad 2\cos^2 A = 1 + \frac{3}{8} = \frac{11}{8}$$

$$\Rightarrow \quad \cos^2 A = \frac{11}{16}$$

$$\Rightarrow \quad \cos A = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4} \text{ where } 0° < A < 90°$$

5. Given $\sin 2A = 2\sin A\cos A$ and $\cos 2A = \cos^2 A - \sin^2 A$

(i) $2\sin 15°\cos 15° = \sin 2(15°) = \sin 30° = \frac{1}{2}$

(ii) $2\sin 75°\cos 75° = \sin 2(75°)$

$$= \sin 150° = \frac{1}{2}$$

(iii) $\cos^2 22\frac{1}{2}° - \sin^2 22\frac{1}{2}°$

$$= \cos 2\left(22\frac{1}{2}°\right)$$

$$= \cos 45° = \frac{1}{\sqrt{2}}$$

6. $$\frac{2\tan 22\frac{1}{2}°}{1 - \tan^2 22\frac{1}{2}°} = \tan 2\left(22\frac{1}{2}°\right)$$

$$= \tan 45° = 1$$

7. $\cos 3A = \cos(2A + A)$

$$= \cos 2A\cos A - \sin 2A\sin A$$
$$= (2\cos^2 A - 1)\cos A - 2\sin A\cos A\sin A$$
$$= 2\cos^3 A - \cos A - 2\cos A\sin^2 A$$
$$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$$
$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$
$$= 4\cos^3 A - 3\cos A$$

8. (i) Prove: $(\sin A + \cos A)^2 = 1 + \sin 2A$

Proof: $(\sin A + \cos A)^2$

$$= \sin^2 A + 2\sin A\cos A + \cos^2 A$$
$$= \cos^2 A + \sin^2 A + 2\sin A\cos A$$
$$= 1 + \sin 2A$$

(ii) Prove: $\dfrac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$

Proof: $\dfrac{\cos 2A}{\cos A + \sin A} = \dfrac{\cos^2 A - \sin^2 A}{\cos A + \sin A}$

$= \dfrac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A + \sin A}$

$= \cos A - \sin A$

9. Show that $1 - (\cos x - \sin x)^2 = \sin 2x$

Proof: $1 - (\cos x - \sin x)^2 = 1 - [\cos^2 x - 2 \sin x \cos x + \sin^2 x]$

$= 1 - [\cos^2 x + \sin^2 x - \sin 2x]$

$= 1 - [1 - \sin 2x]$

$= 1 - 1 + \sin 2x = \sin 2x$

10. $\tan A = \dfrac{1}{2}$

$\Rightarrow \tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A} = \dfrac{2\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2}$

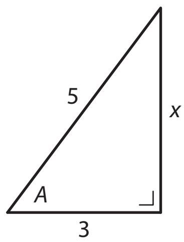
$= \dfrac{1}{1 - \frac{1}{4}} = \dfrac{1}{\frac{3}{4}} = \dfrac{4}{3}$

11. $\cos A = \dfrac{3}{5}$ $\Rightarrow$ Pythagoras: $x^2 + (3)^2 = (5)^2$

$\Rightarrow$ $x^2 + 9 = 25$

$\Rightarrow$ $x^2 = 25 - 9 = 16$

$\Rightarrow$ $x = \sqrt{16} = 4$



Hence, $\sin A = \dfrac{4}{5}$

(i) $\sin 2A = 2 \sin A \cos A = 2\left(\dfrac{4}{5}\right)\left(\dfrac{3}{5}\right) = \dfrac{24}{25}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = \left(\dfrac{3}{5}\right)^2 - \left(\dfrac{4}{5}\right)^2$

$= \dfrac{9}{25} - \dfrac{16}{25}$

$= -\dfrac{7}{25}$

12. Prove that $\dfrac{1 - \cos 2A}{\sin 2A} = \tan A$

Proof: $\dfrac{1 - \cos 2A}{\sin 2A} = \dfrac{1 - \frac{1 - \tan^2 A}{1 + \tan^2 A}}{\frac{2 \tan A}{1 + \tan^2 A}}$

$= \dfrac{\frac{1 + \tan^2 A - (1 - \tan^2 A)}{1 + \tan^2 A}}{\frac{2 \tan A}{1 + \tan^2 A}}$

$= \dfrac{1 + \tan^2 A - 1 + \tan^2 A}{2 \tan A}$

$= \dfrac{2 \tan^2 A}{2 \tan A} = \tan A$

13. Show that $\dfrac{2 \tan A}{1 + \tan^2 A} = \sin 2A$

Proof: $\dfrac{2 \tan A}{1 + \tan^2 A} = \dfrac{\frac{2 \sin A}{\cos A}}{\sec^2 A} = \dfrac{\frac{2 \sin A}{\cos A}}{\frac{1}{\cos^2 A}}$

$$= 2\frac{\sin A}{\cos A} \cdot \frac{\cos^2 A}{1}$$
$$= 2 \sin A \cos A$$
$$= \sin 2A$$

14. $\tan 2\theta = \frac{4}{3}$

$\Rightarrow \quad \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{4}{3}$

$\Rightarrow \quad 6 \tan \theta = 4 - 4 \tan^2 \theta$

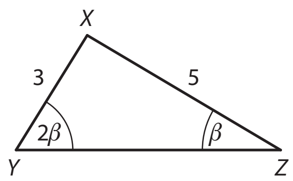
$\Rightarrow \quad 4 \tan^2 \theta + 6 \tan \theta - 4 = 0$

$\Rightarrow \quad 2 \tan^2 \theta + 3 \tan \theta - 2 = 0$

$\Rightarrow (2 \tan \theta - 1)(\tan \theta + 2) = 0$

$\Rightarrow \quad \tan \theta = \frac{1}{2}$ or -2

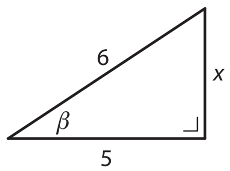
15. (i) sine rule: $\frac{5}{\sin 2\beta} = \frac{3}{\sin \beta}$



$\Rightarrow \quad 3 \sin 2\beta = 5 \sin \beta$

$\Rightarrow \quad \sin 2\beta = \frac{5 \sin \beta}{3} = \frac{5}{3} \sin \beta$

(ii) $\sin 2\beta = 2 \sin \beta \cos \beta = \frac{5 \sin \beta}{3}$



$\Rightarrow \quad 6 \sin \beta \cos \beta = 5 \sin \beta$

$\Rightarrow \quad \cos \beta = \frac{5}{6}$

Pythagoras: $x^2 + (5)^2 = (6)^2$

$\Rightarrow \quad x^2 + 25 = 36$

$\Rightarrow \quad x^2 = 36 - 25 = 11$

$\Rightarrow \quad x = \sqrt{11}$

Hence, $\tan \beta = \frac{\sqrt{11}}{5}$

16. (i) $\tan (A + B) = -1$ and $\tan A = \frac{4}{3}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -1$

$\Rightarrow \quad \frac{\frac{4}{3} + \tan B}{1 - \frac{4}{3} \cdot \tan B} = \frac{-1}{1}$

$\Rightarrow \quad \frac{4}{3} + \tan B = -1 + \frac{4}{3} \tan B$

$\Rightarrow \quad 4 + 3 \tan B = -3 + 4 \tan B$

$\Rightarrow \quad -\tan B = -7$

$\Rightarrow \quad \tan B = 7$

(ii) $\sin 2B = \frac{2 \tan B}{1 + \tan^2 B}$

$$= \frac{2(7)}{1 + (7)^2} = \frac{14}{50} = \frac{7}{25}$$

17. (i) Show that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

Proof: $$\frac{\sin 2A}{1+\cos 2A} = \frac{\frac{2\tan A}{1+\tan^2 A}}{1+\frac{1-\tan^2 A}{1+\tan^2 A}}$$

$$= \frac{\frac{2\tan A}{1+\tan^2 A}}{\frac{1+\tan^2 A+1-\tan^2 A}{1+\tan^2 A}}$$

$$= \frac{2\tan A}{2} = \tan A$$

(ii) Let $A = 22\frac{1}{2}^\circ$ $\Rightarrow$ $\dfrac{\sin 2\left(22\frac{1}{2}^\circ\right)}{1+\cos 2\left(22\frac{1}{2}^\circ\right)^2} = \tan 22\frac{1}{2}^\circ$

$$\Rightarrow \frac{\sin 45^\circ}{1+\cos 45^\circ} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \tan 22\frac{1}{2}^\circ$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}} = \tan 22\frac{1}{2}^\circ$$

$$= \frac{1}{\sqrt{2}+1} = \tan 22\frac{1}{2}^\circ$$

$$\Rightarrow \tan 22\frac{1}{2}^\circ = \frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1$$

18. (i) $\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow$ $\cos 4A = \cos 2(2A) = 1 - 2\sin^2 2A$

(ii) $\cos 2A = 2\cos^2 A - 1$

$\Rightarrow$ $\cos 4A = \cos 2(2A) = 2\cos^2 2A - 1$

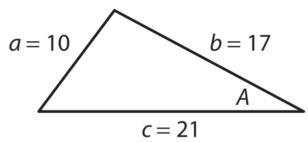
Show that $\dfrac{1-\cos 4A}{1+\cos 4A} = \tan^2 2A$

Proof: $$\frac{1-\cos 4A}{1+\cos 4A} = \frac{1-(1-2\sin^2 2A)}{1+(2\cos^2 2A-1)}$$

$$= \frac{1-1+2\sin^2 2A}{1+2\cos^2 2A-1}$$

$$= \frac{2\sin^2 2A}{2\cos^2 2A} = \tan^2 2A$$

19. (i) $$\cos A = \frac{b^2+c^2-a^2}{2bc}$$



$$= \frac{(17)^2+(21)^2-(10)^2}{2(17)(21)}$$

$$= \frac{289+441-100}{714} = \frac{630}{714} = \frac{15}{17}$$

(ii) $$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\Rightarrow \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\Rightarrow \frac{15}{17} = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

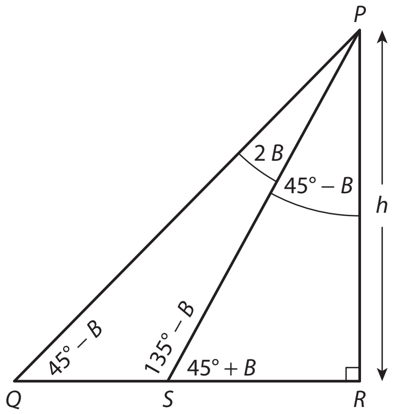
$$\Rightarrow 15 + 15\tan^2\frac{A}{2} = 17 - 17\tan^2\frac{A}{2}$$

$$\Rightarrow \qquad 32\tan^2\frac{A}{2} = 2$$

$$\Rightarrow \qquad \tan^2\frac{A}{2} = \frac{1}{16}$$

$$\Rightarrow \qquad \tan\frac{A}{2} = \sqrt{\frac{1}{16}} = \frac{1}{4} \text{ if } A \text{ is acute}$$

20. (i) In $\triangle SRP$



$$\frac{\tan(45° - B)}{1} = \frac{|SR|}{h}$$

$$\Rightarrow \qquad |SR| = h\tan(45° - B)$$

(ii) $\tan(45° - B) = \dfrac{\tan 45° - \tan B}{1 + \tan 45° \tan B}$

$$= \frac{1 - \tan B}{1 + 1\,.\tan B} = \frac{1 - \tan B}{1 + \tan B}$$

$$\Rightarrow \qquad |SR| = h\left(\frac{1 - \tan B}{1 + \tan B}\right)$$

In $\triangle PQR$: $\tan(45° - B) = \dfrac{h}{|QR|}$

$$\Rightarrow |QR|\tan(45° - B) = h$$

$$\Rightarrow |QR| = \frac{h}{\tan(45° - B)}$$

$$\Rightarrow |QR| = \frac{h}{\dfrac{1 - \tan B}{1 + \tan B}} = \frac{h(1 + \tan B)}{1 - \tan B}$$

Hence, $|QS| = |QR| - |SR|$

$$= \frac{h(1 + \tan B)}{1 - \tan B} - \frac{h(1 - \tan B)}{1 + \tan B}$$

$$= \frac{h(1 + \tan B)(1 + \tan B) - h(1 - \tan B)(1 - \tan B)}{(1 - \tan B)(1 + \tan B)}$$

$$= \frac{h[1 + 2\tan B + \tan^2 B - (1 - 2\tan B + \tan^2 B)]}{1 - \tan^2 B}$$

$$= \frac{h[1 + 2\tan B + \tan^2 B - 1 + 2\tan B - \tan^2 B]}{1 - \tan^2 B}$$

$$= h \cdot \frac{2(2\tan B)}{1 - \tan^2 B} = 2h\tan 2B$$

Exercise 8.4

1. (i) $\sin 5x + \sin 3x = 2\sin\dfrac{5x + 3x}{2}\cos\dfrac{5x - 3x}{2}$

$$= 2\sin 4x\cos x$$

(ii) $\sin 4x - \sin 2x = 2\cos\dfrac{4x + 2x}{2}\sin\dfrac{4x - 2x}{2}$

$$= 2\cos 3x\sin x$$

(iii) $\cos 3x + \cos x = 2\cos\dfrac{3x + x}{2}\cos\dfrac{3x - x}{2}$

$$= 2\cos 2x\cos x$$

(iv) $\cos 7\theta - \cos 5\theta = -2\sin\dfrac{7\theta + 5\theta}{2}\sin\dfrac{7\theta - 5\theta}{2}$

$$= -2\sin 6\theta\sin\theta$$

(v) $\cos 3\theta - \cos\theta = -2\sin\dfrac{3\theta + \theta}{2}\sin\dfrac{3\theta - \theta}{2}$

$$= -2\sin 2\theta\sin\theta$$

(vi) $\sin 3\theta - \sin 7\theta = 2\cos\frac{3\theta + 7\theta}{2}\sin\frac{3\theta - 7\theta}{2}$

$= 2\cos 5\theta \sin(-2\theta)$

$= -2\cos 5\theta \sin 2\theta$

2. (i) $\cos 80° + \cos 40° = 2\cos\frac{80° + 40°}{2}\cos\frac{80° - 40°}{2}$

$= 2\cos 60° \cos 20°$

$= 2 \cdot \frac{1}{2}\cos 20° = \cos 20°$

(ii) $\sin 125° - \sin 55° = 2\cos\frac{125° + 55°}{2}\sin\frac{125° - 55°}{2}$

$= 2\cos 90° \sin 35°$

$= 2(0)\sin 35° = 0$

(iii) $\cos 75° - \cos 15° = -2\sin\frac{75° + 15°}{2}\sin\frac{75° - 15°}{2}$

$= -2\sin 45° \cos 30°$

$= -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{2}}$

3. (i) $\sin 75° - \sin 15° = 2\cos\frac{75° + 15°}{2}\sin\frac{75° - 15°}{2}$

$= 2\cos 45° \sin 30°$

$= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

(ii) $\sin 10° + \sin 80° = 2\sin\frac{10° + 80°}{2}\cos\frac{10° - 80°}{2}$

$= 2\cos 45° \cos(-35°)$

$= 2 \cdot \frac{1}{\sqrt{2}}\cos 35°$

$= \sqrt{2}\cos 35°$

4. (i) $\cos(x + 45°) + \cos(x - 45°)$

$= 2\cos\frac{x + 45° + x - 45°}{2}\cos\frac{x + 45° - (x - 45°)}{2}$

$= 2\cos\frac{2x}{2}\cos\frac{x + 45° - x + 45°}{2}$

$= 2\cos x \cos 45°$

$= 2\cos x \cdot \frac{1}{\sqrt{2}} = \sqrt{2}\cos x$

(ii) $\cos(x + 60°) - \cos(x - 60°)$

$= -2\sin\frac{x + 60° + x - 60°}{2} \cdot \sin\frac{x + 60° - (x - 60°)}{2}$

$= -2\sin\frac{2x}{2} \cdot \sin\frac{x + 60° - x + 60°}{2}$

$= -2\sin x \cdot \sin 60°$

$= -2\sin x \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}\sin x$

5. (i) $2\sin 3A \cos 2A = \sin(3A + 2A) + \sin(3A - 2A)$

$= \sin 5A + \sin A$

(ii) $2\cos 4x \sin x = \sin(4x + x) - \sin(4x - x)$

$= \sin 5x - \sin 3x$

(iii) $2\cos 5A \cos 2A = \cos(5A + 2A) + \cos(5A - 2A)$

$= \cos 7A + \cos 3A$

(iv) $-2\cos 6A \sin 2A = -[\cos(6A - 2A) - \cos(6A + 2A)]$
$= -[\cos 4A - \cos 8A]$
$= \cos 8A - \cos 4A$

(v) $\sin 2A \sin A = \frac{1}{2}[\cos(2A - A) - \cos(2A + A)]$
$= \frac{1}{2}[\cos A - \cos 3A]$
$= -\frac{1}{2}[\cos 3A - \cos A]$

(vi) $\sin x \cos 5x = \frac{1}{2}[\sin(x + 5x) + \sin(x - 5x)]$
$= \frac{1}{2}[\sin 6x + \sin(-4x)]$
$= \frac{1}{2}[\sin 6x - \sin 4x]$

6. (i) $2\sin 75° \cos 45° = \sin(75° + 45°) + \sin(75° - 45°)$
$= \sin 120° + \sin 30°$
$= \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1}{2}(\sqrt{3} + 1)$

(ii) $10 \sin 67\frac{1}{2}° \sin 22\frac{1}{2}° = 5\left[2\sin 67\frac{1}{2}° \sin 22\frac{1}{2}°\right]$
$= 5\left[\cos\left(67\frac{1}{2}° - 22\frac{1}{2}°\right) - \cos\left(67\frac{1}{2}° + 22\frac{1}{2}°\right)\right]$
$= 5[\cos 45° - \cos 90°]$
$= 5\left[\frac{1}{\sqrt{2}} - 0\right]$
$= 5 \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

7. (i) Show: $2\cos(A + 45°)\sin(A - 45°) = \sin 2A - 1$
Proof: $2\cos(A + 45°)\sin(A - 45°)$
$= \sin[A + 45° + (A - 45°)] - \sin[A + 45° - (A - 45°)]$
$= \sin[A + 45° + A - 45°] - \sin[A + 45° - A + 45°)]$
$= \sin 2A - \sin 90°$
$= \sin 2A - 1$

(ii) Show that $\frac{\cos 50° - \cos 70°}{\sin 70° - \sin 50°} = \sqrt{3}$

Proof: $\frac{\cos 50° - \cos 70°}{\sin 70° - \sin 50°}$

$= \frac{-2\sin\frac{50° + 70°}{2}\sin\frac{50° - 70°}{2}}{2\cos\frac{70° + 50°}{2}\sin\frac{70° - 50°}{2}}$

$= \frac{-2\sin 60° \cdot \sin(-10°)}{2\cos 60° \cdot \sin 10°}$

$= \frac{2\sin 60° \sin 10°}{2\cos 60° \sin 10°}$

$= \tan 60° = \sqrt{3}$

8. Show that $\dfrac{\sin(\theta + 15°) + \sin(\theta - 15°)}{\cos(\theta + 15°) + \cos(\theta - 15°)} = \tan\theta$

Proof: $\dfrac{\sin(\theta + 15°) + \sin(\theta - 15°)}{\cos(\theta + 15°) + \cos(\theta - 15°)}$

$$= \frac{2\sin\frac{\theta + 15° + \theta - 15°}{2}\cos\frac{\theta + 15° - (\theta - 15°)}{2}}{2\cos\frac{\theta + 15° + \theta - 15°}{2}\cos\frac{\theta + 15° - (\theta - 15°)}{2}}$$

$$= \frac{2\sin\frac{2\theta}{2}\cos\frac{\theta + 15° - \theta + 15°}{2}}{2\cos\frac{2\theta}{2}\cos\frac{\theta + 15° - \theta + 15°}{2}}$$

$$= \frac{\sin\theta\cos 15°}{\cos\theta\cos 15°} = \tan\theta$$

9. Show that $\dfrac{\sin 4A + \sin 2A}{2\sin 3A} = \cos A$

Proof: $\dfrac{\sin 4A + \sin 2A}{2\sin 3A} = \dfrac{2\sin\frac{4A + 2A}{2}\cos\frac{4A - 2A}{2}}{2\sin 3A}$

$$= \frac{2\sin 3A\cos A}{2\sin 3A} = \cos A$$

10. Show $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$

Proof: $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \dfrac{2\cos\frac{5A + 3A}{2}\sin\frac{5A - 3A}{2}}{2\cos\frac{5A + 3A}{2}\cos\frac{5A - 3A}{2}}$

$$= \frac{\cos 4A\sin A}{\cos 4A\cos A} = \tan A$$

11. Show that $2\sin(135° + A)\sin(45° + A) = \cos 2A$

Proof: $2\sin(135° + A)\sin(45° + A)$

$= \cos[135 + A - (45 + A)] - \cos[135 + A + (45 + A)]$

$= \cos[135 + A - 45 - A] - \cos[135 + A + 45 + A]$

$= \cos 90° - \cos(180 + 2A)$

$= 0 - [\cos 180\cos 2A - \sin 180\sin 2A]$

$= -[(-1).\cos 2A - 0.\sin 2A] = \cos 2A$

12. Given that $\tan 3\theta = 2$,

Evaluate $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \dfrac{(\sin 5\theta + \sin\theta) + \sin 3\theta}{(\cos 5\theta + \cos\theta) + \cos 3\theta}$

$$= \frac{2\sin\frac{5\theta + \theta}{2}\cos\frac{5\theta - \theta}{2} + \sin 3\theta}{2\cos\frac{5\theta + \theta}{2}\cos\frac{5\theta - \theta}{2} + \cos 3\theta}$$

$$= \frac{2\sin 3\theta\cos 2\theta + \sin 3\theta}{2\cos 3\theta\cos 2\theta + \cos 3\theta}$$

$$= \frac{\sin 3\theta(2\cos 2\theta + 1)}{\cos 3\theta(2\cos 2\theta + 1)}$$

$= \tan 3\theta = 2$

Exercise 8.5

1. (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

(ii) $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

(iii) $\tan^{-1}(1) = 45°$

(iv) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$

(v) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60°$

(vi) $\tan^{-1}(-1) = -45°$

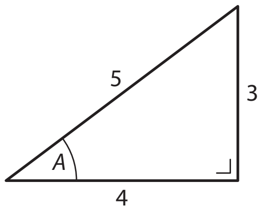
(vii) $\cos^{-1}\left(-\frac{1}{2}\right) = 120°$

(viii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$

2. (i) $\sin^{-1}\left(\frac{3}{5}\right) = A$

and $\tan^{-1}\left(\frac{3}{4}\right) = A$

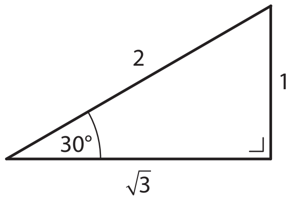
$\Rightarrow \quad \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$



(ii) $\sin^{-1}\left(\frac{1}{2}\right) = 30°$

and $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$

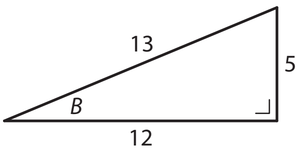
$\Rightarrow \quad \sin^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$



(iii) $\sin^{-1}\left(\frac{5}{13}\right) = B$

and $\tan^{-1}\left(\frac{5}{12}\right) = B$

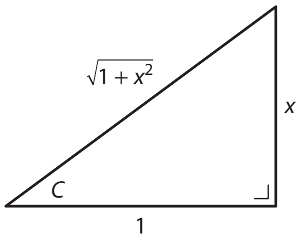
$\Rightarrow \quad \sin^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{5}{12}\right)$



(iv) $\tan^{-1}(x) = C$

$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = C$

$\Rightarrow \quad \tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$



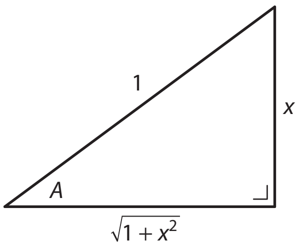
3. (i) $\sin^{-1}\left(\frac{x}{1}\right) = A$

$\Rightarrow \quad \sin(\sin^{-1} x) = \sin A = \frac{x}{1} = x$

(ii) $\sin^{-1}\left(\frac{x}{1}\right) = A$

$\Rightarrow \quad \cos(\sin^{-1} x)$

$= \cos A = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$



$$x^2 + (\sqrt{1-x^2})^2 = 1^2$$

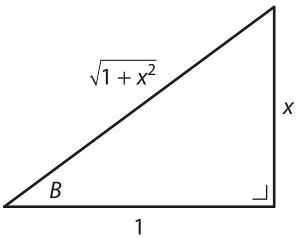
$$\Rightarrow \quad x^2 + 1 - x^2 = 1$$

$$\Rightarrow \quad 1 = 1$$

(iii) $\tan^{-1}\left(\frac{x}{1}\right) = B$

$\Rightarrow \quad \sin\left[\tan^{-1}\left(\frac{x}{1}\right)\right]$

$= \sin B = \frac{x}{\sqrt{1+x^2}}$



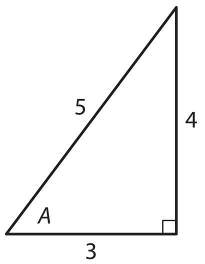
$$x^2 + 1^2 = (\sqrt{1+x^2})^2$$

$$\Rightarrow \quad x^2 + 1 = 1 + x^2$$

4. (i) $\cos^{-1}\left(\frac{3}{5}\right) = A$

$\Rightarrow \quad \sin\left[\cos^{-1}\left(\frac{3}{5}\right)\right]$

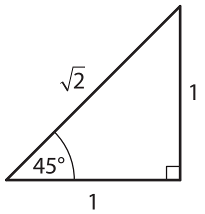
$= \sin A = \frac{4}{5}$



(ii) $\tan^{-1}\left(\frac{1}{1}\right) = 45°$

$\Rightarrow \quad \cos(\tan^{-1} 1)$

$= \cos 45° = \frac{1}{\sqrt{2}}$



(iii) $\tan^{-1}\left(\frac{8}{15}\right) = B$

$\sin\left[\tan^{-1}\left(\frac{8}{15}\right)\right]$

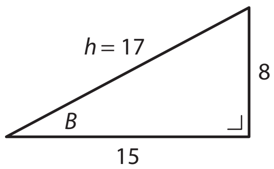
$= \sin B = \frac{8}{17}$

$h^2 = 8^2 + 15^2$

$= 64 + 225$

$= 289$

$\Rightarrow \quad h = \sqrt{289} = 17$

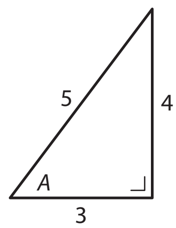


5. (i) $\cos^{-1}\left(\frac{3}{5}\right) = A$

$\Rightarrow \quad \sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

$= \sin 2A$

$= 2\sin A\cos A = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$

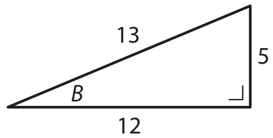


(ii) $\sin^{-1}\left(\frac{5}{13}\right) = B$

$\cos\left[2\sin^{-1}\left(\frac{5}{13}\right)\right] = \cos 2B$

$= \cos^2 B - \sin^2 B$

$= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$



6. (i) $\sin^{-1}\left(\frac{5}{13}\right) = C \quad \Rightarrow \quad \sin C = \frac{5}{13}$

and $\cos C = \frac{12}{13}$

$\sin^{-1}\left(\frac{4}{5}\right) = D \quad \Rightarrow \quad \sin D = \frac{4}{5}$

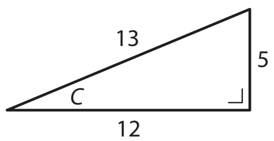
and $\cos D = \frac{3}{5}$

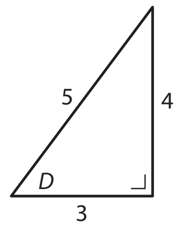
$\sin\left[\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right]$

$= \sin(C + D)$

$= \sin C\cos D + \cos C\sin D$

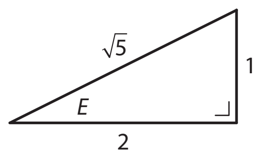
$= \frac{5}{13} \cdot \frac{3}{5} + \frac{12}{13} \cdot \frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$





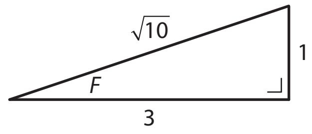
(ii) $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = E \quad \Rightarrow \quad \sin E = \frac{1}{\sqrt{5}}$

$\text{and } \cos E = \frac{2}{\sqrt{5}}$



$\sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = F \quad \Rightarrow \quad \sin F = \frac{1}{\sqrt{10}}$

$\text{and } \cos F = \frac{3}{\sqrt{10}}$



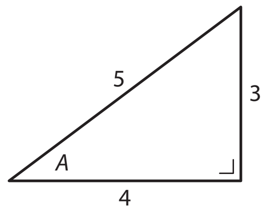
$\sin\left[\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}\right]$

$= \sin(E + F) = \sin E \cos F + \cos E \sin F$

$$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}} = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

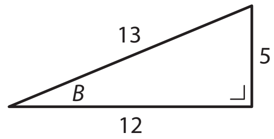
7. $\sin^{-1}\left(\frac{3}{5}\right) = A \quad \Rightarrow \quad \sin A = \frac{3}{5}$

$\text{and } \tan A = \frac{3}{4}$



$\sin^{-1}\left(\frac{5}{13}\right) = B \quad \Rightarrow \quad \sin B = \frac{5}{13}$

$\text{and } \tan B = \frac{5}{12}$



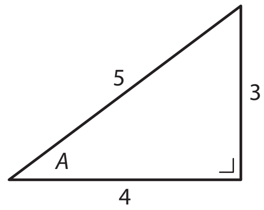
$\tan\left[\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right]$

$= \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{\frac{7}{6}}{1 - \frac{5}{16}} = \frac{\frac{7}{6}}{\frac{11}{16}} = \frac{56}{33}$$

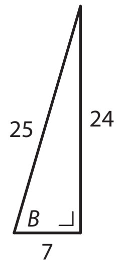
8. $\tan^{-1}\left(\frac{3}{4}\right) = A \quad \Rightarrow \quad \sin A = \frac{3}{5}$

$\text{and } \cos A = \frac{4}{5}$



$\cos^{-1}\left(\frac{7}{25}\right) = B \quad \Rightarrow \quad \sin B = \frac{24}{25}$

$\text{and } \cos B = \frac{7}{25}$



$\sin\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \sin 2A = 2 \sin A \cos A$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

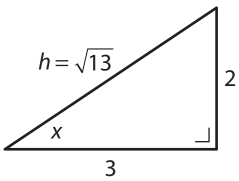
$\sin\left[\cos^{-1}\left(\frac{7}{25}\right)\right] = \sin B = \frac{24}{25}$

Hence, $\sin\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \sin\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$

Revision Exercise 8 (Core)

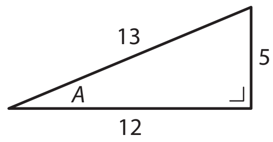
1. $\sin 2x = 2 \sin x \cos x$

$= 2\frac{2}{\sqrt{13}} \cdot \frac{3}{\sqrt{13}} = \frac{12}{13}$



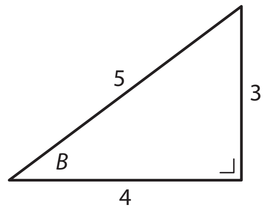
$h^2 = 2^2 + 3^2 = 13$

$\Rightarrow h = \sqrt{13}$



2. $\tan A = \frac{3}{4} \Rightarrow \sin A = \frac{5}{13}$

and $\cos A = \frac{12}{13}$



$\tan B = \frac{3}{4} \Rightarrow \sin B = \frac{3}{5}$

and $\cos B = \frac{4}{5}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= \frac{12}{13} \cdot \frac{4}{5} + \frac{5}{13} \cdot \frac{3}{5} = \frac{48 + 15}{65} = \frac{63}{65}$

3. Show that $(\cos A + \sin A)^2 = 1 + \sin 2A$

Proof: $(\cos A + \sin A)^2 = \cos^2 A + 2 \sin A \cos A + \sin^2 A$

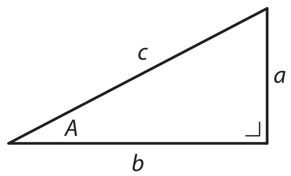
$= 1 + \sin 2A$

4. $\cos x = \frac{4}{5} \Rightarrow \tan x = \frac{3}{4}$

Hence, $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2}$

$= \frac{\frac{6}{4}}{1 - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}$

5. Prove: $\sin^2 A + \cos^2 A = 1$



Proof: $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)$

$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$= \frac{a^2 + b^2}{c^2}$ as $a^2 + b^2 = c^2$(Pythagoras' theorem)

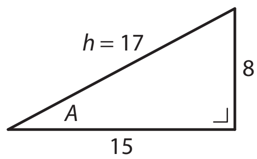
$= \frac{c^2}{c^2} = 1$

6. $\tan A = \frac{8}{15} \Rightarrow$ Pythagoras: $h^2 = 8^2 + 15^2$

$= 64 + 225$

$= 289$

$\Rightarrow h = \sqrt{289} = 17$



(i) $\cos A = \frac{15}{17}$

(ii) $\sin 2A = 2 \sin A \cos A = 2\left(\frac{8}{17}\right)\left(\frac{15}{17}\right) = \frac{240}{289}$

7. (i) $\sin 75° \cos 15° - \cos 75° \sin 15°$

$= \sin(75° - 15°)$

$= \sin 60° = \frac{\sqrt{3}}{2}$

(ii) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

Hence, $2 + 2\cos 2x = 2 + 2(2\cos^2 x - 1)$

$= 2 + 4\cos^2 x - 2 = 4\cos^2 x$

8. $\tan 75° = \tan(45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{3 + \sqrt{3} + \sqrt{3} + 1}{3 + \sqrt{3} - \sqrt{3} - 1}$$

$$= \frac{4 + 2\sqrt{3}}{2}$$

$$= 2 + \sqrt{3} = a + b\sqrt{3}$$

Hence, $a = 2, b = 1$.

9. (i) Show that $\tan\theta \sin\theta + \cos\theta = \sec\theta$

Proof: $\tan\theta \sin\theta + \cos\theta = \dfrac{\sin\theta}{\cos\theta} \cdot \sin\theta + \dfrac{\cos\theta}{1}$

$$= \frac{\sin\theta \sin\theta + \cos\theta \cos\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

(ii) $\cos\theta = \dfrac{5}{13}$

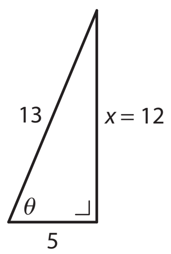
$\Rightarrow \sin\theta = \dfrac{12}{13}$

$$x^2 + (5)^2 = (13)^2$$

$$\Rightarrow x^2 + 25 = 169$$

$$\Rightarrow x^2 = 169 - 25 = 144$$

$$\Rightarrow x = \sqrt{144} = 12$$



Hence, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\left(\frac{12}{13}\right) \cdot \left(\frac{5}{13}\right) = \frac{120}{169}$$

10. (i) $\sin 75° - \sin 15° = 2\cos\dfrac{75° + 15°}{2} \cdot \sin\dfrac{75° - 15°}{2}$

$$= 2\cos 45° \sin 30°$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{k}} \Rightarrow k = 2$$

(ii) $A = \sin^{-1}\dfrac{1}{2} = 30°$

$\Rightarrow \tan 2A = \tan 2(30°) = \tan 60° = \sqrt{3}$

Revision Exercise 8 (Advanced)

1. (i) $\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1$

$\Rightarrow \cos 2A + 1 = 2\cos^2 A$

$\Rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$

(ii) $\sin 40° \cos 20° + \cos 40° \sin 20°$

$= \sin(40° + 20°) = \sin 60° = \frac{\sqrt{3}}{2}$

2. (i) Given $\sin\theta = \frac{4}{5}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= (1 - \sin^2\theta) - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{4}{5}\right)^2 = 1 - \frac{32}{25} = -\frac{7}{25}$

(ii) Show that $2\cos^2 A - \cos 2A - 1 = 0$

Proof: $2\cos^2 A - \cos 2A - 1$

$= 2\cos^2 A - (\cos^2 A - \sin^2 A) - 1$

$= 2\cos^2 A - \cos^2 A + \sin^2 A - 1$

$= (\cos^2 A + \sin^2 A) - 1 = 1 - 1 = 0$

3. (i) $2\sin 4\theta \cos 2\theta = \sin(4\theta + 2\theta) + \sin(4\theta - 2\theta)$

$= \sin 6\theta + \sin 2\theta$

(ii) $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$

$= \cos^2 x + 2\sin x\cos x + \sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x$

$= 2(\cos^2 x + \sin^2 x)$

$= 2(1) = 2$

4. (i) Prove: $\cos(45° + \theta) - \cos(45° - \theta) = -\sqrt{2}\sin\theta$

Proof: $\cos(45° + \theta) - \cos(45° - \theta)$

$= -2\sin\frac{45° + \theta + 45° - \theta}{2} \cdot \sin\frac{45° + \theta - (45° - \theta)}{2}$

$= -2\sin\frac{90°}{2} \cdot \sin\frac{45° + \theta - 45° + \theta}{2}$

$= -2\sin 45° \sin\theta$

$= -2 \cdot \frac{1}{\sqrt{2}}\sin\theta = -\sqrt{2}\sin\theta$

(ii) Prove that $\frac{1}{\cos\theta} - \cos\theta = \tan\theta\sin\theta$

Proof $\frac{1}{\cos\theta} - \cos\theta = \frac{1 - \cos\theta\cos\theta}{\cos\theta}$

$= \frac{1 - \cos^2\theta}{\cos\theta}$

$= \frac{\sin^2\theta}{\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} \cdot \sin\theta = \tan\theta\sin\theta$

5. (i) $\cos^2 15° - \sin^2 15°$

$$= \cos 2(15°) = \cos 30° = \frac{\sqrt{3}}{2}$$

(ii) Prove that $\dfrac{\sin 3\theta}{\sin \theta} - \dfrac{\cos 3\theta}{\cos \theta} = 2$

Proof: $$\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = \frac{\sin 3\theta \cos \theta - \cos 3\theta \sin \theta}{\sin \theta \cos \theta}$$

$$= \frac{\sin (3\theta - \theta)}{\sin \theta \cos \theta}$$

$$= \frac{\sin 2\theta}{\sin \theta \cos \theta}$$

$$= \frac{2 \sin \theta \cos \theta}{\sin \theta \cos \theta} = 2$$

6. (i) Show than $\tan 15° = 2 - \sqrt{3}$

Proof: $\tan 15° = \tan (45° - 35°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3} - 1}{\sqrt{3}}}{\frac{\sqrt{3} + 1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - \sqrt{3} - \sqrt{3} + 1}{3 - \sqrt{3} + \sqrt{3} - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

(ii) Prove that $\dfrac{\cos 5\theta - \cos 3\theta}{\sin 4\theta} = -2 \sin \theta$

Proof: $$\frac{\cos 5\theta - \cos 3\theta}{\sin 4\theta}$$

$$= \frac{-2 \sin \frac{5\theta + 3\theta}{2} \sin \frac{5\theta - 3\theta}{2}}{\sin 4\theta}$$

$$= \frac{-2 \sin 4\theta \sin \theta}{\sin \theta} = -2 \sin \theta$$

7. Prove: $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Proof: $$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

8. $A + B = \dfrac{\pi}{4}$

$$\Rightarrow \tan (A + B) = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A + \tan A \tan B = 1 - \tan B$$

$$\Rightarrow \tan A (1 + \tan B) = 1 - \tan B$$

$$\Rightarrow \tan A = \frac{1 - \tan B}{1 + \tan B}$$

Hence, $(1 + \tan A)(1 + \tan B)$

$$= \left(1 + \frac{1 - \tan B}{1 + \tan B}\right)(1 + \tan B)$$

$$= 1 + \tan B + \left(\frac{1 - \tan B}{1 + \tan B}\right)(1 + \tan B)$$

$$= 1 + \tan B + 1 - \tan B = 2$$

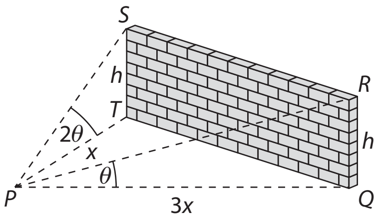
9. $\sin 105° = 2 \cos \frac{105° + 15°}{2} \sin \frac{105° - 15°}{2}$

$$= 2 \cos 60° \sin 45°$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

10. Triangle $PTS \Rightarrow \tan 2\theta = \frac{h}{x}$



Triangle $PQR \Rightarrow \tan \theta = \frac{h}{3x} = \frac{1}{3} \cdot \frac{h}{x} = \frac{1}{3} \tan 2\theta$

$$\Rightarrow 3 \tan \theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\Rightarrow 3 = \frac{2}{1 - \tan^2 \theta}$$

$$\Rightarrow 2 = 3 - 3 \tan^2 \theta$$

$$\Rightarrow 3 \tan^2 \theta = 1$$

$$\Rightarrow \tan^2 \theta = \frac{1}{3}$$

$$\Rightarrow \tan \theta = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} (\theta \text{ is acute})$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Revision Exercise 8 (Extended–Response)

1. Prove that $\cos 2x = 1 - 2 \sin^2 x$

Proof: $\cos 2x = \cos(x + x)$

$= \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

$= 1 - \sin^2 x - \sin^2 x = 1 - 2 \sin^2 x$

Prove that $\sin 3x = 3 \sin x - 4 \sin^3 x$

Proof: $\sin 3x = \sin(2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2 \sin x \cos x \cos x + (1 - 2 \sin^2 x) \sin x$

$= 2 \sin x(\cos^2 x) + \sin x - 2 \sin^3 x$

$= 2 \sin x(1 - \sin^2 x) + \sin x - 2 \sin^3 x$

$= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x$

$= 3 \sin x - 4 \sin^3 x$

2 Show that $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2 + 2 \cos(A - B)$

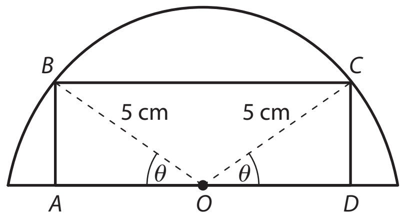
Proof: $(\cos A + \cos B)^2 + (\sin A + \sin B)^2$

$= \cos^2 A + 2 \cos A \cos B + \cos^2 B + \sin^2 A + 2 \sin A \sin B + \sin^2 B$

$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) + 2(\cos A \cos B + \sin A \sin B)$

$= 1 + 1 + 2 \cos(A - B) = 2 + 2 \cos(A - B)$

3 (i) Triangle AOB: $\sin\theta = \dfrac{|AB|}{5}$



$\Rightarrow |AB| = 5\sin\theta = |CD|$

and $\cos\theta = \dfrac{|AO|}{5}$

$\Rightarrow |AO| = 5\cos\theta = |OD|$

$\Rightarrow |AD| = 5\cos\theta + 5\cos\theta = 10\cos\theta$

Hence, perimeter $p = 2(|AD| + |AB|)$

$= 2(10\cos\theta + 5\sin\theta)$

$= 20\cos\theta + 10\sin\theta$

(ii) Area rectangle $= |AB|\,.\,|AD|$

$= 5\sin\theta\,.\,10\cos\theta$

$= 50\sin\theta\cos\theta$

$= 25\,.\,2\sin\theta\cos\theta$

$\Rightarrow k\sin 2\theta = 25\sin 2\theta \Rightarrow k = 25$

4. (i) Given $\cos(A + B) = \cos A\cos B - \sin A\sin B$

Then, replacing B with $-B$,

$\cos(A - B) = \cos A\cos(-B) - \sin A\sin(-B)$

$= \cos A\cos B + \sin A\sin B$, as $\sin(-B) = \sin B$

(ii) Thus $\sin(A + B) = \cos[90° - (A + B)]$

$= \cos[(90° - A) - B)]$

$= \cos(90° - A)\cos B + \sin(90° - A)\sin B$

$= \sin A\cos B + \cos A\cos B$

(iii) (a) $\cos(30°) = \dfrac{\sqrt{3}}{2}$ (b) $\sin(30°) = \dfrac{1}{2}$

(iv) $\sin(A - 30°) = 3\cos A$

$\sin A\cos 30° - \cos A\sin 30° = 3\cos A$

$\sin A\left(\dfrac{\sqrt{3}}{2}\right) - \cos A\left(\dfrac{1}{2}\right) = 3\cos A$

$\sqrt{3}\sin A - \cos A = 6\cos A$

$\sqrt{3}\sin A = 7\cos A$

$\dfrac{\sin A}{\cos A} = \dfrac{7}{\sqrt{3}}$

$\tan A = \dfrac{7}{\sqrt{3}}$

5. (i) Show that $\sqrt{2\sin^2\theta + 6\cos^2\theta - 2} = 2\cos\theta$

Proof: $\sqrt{2\sin^2\theta + 6\cos^2\theta - 2} = \sqrt{2(1 - \cos^2\theta) + 6\cos^2\theta - 2}$

$= \sqrt{2 - 2\cos^2\theta + 6\cos^2\theta - 2}$

$= \sqrt{4\cos^2\theta}$

$= 2\cos\theta$

(ii) Equation: $a\sin^2 2x + \cos 2x - b = 0$

$x = 0°$ $\Rightarrow a\sin^2 2(0°) + \cos 2(0°) - b = 0$

$\Rightarrow a(0) + \cos 0° - b = 0$

$\Rightarrow 0 + 1 - b = 0 \Rightarrow b = 1$

$x = 60°$ $\Rightarrow a\sin^2 2(60°) + \cos 2(60°) - b = 0$

$\Rightarrow a[\sin 120°]^2 + \cos 120° - b = 0$

$\Rightarrow a\left(\dfrac{3}{4}\right) - \dfrac{1}{2} - 1 = 0$

$\Rightarrow 3a - 2 - 4 = 0$

$\Rightarrow 3a = 6$

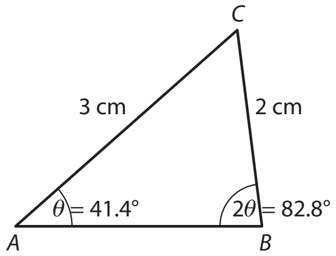
$\Rightarrow a = 2$

6. $\frac{3}{\sin 2\theta} = \frac{2}{\sin \theta}$ $\Rightarrow 2 \sin 2\theta = 3 \sin \theta$

$\Rightarrow 2 \,.\, 2 \sin \theta \cos \theta = 3 \sin \theta$

$\Rightarrow \cos \theta = \frac{3}{4} = 0.75$

$\Rightarrow \theta = \cos^{-1}(0.75) = 41.409^\circ = 41.4^\circ$



$\theta = 41.4^\circ$ $\Rightarrow 2\theta = 2(41.4^\circ) = 82.8^\circ$

$\Rightarrow$ angle $ACB = 180^\circ - (41.4^\circ + 82.8^\circ)$

$= 180^\circ - 124.2^\circ$

$= 55.8^\circ$

Since $55.8^\circ > 41.4^\circ \Rightarrow |AB| > 2$ cm as bigger side is opposite greater angle.

7. (i) $\sin 2\theta = 1 \Rightarrow 2\theta = \sin^{-1}(1) = 90^\circ$

$\Rightarrow \theta = \frac{90^\circ}{2} = 45^\circ$

(a) $\sin \theta = \sin 45^\circ = \frac{1}{\sqrt{2}}$

(b) $\tan \theta = \tan 45^\circ = 1$

(ii) Show that $\frac{\sin 4\theta(1 - \cos 2\theta)}{\cos 2\theta(1 - \cos 4\theta)} = \tan \theta$

Proof: $\frac{\sin 4\theta(1 - \cos 2\theta)}{\cos 2\theta(1 - \cos 4\theta)} = \frac{2 \sin 2\theta \cos 2\theta(1 - \cos 2\theta)}{\cos 2\theta(1 - \cos 4\theta)}$

$= \frac{2 \sin 2\theta(1 - \cos 2\theta)}{1 - (1 - 2\sin^2 2\theta)}$

$= \frac{2 \sin 2\theta(1 - \cos 2\theta)}{1 - 1 + 2\sin^2 2\theta}$

$= \frac{2 \sin 2\theta(1 - \cos 2\theta)}{2\sin^2 2\theta}$

$= \frac{1 - \cos 2\theta}{\sin 2\theta}$

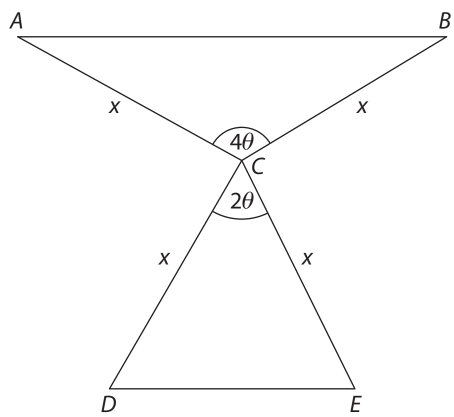
$= \frac{1 - \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}}{\frac{2 \tan \theta}{1 + \tan^2 \theta}}$

$= \frac{\frac{1 + \tan^2 \theta - (1 - \tan^2 \theta)}{1 + \tan^2 \theta}}{\frac{2 \tan \theta}{1 + \tan^2 \theta}}$

$= \frac{1 + \tan^2 \theta - 1 + \tan^2 \theta}{2 \tan \theta}$

$= \frac{2 \tan^2 \theta}{2 \tan \theta} = \tan \theta$

8. (i) Area ΔACB = Area ΔDCE



$\Rightarrow \quad \frac{1}{2}x \,.\, x \sin 4\theta = \frac{1}{2}x \,.\, x \sin 2\theta$

$\Rightarrow \quad \sin 4\theta = \sin 2\theta$

$\Rightarrow \quad 2 \sin 2\theta \cos 2\theta = \sin 2\theta$

$\Rightarrow \quad \cos 2\theta = \frac{1}{2}$

$\Rightarrow \quad 2\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

$\Rightarrow \quad \theta = \frac{60^\circ}{2} = 30^\circ$

(ii) $\theta = 30° \Rightarrow 2\theta = 60°$ and $4\theta = 120°$

Triangle ABC: $|AB|^2 = x^2 + x^2 - 2x \,.\, x \cos 60°$

$$= 2x^2 - 2x\,.\,x\left(\frac{1}{2}\right)$$

$$= 2x^2 - x^2 = x^2$$

Triangle DCE: $|DE|^2 = x^2 + x^2 - 2x\,.\,x\cos 120°$

$$= 2x^2 - 2x^2\left(-\frac{1}{2}\right)$$

$$= 2x^2 + x^2 = 3x^2$$

Given $|AB|^2 + |DE|^2 = 24$

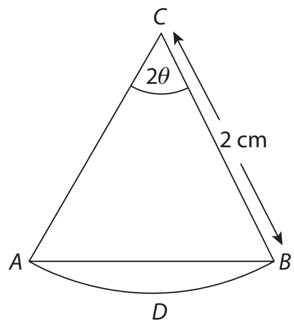
$\Rightarrow \quad x^2 + 3x^2 = 24$

$\Rightarrow \quad 4x^2 = 24$

$\Rightarrow \quad x^2 = 6$

$\Rightarrow \quad x = \sqrt{6}$

9. (i) Area sector $ADBC = \frac{1}{2}(2)^2\, 2\theta = 4\theta$ radians



(ii) Area triangle $ABC = \frac{1}{2}(2)(2)\sin 2\theta = 2\sin 2\theta$

Area $\Delta\, ABC = \frac{3}{4}$Area sector $ADBC$

$\Rightarrow 2\sin 2\theta = \frac{3}{4}(4\theta)$

$\Rightarrow 2\sin 2\theta = 3\theta$

Area $\Delta\, ABC = \sqrt{3}$

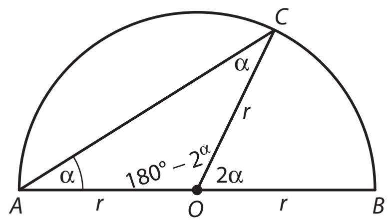
$\Rightarrow \quad 2\sin 2\theta = \sqrt{3}$

$\Rightarrow \quad \sin 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad 2\theta = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$radians

10. (i) Triangle ACO: $\dfrac{|AC|}{\sin 180° - 2\alpha} = \dfrac{r}{\sin\alpha}$



$\Rightarrow \quad \dfrac{|AC|}{\sin 2\alpha} = \dfrac{r}{\sin\alpha}$

$\Rightarrow \quad |AC| = \dfrac{r\sin 2\alpha}{\sin\alpha}$

$$= \frac{r\,.\,2\sin\alpha\cos\alpha}{\sin\alpha}$$

$$= 2r\cos\alpha$$

(ii) $[AC]$ bisects the area of the semicircular region

$\Rightarrow$ 2[Area ΔAOC + Area sector COB] = Area semicircle

$\Rightarrow 2\left[\frac{1}{2}r\cdot r\sin(180° - 2\alpha) + \frac{1}{2}r^2\cdot 2\alpha\right] = \frac{1}{2}\pi r^2$

$\Rightarrow \quad 2\cdot\frac{1}{2}r^2\sin 2\alpha + 2\cdot\frac{1}{2}r^2\cdot 2\alpha = \frac{\pi}{2}r^2$

$\Rightarrow \quad \sin 2\alpha + 2\alpha = \frac{\pi}{2}$

Chapter 9

Exercise 9.1

1. (i) 6, 12, 18, 24, 30, 36, 42

(ii) 7, 12, 17, 22, 27, 32, 37

(iii) 4.7, 5.9, 7.1, 8.3, 9.5, 10.7, 11.9

(iv) 2, −1, −4, −7, −10, −13, −16

(v) 2, 3, 6, 11, 18, 27, 38, 51, 66

(vi) 78, 70, 62, 54, 46, 38, 30

(vii) 10, 5, 0, −5, −10, −15, −20, −25

(viii) −64, −55, −46, −37, −28, −19, −10

(ix) 2, 6, 18, 54, 162, 486

(x) 2, 6, 12, 20, 30, 42, 56

(xi) $\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}$

(xii) 1, 2, 4, 7, 11, 16, 22, 29

(xiii) 0, 3, 8, 15, 24, 35, 48, 63

(xiv) 3, −6, 12, −24, 48, −96, 192

(xv) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \frac{1}{56}$

2. (i) $T_n = 4n - 2$

$$\left.\begin{aligned} \Rightarrow T_1 &= 4(1) - 2 = 2 \\ T_2 &= 4(2) - 2 = 6 \\ T_3 &= 4(3) - 2 = 10 \\ T_4 &= 4(4) - 2 = 14 \end{aligned}\right\} \quad 2, 6, 10, 14$$

(ii) $T_n = (n+1)^2$

$$\left.\begin{aligned} \Rightarrow T_1 &= (1+1)^2 = 4 \\ T_2 &= (2+1)^2 = 9 \\ T_3 &= (3+1)^2 = 16 \\ T_4 &= (4+1)^2 = 25 \end{aligned}\right\} \quad 4, 9, 16, 25$$

(iii) $T_n = n^2 - 2n$

$$\left.\begin{aligned} \Rightarrow T_1 &= 1^2 - 2(1) = -1 \\ T_2 &= 2^2 - 2(2) = 0 \\ T_3 &= 3^2 - 2(3) = 3 \\ T_4 &= 4^2 - 2(4) = 8 \end{aligned}\right\} \quad -1, 0, 3, 8$$

(iv) $T_n = (n+3)(n+1)$

$$\left.\begin{aligned} \Rightarrow T_1 &= (1+3)(1+1) = 8 \\ T_2 &= (2+3)(2+1) = 15 \\ T_3 &= (3+3)(3+1) = 24 \\ T_4 &= (4+3)(4+1) = 35 \end{aligned}\right\} \quad 8, 15, 24, 35$$

(v) $T_n = n^3 - 1$

$$\left.\begin{aligned} \Rightarrow T_1 &= (1)^3 - 1 = 0 \\ T_2 &= (2)^3 - 1 = 7 \\ T_3 &= (3)^3 - 1 = 26 \\ T_4 &= (4)^3 - 1 = 63 \end{aligned}\right\} \quad 0, 7, 26, 63$$

(vi) $T_n = \frac{n}{n+2}$

$$\left.\begin{aligned} \Rightarrow T_1 &= \frac{1}{1+2} = \frac{1}{3} \\ T_2 &= \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2} \\ T_3 &= \frac{3}{3+2} = \frac{3}{5} \\ T_4 &= \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3} \end{aligned}\right\} \quad \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$$

(vii) $T_n = 2^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= 2^1 = 2 \\ T_2 &= 2^2 = 4 \\ T_3 &= 2^3 = 8 \\ T_4 &= 2^4 = 16 \end{aligned}\right\} \quad 2, 4, 8, 16$$

(viii) $T_n = (-3)^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= (-3)^1 = -3 \\ T_2 &= (-3)^2 = 9 \\ T_3 &= (-3)^3 = -27 \\ T_4 &= (-3)^4 = 81 \end{aligned}\right\} \quad -3, 9, -27, 81$$

(ix) $T_n = n \,.\, 2^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= 1 \,.\, 2^1 = 2 \\ T_2 &= 2 \,.\, 2^2 = 8 \\ T_3 &= 3 \,.\, 2^3 = 24 \\ T_4 &= 4 \,.\, 2^4 = 64 \end{aligned}\right\} \quad 2, 8, 24, 64$$

3. (i) 5 cm, 9 cm, 13 cm, 17 cm, etc.

(ii) $T_1 = 5$

$T_2 = 5 + 4$

$T_3 = 5 + 2(4)$

$T_4 = 5 + 3(4)$ etc.

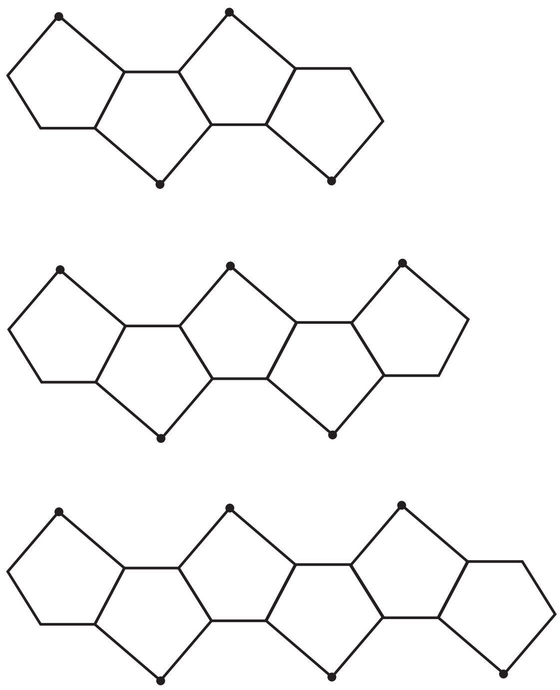
$\Rightarrow T_n = 5 + (n-1)4 = 5 + 4n - 4 = 4n + 1$

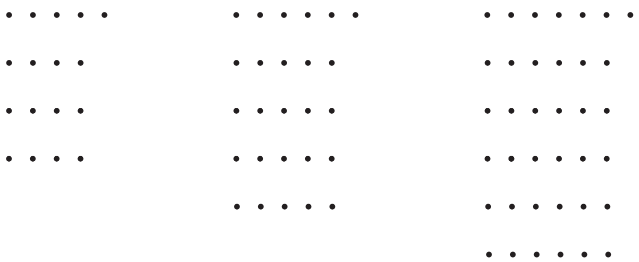
$\Rightarrow T_5 = 4(5) + 1 = 21$ cm.

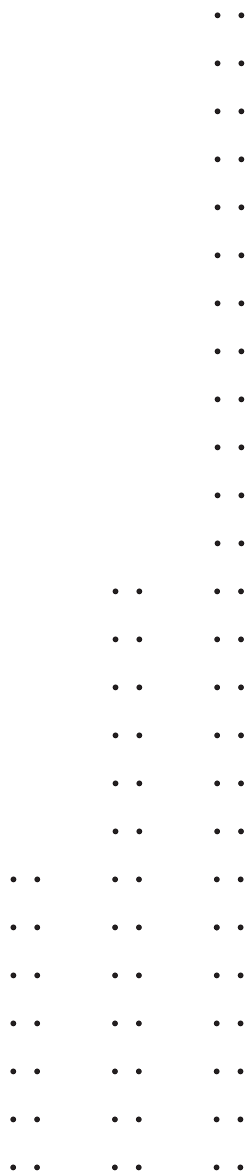
4. (i) $w_1 = 1$ km
$w_2 = (1 + 1)$ km $= 2$ km
$w_3 = (2 + 2)$ km $= 4$ km
$w_4 = (4 + 3)$ km $= 7$ km
$w_5 = (7 + 4)$ km $= 11$ km
$w_6 = (11 + 5)$ km $= 16$ km
(ii) $w_7 = (16 + 6)$ km $= 22$ km
$w_8 = (22 + 7)$ km $= 29$ km
In week 8, she ran 29 km.

5. $T_n = 4n - 3$
$\Rightarrow T_1 = 4(1) - 3 \quad = 1$
$T_5 = 4(5) - 3 \quad = 17$
$T_{10} = 4(10) - 3 = 37.$

6. $T_n = (-2)^{n+1}$
$\Rightarrow T_1 = (-2)^{1+1} \quad = 4$
$T_6 = (-2)^{6+1} \quad = -128$
$T_{11} = (-2)^{11+1} = 4096.$

7. (i)


(ii)


(iii)


8. (i) $T_n = 4n - 2 = 2, 6, 10, 14, \ldots$ C
(ii) $T_n = 2n^2 = 2, 8, 18, 32, \ldots$ B
(iii) $T_n = n(n + 1) = 2, 6, 12, 20, \ldots.$ D
(iv) $T_n = 2^n = 2, 4, 8, 16, \ldots.$ A

9. (i) 5, 6, 7, 8, 9

$\Rightarrow T_1 = 5$
$T_2 = 6 = 5 + 1$
$T_3 = 7 = 5 + 2$
$T_4 = 8 = 5 + 3$
$T_5 = 9 = 5 + 4$
$\Rightarrow T_n = 5 + (n - 1) = 5 + n - 1 = n + 4.$

(ii) 2, 4, 6, 8, 10

$\Rightarrow T_1 = 2$
$T_2 = 4 = 2 \times 2$
$T_3 = 6 = 2 \times 3$
$T_4 = 8 = 2 \times 4$
$T_5 = 10 = 2 \times 5$
$\Rightarrow T_n = 2 \times n = 2n.$

(iii) 2, 5, 8, 11, 14

$\Rightarrow T_1 = 2$
$T_2 = 5 = 2 + 3$
$T_3 = 8 = 2 + 2(3)$
$T_4 = 11 = 2 + 3(3)$
$T_5 = 14 = 2 + 4(3)$
$\Rightarrow T_n = 2 + (n - 1)3 = 2 + 3n - 3 = 3n - 1.$

(iv) 1, 4, 9, 16, 25

$\Rightarrow T_1 = 1 = 1^2$
$T_2 = 4 = 2^2$
$T_3 = 9 = 3^2$
$T_4 = 16 = 4^2$
$T_5 = 25 = 5^2$
$\Rightarrow T_n = n^2.$

(v) 2, 5, 10, 17, 26

$\Rightarrow T_1 = 2 = 1 + 1 = 1^2 + 1$
$T_2 = 5 = 4 + 1 = 2^2 + 1$
$T_3 = 10 = 9 + 1 = 3^2 + 1$
$T_4 = 17 = 16 + 1 = 4^2 + 1$
$T_5 = 26 = 25 + 1 = 5^2 + 1$
$\Rightarrow T_n = n^2 + 1.$

(vi) $-1, 1, -1, 1, -1$

$\Rightarrow T_1 = -1$
$T_2 = 1 = (-1)^2$
$T_3 = -1 = (-1)^3$
$T_4 = 1 = (-1)^4$
$T_5 = -1 = (-1)^5$
$\Rightarrow T_n = (-1)^n.$

(vii) 1, 5, 9, 13, 17

$\Rightarrow T_1 = 1$

$T_2 = 5 \quad = 1 + 4$

$T_3 = 9 \quad = 1 + 2(4)$

$T_4 = 13 = 1 + 3(4)$

$T_5 = 17 = 1 + 4(4)$

$\Rightarrow T_n \qquad = 1 + (n - 1)4 = 1 + 4n - 4 = 4n - 3.$

(viii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

$\Rightarrow T_1 = 1$

$T_2 = \frac{1}{2}$

$T_3 = \frac{1}{3}$

$T_4 = \frac{1}{4}$

$T_5 = \frac{1}{5}$

$\Rightarrow T_n = \frac{1}{n}.$

(ix) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$

$\Rightarrow T_1 = \frac{2}{3} = \frac{n+1}{n+2}$

$T_2 = \frac{3}{4}$

$T_3 = \frac{4}{5}$

$T_4 = \frac{5}{6}$

$\Rightarrow T_n = \frac{n+1}{n+2}.$

(x) $(2 \times 3), (3 \times 4), (4 \times 5), (5 \times 6), ...$

$\Rightarrow T_1 = 2 \times 3 = (n + 1)(n + 2)$

$T_2 = 3 \times 4$

$T_3 = 4 \times 5$

$T_4 = 5 \times 6$

$\Rightarrow T_n = (n + 1)(n + 2).$

10. 0, 1, 1, 2, 3, 5, 8, 13, 21 ....

The Fibonacci series is formed by adding the two previous terms to give the next term.

... 34, 55, 89, 144 ...

11.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

(i) 1, 2, 3, 4, 5, ... $T_n = n$

(ii) 1, 3, 6, 10, 15, 21, ... $T_n = \frac{1}{2}(n^2 + n)$

(iii) 1, 2, 4, 8, 16, 32 ..... $T_n = \frac{2^n}{2} = 2^{n-1}$

(iv) 3, 6, 10, 15, 21, 28, ...

$= \frac{1}{2}(6, 12, 20, 30 .....)$ $T_n = \frac{1}{2}(n + 1)(n + 2)$

Exercise 9.2

1. (i) 8, 13, 18, 23, ...

$\Rightarrow a = 8, d = 5$

$\Rightarrow \quad T_n = a + (n - 1)d$

$= 8 + (n - 1)5$

$= 8 + 5n - 5 = 5n + 3$

Also, $T_{22} = 5(22) + 3 = 113$

(ii) 16, 36, 56, 76, ...

$\Rightarrow a = 16, d = 20$

$\Rightarrow \quad T_n = a + (n - 1)d$

$= 16 + (n - 1)20$

$= 16 + 20n - 20 = 20n - 4$

Also, $T_{22} = 20(22) - 4 = 436$

(iii) 10, 7, 4, 1, ...

$\Rightarrow a = 10, d = -3$

$\Rightarrow \quad T_n = a + (n - 1)d$

$= 10 + (n - 1)(-3)$

$= 10 - 3n + 3 = -3n + 13$

Also, $T_{22} = -3(22) + 13 = -53$

2. $T_n = 5n - 2$

$$\left.\begin{aligned} \Rightarrow T_1 &= 5(1) - 2 = 3 \\ T_2 &= 5(2) - 2 = 8 \\ T_3 &= 5(3) - 2 = 13 \\ T_4 &= 5(4) - 2 = 18 \end{aligned}\right\} \text{3, 8, 13, 18 ...}$$

3. (i) $-5, -1, 3, 7, \ldots\ldots 75$

$\Rightarrow a = -5, d = 4$

$\Rightarrow T_n = -5 + (n - 1)4$

$= -5 + 4n - 4 = 4n - 9$

When $T_n = 75 \quad \Rightarrow 75 = 4n - 9$

$4n = 75 + 9 = 84$

$n = \dfrac{84}{4} = 21.$

(ii) $2, 5, 8, 11, \ldots\ldots 59$

$\Rightarrow a = 2, d = 3$

$\Rightarrow T_n = a + (n - 1)d$

$= 2 + (n - 1)3$

$= 2 + 3n - 3 = 3n - 1$

When $T_n = 59 \quad \Rightarrow 59 = 3n - 1$

$3n = 59 + 1 = 60$

$n = \dfrac{60}{3} = 20.$

(iii) $-\dfrac{3}{2}, -1, -\dfrac{1}{2}, 0 \ldots\ldots. 14$

$\Rightarrow a = -\dfrac{3}{2}, d = \dfrac{1}{2}$

$\Rightarrow T_n = a + (n - 1)d$

$= -\dfrac{3}{2} + (n - 1)\dfrac{1}{2}$

$= -\dfrac{3}{2} + \dfrac{n}{2} - \dfrac{1}{2} = \dfrac{n}{2} - 2$

When $T_n = 14 \quad \Rightarrow 14 = \frac{n}{2} - 2$

$$\frac{n}{2} = 14 + 2 = 16$$

$$\Rightarrow \quad n = 16 \times 2 = 32.$$

4. (i) $T_1 = 4, \quad T_7 = 22$

$\Rightarrow$ since $T_n = a + (n - 1)d$

$T_1 = a + (1 - 1)d = a$

Also, $T_7 = a + (7 - 1)d = a + 6d$

$\therefore \quad a = 4$

and $\quad a + 6d = 22$

$\Rightarrow \quad 4 + 6d = 22$

$6d = 18$

$\therefore \quad d = 3$

(ii) $\Rightarrow T_n = 4 + (n - 1)3$

$= 4 + 3n - 3 = 3n + 1.$

$$\therefore \left.\begin{aligned} T_1 &= 3(1) + 1 = 4 \\ T_2 &= 3(2) + 1 = 7 \\ T_3 &= 3(3) + 1 = 10 \\ T_4 &= 3(4) + 1 = 13 \\ T_5 &= 3(5) + 1 = 16 \end{aligned}\right\} \; 4, 7, 10, 13, 16 \ldots$$

(iii) Also, $T_{20} = 3(20) + 1 = 61.$

5. (i) Red tiles sequence: $\quad 1, 2, 3, 4, \ldots\ldots T_n = n.$

Orange tiles sequence: $\quad 8, 10, 12, \ldots\ldots \quad \Rightarrow a = 8, d = 2.$

$$\Rightarrow T_n = 8 + (n - 1)2$$

$$= 8 + 2n - 2$$

$$= 2n + 6.$$

$\therefore$ For design 8, $T_8 = 8$ red tiles

$T_8 = (2(8) + 6)$ orange tiles

$= 22$ orange tiles.

(ii) Total number of tiles needed

$= 9, 12, 15, 18, \ldots\ldots\ldots T_n = 3(n + 2),$

i.e. the number of tiles is always a multiple of 3.

Since 38 is not a multiple of 3, no design will need 38 tiles.

6. $T_n \;= a + (n - 1)d$

$\Rightarrow T_2 \;= a + (2 - 1)d = a + d$

$T_7 \;= a + (7 - 1)d = a + 6d$

$T_{13} = a + (13 - 1)d = a + 12d$

$T_{13} = 27 \Rightarrow a + 12d = 27$

$T_7 = 3T_2 \Rightarrow a + 6d = 3(a + d)$

$\Rightarrow \quad a + 6d = 3a + 3d$

$\Rightarrow -2a + 3d = 0.$

Since $\quad a + 12d = 27 \ldots\ldots$ A

and $\quad -2a + 3d = \;\; 0 \ldots\ldots$ B

$\Rightarrow$ 2A: $\quad 2a + 24d = 54$

B: $\quad \underline{-2a + \;\; 3d = \;\; 0}$

$\Rightarrow \quad 27d = 54 \quad$ (adding)

$d = 2$

Also, $-2a + 3(2) = 0$

$-2a = -6$

$a = 3$

$\therefore$ 3, 5, 7, 9, 11, 13 are the first six terms.

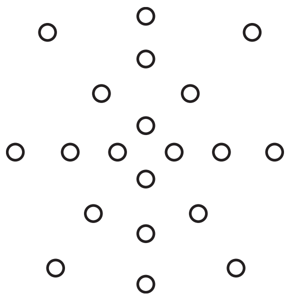
7. (i) $2k + 2, 5k - 3, 6k$ are three consecutive terms.

$$\begin{aligned} \Rightarrow \quad 5k - 3 - (2k + 2) &= 6k - (5k - 3) \\ 5k - 3 - 2k - 2 &= 6k - 5k + 3 \\ 3k - 5 &= k + 3 \\ 2k &= 8 \\ k &= 4. \end{aligned}$$

(ii) $4p, -3 - p, 5p + 16$ are three consecutive terms.

$$\begin{aligned} \Rightarrow \quad -3 - p - 4p &= 5p + 16 - (-3 - p) \\ -3 - 5p &= 5p + 16 + 3 + p \\ -3 - 5p &= 6p + 19 \\ -11p &= 22 \\ p &= -2. \end{aligned}$$

8.



(i) 12, 20, 28

$\Rightarrow a = 12, d = 8$

$$\begin{aligned} \Rightarrow T_n &= a + (n - 1)d \\ &= 12 + (n - 1)8 \\ &= 12 + 8n - 8 = 8n + 4 \end{aligned}$$

(ii) $T_{15} = 8(15) + 4 = 124$

(iii) $T_n = 164 \quad \Rightarrow \quad 8n + 4 = 164$

$$\begin{aligned} 8n &= 160 \\ n &= 20 \end{aligned}$$

9. $T_n = 4n - 2$

$$\begin{aligned} \Rightarrow \quad T_{n+1} &= 4(n + 1) - 2 \\ &= 4n + 4 - 2 = 4n + 2. \end{aligned}$$

$$\begin{aligned} \therefore \quad T_{n+1} - T_n &= (4n + 2) - (4n - 2) \\ &= \cancel{4n} + 2 - \cancel{4n} + 2 = 4, \text{ a constant.} \end{aligned}$$

Since there is a constant difference between terms, the sequence is arithmetic.

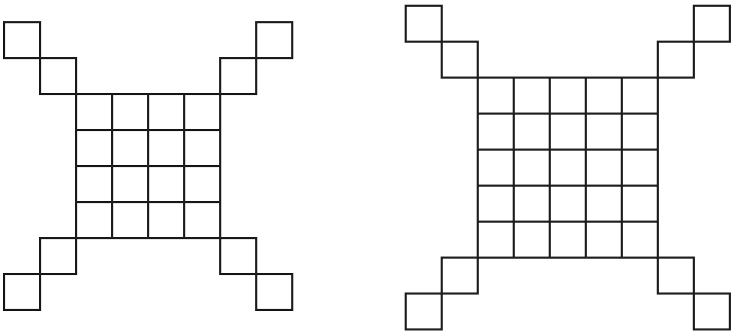
10. $T_n = n(n + 2)$

$$\begin{aligned} \Rightarrow \quad T_{n+1} &= (n + 1)((n + 1) + 2) \\ &= (n + 1)(n + 3) \end{aligned}$$

$$\begin{aligned} \therefore \quad T_{n+1} - T_n &= (n + 1)(n + 3) - (n)(n + 2) \\ &= \cancel{n^2} + 3n + n + 3 - \cancel{n^2} - 2n \\ &= 2n + 3 \text{ which depends on } n. \end{aligned}$$

$\therefore$ Since the difference is not constant, the sequence is not arithmetic.

11.



(i) Shape 7 will need 8 light-coloured tiles, since each design has 8 light-coloured tiles.

(ii) Sequence for dark-coloured tiles is 1, 4, 9, ... $T_n = n^2$

$\Rightarrow$ for shape 7, $T_7 = 7^2$ dark tiles

$= 49$ dark tiles.

(iii) $T_n = n^2 + 8$

(iv) $T_{n+1} = (n + 1)^2 + 8$

$= n^2 + 2n + 1 + 8$

$= n^2 + 2n + 9$

$\Rightarrow T_{n+1} - T_n = n^2 + 2n + 9 - (n^2 + 8)$

$= \cancel{n^2} + 2n + 9 - \cancel{n^2} - 8$

$= 2n + 1$ which depends on the value of n and is therefore not arithmetic.

12.

Number of hexagons	1	2	3	4	10	(20)	30
Perimeter	6	10	(14)	(18)	(42)	82	(122)

(i) 87 matchsticks left over

Sequence: 6, 10, 14, 18, ...

$\Rightarrow a = 6, d = 4$

$\Rightarrow T_n = 6 + (n - 1)4$

$= 6 + 4n - 4$

$= 4n + 2.$

If $T_n = 4n + 2 = 87,$

$\Rightarrow 4n = 85$

$n = \frac{85}{4}$ which is not a whole number.

$\therefore$ Sandrine will have matchsticks left over.

(ii) $T_n = 4n + 2$

$T_{n+1} = 4(n + 1) + 2$

$= 4n + 4 + 2 = 4n + 6$

$\Rightarrow T_{n+1} - T_n = 4n + 6 - (4n + 2)$

$= \cancel{4n} + 6 - \cancel{4n} - 2 = 4,$ a constant

$\therefore$ The sequence is arithmetic.

(iii) New design sequence: 6, 12, 18, ...

$\Rightarrow a = 6, d = 6.$

$\Rightarrow T_n = a + (n - 1)d$

$= 6 + (n - 1)6$

$= 6 + 6n - 6$

$= 6n$

Number of matches in each design: 6, 12, 18, 24, 30, 36, ...

Total number of matches used: 18, 36, 60, 90, 126, ...

$\therefore$ A total of 5 completed levels is possible using a total of 90 matchsticks with $122 - 90 = 32$ left over.

13. $a = 12$ min

$d = 6$ min

$\Rightarrow T_n = 12 + (n - 1)6$

$= 12 + 6n - 6$

$= 6n + 6.$

If $T_n = 60 \quad \Rightarrow 6n + 6 = 60$

$\therefore 6n = 54$

$n = 9$ weeks.

Exercise 9.3

1. (i) $1 + 5 + 9 + 13$

$\Rightarrow \quad a = 1, d = 4$

$\therefore \quad S_n = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{n}{2}(2(1) + (n - 1)4)$

$= \frac{n}{2}(2 + 4n - 4)$

$= \frac{n}{2}(4n - 2)$

$\Rightarrow S_{20} = \frac{20}{2}(4(20) - 2)$

$= 780$

(ii) $50 + 48 + 46 + 44$

$\Rightarrow \quad a = 50, d = -2$

$\therefore \quad S_n = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{n}{2}(2(50) + (n - 1)(-2))$

$= \frac{n}{2}(100 - 2n + 2)$

$= \frac{n}{2}(-2n + 102)$

$\Rightarrow S_{20} = \frac{20}{2}(-2(20) + 102)$

$= 620$

(iii) $1 + 1.1 + 1.2 + 1.3 + ...$

$\Rightarrow \quad a = 1, d = 0.1$

$\Rightarrow S_n = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{n}{2}(2(1) + (n - 1)(0.1))$

$= \frac{n}{2}(2 + 0.1n - 0.1)$

$= \frac{n}{2}(0.1n + 1.9)$

$\Rightarrow S_{20} = \frac{20}{2}((0.1)(20) + (1.9))$

$= 39$

(iv) $-7 - 3 + 1 + 5 + ...$

$\Rightarrow \quad a = -7, d = 4$

$\therefore \quad S_n = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{n}{2}(2(-7) + (n - 1)4)$

$= \frac{n}{2}(-14 + 4n - 4)$

$= \frac{n}{2}(4n - 18)$

$\Rightarrow S_{20} = \frac{20}{2}(4(20) - 18)$

$= 620$

2. (i) $6 + 10 + 14 + 18 + ...\ 50$

$\Rightarrow a = 6, d = 4$

$\Rightarrow T_n = a + (n - 1)d$

$= 6 + (n - 1)4$

$= 6 + 4n - 4$

$= 4n + 2.$

$T_n = 50 \Rightarrow \quad 4n + 2 = 50$

$4n = 48$

$n = 12.$

$\therefore \quad S_n = \frac{n}{2}(2a + (n - 1)d)$

$\Rightarrow S_{12} = \frac{12}{2}(2(6) + (12 - 1)4)$

$= 6(12 + 44)$

$= 336$

(ii) $1 + 2 + 3 + 4 + ...\ 100$

It is obvious that $n = 100$ (i.e. there are 100 terms in this sequence)

$a = 1, d = 1$

$\therefore \quad S_n = \frac{n}{2}(2a + (n - 1)d)$

$\Rightarrow S_{100} = \frac{100}{2}(2(1) + (100 - 1)(1))$

$= 50(2 + 99)$

$= 5050$

(iii) $80 + 74 + 68 + 62 + \ldots\ldots -34$

$\Rightarrow a = 80, d = -6$

$\Rightarrow T_n = a + (n-1)d$

$= 80 + (n-1)(-6)$

$= 80 - 6n + 6$

$= -6n + 86$

$T_n = -34 \Rightarrow \quad -6n + 86 = -34$

$-6n = -120$

$n = 20$

$\therefore \quad S_n = \frac{n}{2}(2a + (n-1)d)$

$\Rightarrow S_{20} = \frac{20}{2}(2(80) + (20-1)(-6))$

$= 10(160 - 114)$

$= 460$

3. $5 + 8 + 11 + 14 + \ldots\ldots = 98$

$\Rightarrow a = 5, d = 3$

$\therefore \quad S_n = \frac{n}{2}(2a + (n-1)d)$

$\Rightarrow 98 = \frac{n}{2}(2(5) + (n-1)3)$

$= \frac{n}{2}(10 + 3n - 3)$

$= \frac{n}{2}(3n + 7)$

$\therefore \; 196 = 3n^2 + 7n$

$\therefore \; 3n^2 + 7n - 196 = 0$

To find n;

$a = 3, b = +7, c = -196$

$\therefore \; n = \dfrac{-7 \pm \sqrt{(-7)^2 - 4(3)(-196)}}{2(3)}$

$= \dfrac{-7 \pm \sqrt{2401}}{6} = \dfrac{-7 \pm 49}{6}$

$= 7 \text{ or } \dfrac{-56}{6}\left(= -9\frac{2}{3}\right)$

Since n must be a natural number, $n = 7$.

4. $T_n = 5 - 3n$

$\Rightarrow \quad T_1 = 5 - 3(1) \quad = 2 \qquad = a$

Also, $T_2 = 5 - 3(2) \quad = -1$

$\therefore \quad T_2 - T_1 = d \quad = -1 - 2 \quad = -3$

$\Rightarrow S_n = \frac{n}{2}(2a + (n-1)d)$

$\therefore \; S_{10} = \frac{10}{2}(2(2) + (10-1)(-3))$

$= 5(4 - 27)$

$= -115$

5. $a = 10, d = 2, S_n = 190$

$\therefore \quad S_n = \frac{n}{2}(2a + (n-1)d)$

$\Rightarrow 190 = \frac{n}{2}(2(10) + (n-1)2)$

$\Rightarrow 380 = n(20 + 2n - 2)$

$380 = n(2n + 18)$

$= 2n^2 + 18n$

$\therefore \; 2n^2 + 18n - 380 = 0$

$\Rightarrow \quad n^2 + 9n - 190 = 0$

$(n + 19)(n - 10) = 0$

$\therefore n = -19 \quad \text{or} \quad n = 10$

$\therefore n = 10 \quad \text{since } n \in N.$

6. (i) $\sum_{r=1}^{6}(3r + 1) = (3(1) + 1) + (3(2) + 1) + (3(3) + 1) + (3(4) + 1) + (3(5) + 1) + (3(6) + 1)$

$= 4 + 7 + 10 + 13 + 16 + 19$

$= 69$

(ii) $\sum_{r=0}^{5}(4r - 1) = (4(0) - 1) + (4(1) - 1) + (4(2) - 1) + (4(3) - 1) + (4(4) - 1) + (4(5) - 1)$

$= -1 + 3 + 7 + 11 + 15 + 19$

$= 54$

(iii) $\sum_{r=1}^{100} r = 1 + 2 + 3 + 4 + \ldots\ldots 100$

$\Rightarrow a = 1, d = 1, n = 100$

$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{100}{2}(2(1) + (100 - 1)(1))$

$= 50(2 + 99)$

$= 50(101)$

$= 5050$

7. (i) $4 + 8 + 12 + 16 + \ldots. 124$

$\Rightarrow \; a = 4, d = 4$

$\therefore \; T_n = a + (n-1)d$

$\Rightarrow \quad = 4 + (n-1)4$

$= \cancel{4} + 4n - \cancel{4}$

$= 4n$

When $T_n = 124$: $\quad 4n = 124$

$n = 31$

$\therefore \; 4 + 8 + 12 + 16 + \ldots. 124 = \sum_{n=1}^{31} 4n$

(ii) $-10 - 9\frac{1}{2} - 8 - 7\frac{1}{2} + \ldots\ldots 4$

$\Rightarrow \; a = -10, d = +\frac{1}{2}$

$\therefore \; T_n = a + (n-1)d$

$= -10 + (n-1)\frac{1}{2}$

$= -10 + \frac{n}{2} - \frac{1}{2}$

$= \frac{n}{2} - \frac{21}{2}$

When $T_n = 4$: $\frac{n}{2} - \frac{21}{2} = 4$

$\Rightarrow \quad n - 21 = 8$

$\therefore \quad n = 29$

$\therefore \quad -10 - 9\frac{1}{2} - 8 - 7\frac{1}{2} + \ldots\ldots + 4 = \sum_{n=1}^{29} \frac{1}{2}(n - 21)$

(iii) $10 + 10.1 + 10.2 + 10.3 + \ldots\ldots 50$

$\Rightarrow \; a = 10, d = 0.1$

$\Rightarrow T_n = a + (n - 1)d$

$T_n = 10 + (n - 1)(0.1)$

$= 10 + 0.1n - 0.1$

$= 0.1n + 9.9$

When $T_n = 50$: $\quad 0.1n + 9.9 = 50$

$0.1n = 40.1$

$n = 401$

$\therefore 10 + 10.1 + 10.2 + \ldots\ldots 50 = \sum_{n=1}^{401} 0.1n + 9.9$

8. $T_4 = 15$ and also, $S_5 = 55$

$T_n = a + (n - 1)d$

$\Rightarrow T_4 = a + (4 - 1)d$

$\Rightarrow T_4 = a + 3d$

$\therefore a + 3d = 15 \ldots\ldots A$

$S_n = \frac{n}{2}(2a + (n - 1)d)$

$\Rightarrow S_5 = \frac{5}{2}(2a + (5 - 1)d)$

$\Rightarrow S_5 = \frac{5}{2}(2a + 4d)$

$S_5 = 5a + 10d$

$\therefore 5a + 10d = 55 \ldots\ldots B$

$\therefore$ 5A: $5a + 15d = 75$

B: $\underline{5a + 10d = 55}$

$5d = 20$

$d = 4$

$\Rightarrow a + 3(4) = 15$

$\therefore \; a = 3$

$\therefore$ The first five terms are 3, 7, 11, 15, 19.

9. Third term: $T_3 = 18$

Seventh term: $T_7 = 30$

$T_n = a + (n - 1)d$

$\Rightarrow T_3 = a + (3 - 1)d \quad$ also, $\quad T_7 = a + (7 - 1)d$

$\Rightarrow T_3 = a + 2d \qquad\qquad T_7 = a + 6d$

$T_3 = 18 \quad \Rightarrow \quad a + 2d = 18$

$T_7 = 30 \quad \Rightarrow \quad \underline{a + 6d = 30}$

$-4d = -12$ (subtracting)

$\therefore \; d = 3$

$\Rightarrow \; a + 2(3) = 18$

$\therefore \; a = 12$

$\Rightarrow S_n = \frac{n}{2}(2a + (n - 1)d)$

$\therefore S_{33} = \frac{33}{2}(2(12) + (33 - 1)3)$

$= \frac{33}{2}(24 + 96)$

$= 1980$

10. (i) Number sequence for rings = 6, 11, 16, ...

$\Rightarrow a = 6, d = 5$

$T_n = a + (n - 1)d$

$T_{10} = 6 + (10 - 1)5$

= 51 rings needed for design 10.

(ii) $T_{20} = 6 + (20 - 1)5$

= 101 rings needed for design 20.

$S_n = \frac{n}{2}(2a + (n - 1)d)$

$\Rightarrow S_{20} = \frac{20}{2}(2(6) + (20 - 1)5)$

$= 10(107)$

= 1070 rings needed for all 20 designs.

11. $-12, \ldots\ldots\ldots 40, \quad S_n = 196,$

$\Rightarrow a = -12.$

Also, $T_n = a + (n - 1)d$

$\Rightarrow \quad 40 = -12 + (n - 1)d$

$\therefore \quad 52 = (n - 1)d$

$S_n = \frac{n}{2}(2a + (n - 1)d)$

$\Rightarrow 196 = \frac{n}{2}(2(-12) + (n - 1)d)$

$= \frac{n}{2}(-24 + (n - 1)d)$

$= \frac{n}{2}(-24 + 52)$

$\therefore 392 = 28n$

$\therefore n = 14.$

Also, $52 = (14 - 1)d$

$52 = 13d$

$\therefore \quad d = 4.$

12. $1 + 2 + 3 + \ldots\ldots n$

$\Rightarrow \ a = 1, d = 1$

$\therefore \ S_n = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{n}{2}(2(1) + (n - 1)(1))$

$= \frac{n}{2}(2 + n - 1)$

$= \frac{n}{2}(n + 1)$

$\therefore 1 + 2 + 3 + \ldots\ldots 99 \qquad \Rightarrow n = 99.$

$\therefore \quad S_n = \frac{n}{2}(n + 1)$

$\Rightarrow S_{99} = \frac{99}{2}(99 + 1)$

$= 4950$

13. $T_{21} = 5\frac{1}{2}$

$S_{21} = 94\frac{1}{2}$

$T_n = a + (n-1)d \qquad \Rightarrow T_{21} = a + (21-1)d = 5\frac{1}{2}$

$\therefore \quad a + 20d = 5\frac{1}{2}$

$\therefore \quad 2a + 40d = 11$ ..... A

Also, $S_n = \frac{n}{2}(2a + (n-1)d) \qquad \Rightarrow S_{21} = \frac{21}{2}(2a + (21-1)d) = 94\frac{1}{2}$

$\therefore \quad \frac{21}{2}(2a + 20d) = 94\frac{1}{2}$

$\therefore \quad 42a + 420d = 189$

$\therefore \quad 14a + 140d = 63$ ..... B

$$\begin{aligned} 7A:\ 14a + 280d &= 77 \\ B:\ \underline{14a + 140d} &\underline{= 63} \\ 140d &= 14 \quad \text{(subtracting)} \\ \Rightarrow \quad d &= 0.1 \end{aligned}$$

$\therefore \quad 2a + 40d = 11$

$\Rightarrow 2a + 40(0.1) = 11$

$\Rightarrow \quad a = 3.5$

$S_{30} = \frac{30}{2}(2(3.5) + (30-1)(0.1))$

$= 148.5$

14. $T_{21} = 37$

$S_{20} = 320$

$T_n = a + (n-1)d \quad \Rightarrow \quad T_{21} = a + (21-1)d = 37$

$\Rightarrow \quad a + 20d = 37$ ...... A

$S_n = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow \quad S_{20} = \frac{20}{2}(2a + (20-1)d) = 320$

$\Rightarrow \quad 10(2a + 19d) = 320$

$\Rightarrow \quad 2a + 19d = 32$ ..... B

$$\begin{aligned} 2A:\ 2a + 40d &= 74 \\ B:\ \underline{2a + 19d} &\underline{= 32} \\ 21d &= 42 \quad \text{(subtracting)} \\ \Rightarrow \quad d &= 2 \end{aligned}$$

$\therefore \ a + 20d = 37$

$\Rightarrow a + 20(2) = 37$

$a = 37 - 40 = -3$

$\therefore \ S_{10} = \frac{10}{2}(2(-3) + (10-1)(2))$

$= 5(-6 + 18)$

$= 60$

15. $T_1, T_2, T_3, \ldots\ldots l$

$\therefore T_n = a + (n-1)d = l.$

$\Rightarrow \qquad (n-1)d = l - a$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(2a + l - a)$

$= \frac{n(a+l)}{2} \quad \left(\text{i.e. } \frac{n}{2}(a+l)\right)$

16. S_∞ of an arithmetic sequence cannot be found since

$S_n = \frac{n}{2}(a+l)$

$\Rightarrow S_\infty = \frac{\infty}{2}(a+l)$

But in an infinite sequence, there is no last term

$\therefore$ The sum cannot be evaluated.

Exercise 9.4

1. (i) 3, 9, 27, 81, ...

$\frac{9}{3} = \frac{27}{9} = \frac{81}{27} = 3 = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} 81 \times 3 = 243 \\ 243 \times 3 = 729 \end{array}\right\}$ 3, 9, 27, 81, 243, 729

(ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

$\frac{\frac{1}{3}}{1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{\frac{1}{27}}{\frac{1}{9}} = \frac{1}{3} = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} \frac{1}{27} \times \frac{1}{3} = \frac{1}{81} \\ \frac{1}{81} \times \frac{1}{3} = \frac{1}{243} \end{array}\right\}$ $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$

(iii) $-1, 2, -4, 8, \ldots.$

$\frac{2}{-1} = \frac{-4}{2} = \frac{8}{-4} = -2 = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} 8 \times -2 = -16 \\ -16 \times -2 = 32 \end{array}\right\}$ $-1, 2, -4, 8, -16, 32$

(iv) $1, -1, 1, -1, \ldots$

$\frac{-1}{1} = \frac{1}{-1} = \frac{-1}{1} = -1 = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} -1 \times -1 = 1 \\ 1 \times -1 = -1 \end{array}\right\}$ $1, -1, 1, -1, 1, -1$

(v) $1, 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{8}, \ldots$

$= 1, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$

$\Rightarrow \frac{\frac{3}{2}}{1} = \frac{3}{2}, \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{6} \quad \therefore$ Sequence is not geometric

(vi) $a, a^2, a^3, a^4, \ldots..$

$\Rightarrow \frac{a^2}{a} = \frac{a^3}{a^2} = \frac{a^4}{a^3} = a = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} a^4 \times a = a^5 \\ a^5 \times a = a^6 \end{array}\right\}$ $a, a^2, a^3, a^4, a^5, a^6$

(vii) 1, 1.1, 1.21, 1.331, ...

$\Rightarrow \frac{1.1}{1} = \frac{1.21}{1} = \frac{1.331}{1.21} = 1.1 = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} 1.331 \times 1.1 = 1.4641 \\ 1.4641 \times 1.1 = 1.61051 \end{array}\right\}$ 1, 1.1, 1.21, 1.331, 1.4641, 1.61051

(viii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{36}, \ldots$

$\Rightarrow \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}, \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2} \quad \therefore$ Sequence is not geometric

(ix) 2, 4, −8, −16, ...

$\Rightarrow \frac{4}{2} \neq \frac{-8}{4} \neq \frac{-16}{-8} \quad \therefore$ Sequence is not geometric

(x) $\frac{3}{4}, \frac{9}{2}, 27, 162, \ldots$

$\Rightarrow \frac{\frac{9}{2}}{\frac{3}{4}} = \frac{27}{\frac{9}{2}} = \frac{162}{27} = 6 = r \quad \therefore$ Sequence is geometric

$\left.\begin{array}{r} 162 \times 6 = \quad 972 \\ 972 \times 6 = 5{,}832 \end{array}\right\}$ $\frac{3}{4}, \frac{9}{2}$, 27, 162, 972, 5832

2. (i) 5, 10, ...

$\Rightarrow a = 5, r = \frac{10}{5} = 2$

Also, $T_n = a \,.\, r^{n-1}$

$\therefore T_{11} = 5.2^{(11-1)} = 5120$

(ii) 10, 25, ...

$\Rightarrow a = 10, r = \frac{25}{10} = 2.5$

Also, $T_n = a \,.\, r^{n-1}$

$\therefore \quad T_7 = 10 \,.\, (2.5)^{7-1}$

$= 10 \,.\, (2.5)^6 = 2441.41$

(iii) 1.1, 1.21, ...

$\Rightarrow a = 1.1, r = \frac{1.21}{1.1} = 1.1$

Also, $T_n = a \,.\, r^{n-1}$

$\therefore \quad T_8 = 1.1(1.1)^{8-1}$

$= 1.1(1.1)^7 = 2.14358881$

(iv) 24, −12, 6, ...

$\Rightarrow a = 24, r = \frac{-12}{24} = -\frac{1}{2}$

Also, $T_n = a \,.\, r^{n-1}$

$\therefore \quad T_{10} = 24\left(-\frac{1}{2}\right)^{10-1}$

$= 24\left(-\frac{1}{2}\right)^9 = \left(-\frac{3}{64}\right)$

3. $T_2 = 12, T_5 = 324$

$T_n = a \,.\, r^{n-1}$

$\Rightarrow T_2 = a \,.\, r^{2-1} = ar = 12$

$\Rightarrow T_5 = a \,.\, r^{5-1} = ar^4 = 324.$

$\therefore \frac{\cancel{a}\cancel{r^4}^{3}}{\cancel{a}\cancel{r}} = \frac{324}{12}$

$\Rightarrow \quad r^3 = 27$

$\Rightarrow \quad r = 3$

$\therefore \ a(3) = 12$

$\Rightarrow a \quad = \frac{12}{3} = 4$

$\therefore$ Sequence is $4, 4 \times 3, 4 \times 3^2, 4 \times 3^3, 4 \times 3^4$

$= 4, \quad 12, \quad 36, \quad 108, \quad 324$

4. $T_3 = 6, T_8 = 1458$

$$T_n = a\,.\,r^{n-1}$$

$$\Rightarrow T_3 = a\,.\,r^{3-1} = ar^2 = 6$$

$$T_8 = a\,.\,r^{8-1} = ar^7 = 1458$$

$$\therefore \frac{\cancel{a}r^{\cancel{7}\,5}}{\cancel{a}\cancel{r^2}} = \frac{1458}{6}$$

$$\Rightarrow r^5 = 243$$

$$\Rightarrow \quad r = \sqrt[5]{243} = 3$$

5. $T_2 = 4, T_5 = -\frac{1}{16}$

$$T_n = a\,.\,r^{n-1}$$

$$\Rightarrow T_2 = a\,.\,r^{2-1} = ar = 4$$

$$T_5 = a\,.\,r^{5-1} = ar^4 = -\frac{1}{16}$$

$$\therefore \frac{\cancel{a}r^{\cancel{4}\,3}}{\cancel{a}\cancel{r}} = \frac{-\frac{1}{16}}{4} = -\frac{1}{64}$$

$$\therefore \quad r^3 = -\frac{1}{64}$$

$$\Rightarrow \quad r = \sqrt[3]{-\frac{1}{64}} = -\frac{1}{4}$$

$$\therefore \quad a\left(-\frac{1}{4}\right) = 4$$

$$\Rightarrow a \qquad = -16$$

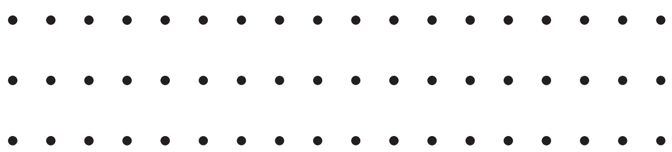
$\therefore$ Sequence is $-16, \left(-16 \times -\frac{1}{4}\right), \left(-16 \times \left(-\frac{1}{4}\right)^2\right), \left(-16 \times \left(-\frac{1}{4}\right)^3\right), \left(-16 \times \left(-\frac{1}{4}\right)^4\right)$

$$= -16, 4, -1, \frac{1}{4}, -\frac{1}{6}.$$

6. A: 2, 6, 8, ...

$\frac{6}{2} = \frac{18}{6} = 3 = r \quad \therefore$ Sequence is geometric

$T_4 = 18 \times 3 = 54$ dots



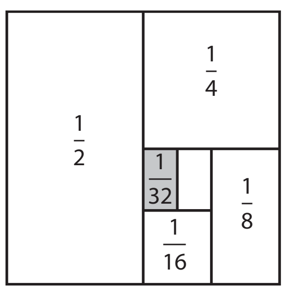
B: 3, 6, 10, ...

$\frac{6}{3} \neq \frac{10}{6} \qquad \therefore$ Sequence in not geometric

C: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, ...

$\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \qquad \therefore$ Sequence is geometric

$\therefore T_6 = \frac{1}{16} \times \frac{1}{2} = \frac{1}{32}$



D: $1, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81} \ldots$

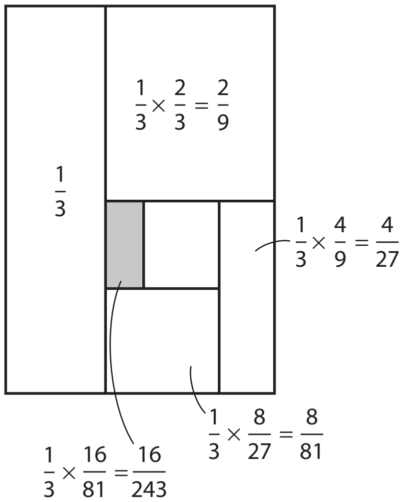
$\frac{\frac{1}{3}}{1} \neq \frac{\frac{2}{9}}{\frac{1}{3}}$ $\quad \therefore$ Sequence is not geometric

i.e $\frac{1}{3} \neq \frac{2}{3}$

[Note: if 1 is left out of this sequence,

$\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}$ is geometric;

$r = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{\frac{4}{27}}{\frac{2}{9}} = \frac{\frac{8}{81}}{\frac{4}{27}} = \frac{2}{3}$]



7. $n - 2, n, n + 3$ .... geometric $\Rightarrow \frac{n}{n-2} = \frac{n+3}{n}$

$$\therefore n^2 = (n + 3)(n - 2)$$

$$\cancel{n^2} = \cancel{n^2} + 3n - 2n - 6$$

$$\Rightarrow \quad n = 6$$

$$\Rightarrow \quad r = \frac{6}{6-2} = \frac{6}{4} = \frac{3}{2}$$

$\therefore (6 - 2), 6, 6 + 3$

$\Rightarrow 4, 6, 9, \frac{27}{2}$

8. (i) $T_3 = -63, T_4 = 189$

$T_n = a \,.\, r^{n-1}$

$\Rightarrow T_3 = a \,.\, r^{3-1} = ar^2 = -63$

$T_4 = a \,.\, r^{4-1} = ar^3 = 189$

$\therefore \frac{\cancel{a}r^{\cancel{3}\,1}}{\cancel{a}\cancel{r^2}} = \frac{189}{-63}$

$\therefore \quad r = -3$

$\Rightarrow a(-3)^2 = -63$

$a \quad = \frac{-63}{9} = -7$

(ii) $\therefore T_n = -7(-3)^{n-1}$

9. $T_1 = 16, T_5 = 9$

$T_1 = a = 16$

$T_n = a \,.\, r^{n-1}$

$\Rightarrow T_5 = 16 \,.\, r^{5-1} = 16 \,.\, r^4 = 9$

$$r^4 = \frac{9}{16}$$

$$\Rightarrow \quad r = \sqrt[4]{\frac{9}{16}} = \frac{\sqrt{3}}{2}$$

$\therefore T_7 = 16\left(\frac{\sqrt{3}}{2}\right)^{7-1} = \frac{16 \,.\, (\sqrt{3})^6}{2^6} = \frac{27}{4}$

10. Let the first three terms be $\frac{a}{r}, a, ar$.

$$\Rightarrow \frac{a}{\cancel{r}} . a . a\cancel{r} = a^3 = 27$$

$$\Rightarrow a = \sqrt[3]{27} = 3$$

Also, $\frac{a}{r} + a + ar = 13$

$$\Rightarrow \quad a + ar + ar^2 = 13r$$

$$a = 3: \quad 3 + 3r + 3r^2 = 13r$$

$$\Rightarrow \quad 3r^2 - 10r + 3 = 0$$

$$(3r - 1)(r - 3) = 0$$

$$\therefore \quad r = \frac{1}{3} \text{ or } r = 3$$

If $r = 3$ and $a = 3 \quad \Rightarrow \quad \frac{3}{3}, 3, 3 \times 3, 3 \times 3^2$

$$\Rightarrow \quad 1, 3, 9, 27$$

If $r = \frac{1}{3}$ and $a = 3 \quad \Rightarrow \quad \frac{3}{\frac{1}{3}}, 3, 3 \times \left(\frac{1}{3}\right), 3 \times \left(\frac{1}{3}\right)^2$

$$\Rightarrow \quad 9, 3, 1, \frac{1}{3}$$

11. $T_n = 3 \times 2^{n-1}$

$$\Rightarrow \left.\begin{aligned} T_1 &= 3 \times 2^{1-1} = 3 \times 2^0 = 3 \\ T_2 &= 3 \times 2^{2-1} = 3 \times 2^1 = 6 \\ T_3 &= 3 \times 2^{3-1} = 3 \times 2^2 = 12 \\ T_4 &= 3 \times 2^{4-1} = 3 \times 2^3 = 24 \\ T_5 &= 3 \times 2^{5-1} = 3 \times 2^4 = 48 \end{aligned}\right\} \ 3, 6, 12, 24, 48$$

12. $T_n = 8\left(\frac{3}{4}\right)^n$

$$\Rightarrow T_1 = 8\left(\frac{3}{4}\right)^1 = 6$$

$$T_2 = 8\left(\frac{3}{4}\right)^2 = \frac{9}{2}$$

$$T_3 = 8\left(\frac{3}{4}\right)^3 = \frac{27}{8}$$

$$T_4 = 8\left(\frac{3}{4}\right)^4 = \frac{81}{32}$$

13. $T_n = (-1)^{n+1} \times \frac{5}{2^{n-4}}$

$$\Rightarrow \left.\begin{aligned} T_1 &= (-1)^{1+1} \times \frac{5}{2^{1-4}} = \frac{5}{2^{-3}} = 40 \\ T_2 &= (-1)^{2+1} \times \frac{5}{2^{2-4}} = \frac{-5}{2^{-2}} = -20 \\ T_3 &= (-1)^{3+1} \times \frac{5}{2^{3-4}} = \frac{5}{2^{-1}} = 10 \\ T_4 &= (-1)^{4+1} \times \frac{5}{2^{4-4}} = \frac{-5}{2^0} = -5 \end{aligned}\right\} \ 40, -20, 10, -5$$

14. (i) $x - 3, x, 3x + 4 \quad \Rightarrow \quad \dfrac{x}{x-3} = \dfrac{3x+4}{x}$

$$\therefore \quad x^2 = (3x+4)(x-3)$$
$$x^2 = 3x^2 - 9x + 4x - 12$$
$$\Rightarrow 2x^2 - 5x - 12 = 0$$
$$(2x+3)(x-4) = 0$$
$$\therefore x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

If $x = 4$; $\quad 1, 4, 16, ...$

If $x = -\frac{3}{2}$; $\quad -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2}, ...$

(ii) $x + 1, x + 4, 3x + 2 \quad \Rightarrow \quad \dfrac{x+4}{x+1} = \dfrac{3x+2}{x+4}$

$$\therefore \quad (x+4)(x+4) = (3x+2)(x+1)$$
$$\therefore \quad x^2 + 8x + 16 = 3x^2 + 5x + 2$$
$$\therefore \quad 2x^2 - 3x - 14 = 0$$
$$(2x-7)(x+2) = 0$$
$$\therefore \quad x = \frac{7}{2} \quad \text{or} \quad -2$$

If $x = -2$; $\quad -1, +2, -4, ...$

If $x = \frac{7}{2}$; $\quad \frac{9}{2}, \frac{15}{2}, \frac{25}{2}$

(iii) $x - 2, x, x + 3 \quad \Rightarrow \quad \dfrac{x}{x-2} = \dfrac{x+3}{x}$

$$\therefore \quad x^2 = (x+3)(x-2)$$
$$x^2 = x^2 + x - 6$$
$$\therefore \quad x - 6 = 0$$
$$x = 6.$$

When $x = 6$; $\quad 4, 6, 9, ...$

(iv) $x - 6, 2x, x^2 \quad \Rightarrow \quad \dfrac{2x}{x-6} = \dfrac{x^2}{2x}$

$$\Rightarrow \quad 4x^2 = x^2(x-6)$$
$$\Rightarrow \quad 4x^2 = x^3 - 6x^2$$
$$\therefore x^3 - 10x^2 = 0$$
$$x^2(x-10) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 10$$

When $x = 10$; $\quad 4, 20, 100, ...$

When $x = 0$; $\quad$ no sequence is formed.

15. $T_n = 2 \times 3^n$

$\Rightarrow T_{n+1} = 2 \times 3^{n+1}$

$\therefore \dfrac{T_{n+1}}{T_n} = \dfrac{\cancel{2} \times 3^{n+1}}{\cancel{2} \times 3^n} = 3$, a constant

$\therefore$ Sequence is geometric.

16. $T_n = 3 \times n^2$

$\Rightarrow T_{n+1} = 3 \times (n+1)^2$

$$\therefore \frac{T_{n+1}}{T_n} = \frac{3 \times (n+1)^2}{3 \times n^2} = \frac{3(n^2 + 2n + 1)}{3n^2} = \frac{3n^2 + 6n + 3}{3n^2}$$
$$= 1 + \frac{2}{n} + \frac{1}{n^2}$$

Which is not constant.

$\therefore$ Sequence is not geometric.

17. (i) $5, 15, 45, \ldots\ldots 3645$

$\Rightarrow a = 5, r = \dfrac{15}{5} = 3$

Also, $T_n = a \,.\, r^{n-1}$

$= 5 \,.\, 3^{n-1}$

$\Rightarrow 5 \,.\, 3^{n-1} = 3645$

$3^{n-1} = 729 = 3^6$

$\Rightarrow \quad n - 1 = 6$

$n \quad = 7$

(ii) $48, 6, \dfrac{3}{4}, \ldots\ldots \dfrac{3}{2048}$

$\Rightarrow a = 48, r = \dfrac{6}{48} = \dfrac{1}{8}$

Also, $T_n = a \,.\, r^{n-1}$

$= 48\left(\dfrac{1}{8}\right)^{n-1}$

$\Rightarrow 48\left(\dfrac{1}{8}\right)^{n-1} = \dfrac{3}{2048}$

$\left(\dfrac{1}{8}\right)^{n-1} = \dfrac{3}{48 \times 2048} = \dfrac{1}{32\,768} = \dfrac{1}{8^5}$

$\therefore \quad n - 1 = 5$

$n \quad = 6$

18. (i) $27 \times \left(\dfrac{2}{3}\right)^1, 27 \times \left(\dfrac{2}{3}\right)^2, 27 \times \left(\dfrac{2}{3}\right)^3, 27 \times \left(\dfrac{2}{3}\right)^4$

$= 18, 12, 8, \dfrac{16}{3}$

(ii) $T_n = 27\left(\dfrac{2}{3}\right)^n$ = height of the n^{th} bounce

(iii) $T_{12} = 27\left(\dfrac{2}{3}\right)^{12} = 0.21\text{m}$

19. (i) $A = €4000(1.03)^t$

To find sum of money on deposit, let $t = 0$

$\Rightarrow \quad A = €4000(1.03)^0 = €4000$

(ii) Year 1: $A = €4000(1.03)^1 = €4120$

Year 2: $A = €4000(1.03)^2 = €4243.60$

Year 3: $A = €4000(1.03)^3 = €4370.91$

Year 4: $A = €4000(1.03)^4 = €4502.04$

(iii) Year 10: $A = €4000(1.03)^{10} = €5375.67$

(iv) Double value $= €8000$

$\Rightarrow \quad €8000 = €4000(1.03)^t$

$\Rightarrow \quad 2 = (1.03)^t$

$t =$ between 23 and 24 years

$t = 23.45$

$$\left[\begin{array}{l}\text{Using logs: } \log 2 = \log 1.03^t = t \log 1.03 \\ \qquad\qquad t = \dfrac{\log 2}{\log 1.03} = 23.45\end{array}\right]$$

20. $A = P(1 + i)^t$

$t = 10$ years, $P = €2500$, $A = €3047$

$\Rightarrow \quad 3{,}047 = 2{,}500\,(1 + i)^{10}$

$1.2188 = (1 + i)^{10}$

$(1 + i) = \sqrt[10]{1.2188} = (1.2188)^{\frac{1}{10}} = 1.02$

$\Rightarrow \quad i = 0.02 \quad = 2\%.$

Exercises 9.5

1. (i) $\lim\limits_{n\to\infty} \dfrac{n+2}{3n-4} = \lim\limits_{n\to\infty} \dfrac{1+\frac{2}{n}}{3-\frac{4}{n}} = \dfrac{1+0}{3-0} = \dfrac{1}{3}$

(ii) $\lim\limits_{n\to\infty} \dfrac{5n-4}{n-4} = \lim\limits_{n\to\infty} \dfrac{5-\frac{4}{n}}{1-\frac{4}{n}} = \dfrac{5-0}{1-0} = 5$

(iii) $\lim\limits_{n\to\infty} \dfrac{n^2+2}{n^2-4} = \lim\limits_{n\to\infty} \dfrac{1+\frac{2}{n^2}}{1-\frac{4}{n^2}} = \dfrac{1+0}{1-0} = 1$

2. (i) $\lim\limits_{n\to\infty} \dfrac{n^3+2n}{2n^3-4n^2+3n} = \lim\limits_{n\to\infty} \dfrac{1+\frac{2}{n^2}}{2-\frac{4}{n}+\frac{3}{n^2}} = \dfrac{1+0}{2-0+0} = \dfrac{1}{2}$

(ii) $\lim\limits_{n\to\infty} \dfrac{8}{n(n-6)} = \lim\limits_{n\to\infty} \dfrac{8}{n^2-6n} = \lim\limits_{n\to\infty} \dfrac{\frac{8}{n^2}}{1-\frac{6}{n}} = \dfrac{0}{1-0} = 0$

(iii) $\lim\limits_{n\to\infty} \dfrac{(n+3)(n-1)}{n^2-9} = \lim\limits_{n\to\infty} \dfrac{(n+3)(n-1)}{(n+3)(n-3)}$

$= \lim\limits_{n\to\infty} \dfrac{n-1}{n-3}$

$= \lim\limits_{n\to\infty} \dfrac{1-\frac{1}{n}}{1-\frac{3}{n}} = \dfrac{1-0}{1-0} = 1$

3. (i) $\lim\limits_{n\to\infty} \dfrac{n^3+2n^2-3n}{n^2} = \lim\limits_{n\to\infty} \left(n+2-\frac{3}{n}\right) = \infty$ (limit does not exist)

(ii) $\lim\limits_{n\to\infty} \dfrac{4n^2+2}{n^2} = \lim\limits_{n\to\infty} \dfrac{4+\frac{2}{n^2}}{1} = \dfrac{4+0}{1} = 4$

(iii) $\lim\limits_{n\to\infty} \dfrac{6n^2+5n}{n^3} = \lim\limits_{n\to\infty} \dfrac{\frac{6}{n}+\frac{5}{n^2}}{1} = \dfrac{0+0}{1} = 0$

4. (i) $\lim\limits_{n\to\infty} \left(4+\dfrac{5n-4}{n-4}\right) = \lim\limits_{n\to\infty} 4 + \lim\limits_{n\to\infty} \dfrac{5-\frac{4}{n}}{1-\frac{4}{n}}$

$= 4 + \dfrac{5-0}{1-0} = 4+5 = 9$

(ii) $\lim\limits_{n\to\infty} \left(\dfrac{1}{2}-\dfrac{n^2-2n}{3n^2}\right) = \lim\limits_{n\to\infty} \dfrac{1}{2} - \lim\limits_{n\to\infty} \dfrac{1-\frac{2}{n}}{3}$

$= \dfrac{1}{2} - \dfrac{1-0}{3} = \dfrac{1}{2}-\dfrac{1}{3} = \dfrac{1}{6}$

(iii) $\lim\limits_{n\to\infty} (3n)\left(\dfrac{n+4}{n-1}\right) = \lim\limits_{n\to\infty} 3n \times \lim\limits_{n\to\infty} \dfrac{1+\frac{4}{n}}{1-\frac{1}{n}}$

$= (\infty) \times \dfrac{1+0}{1-0} = \infty$ (limit does not exist)

5. (i) $\lim\limits_{n\to\infty} \dfrac{5n+2}{3n^2+2} = \lim\limits_{n\to\infty} \dfrac{\frac{5}{n}+\frac{2}{n^2}}{3+\frac{2}{n^2}} = \dfrac{0+0}{3+0} = 0$

(ii) $\lim\limits_{n\to\infty} \dfrac{n-2n^3}{n^2+2n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^2}-2}{\frac{1}{n}+\frac{2}{n^2}} = \dfrac{0-2}{0+0} = \infty$ (limit does not exist)

(iii) $\lim\limits_{n\to\infty} \dfrac{7n^3}{n^2+3n-4} = \lim\limits_{n\to\infty} \dfrac{7}{\frac{1}{n}+\frac{3}{n^2}-\frac{4}{n^3}} = \dfrac{7}{0+0-0} = \infty$ (limit does not exist)

6. The highest power of n is n^p. This is what we divide above and below by.

$$\lim_{n\to\infty}\frac{an^p+cn^q}{bn^p+dn^s}=\lim_{n\to\infty}\frac{a+\frac{cn^q}{n^p}}{b+\frac{dn^s}{n^p}}$$

$$=\lim_{n\to\infty}\frac{a+\frac{c}{n^{p-q}}}{b+\frac{d}{n^{p-s}}}=\frac{a+0}{b+0}=\frac{a}{b}.$$

Exercise 9.6

1. $2+6+18+54+\ldots$ for 10 terms.

$\Rightarrow a=2, r=\dfrac{6}{2}=3$

Also, $S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_{10}=\frac{2(1-3^{10})}{1-3}=\frac{2(1-3^{10})}{-2}=3^{10}-1$$
$$=59048$$

2. $1024+512+256+\ldots\ldots 32.$

$\Rightarrow a=1024,\quad r=\dfrac{512}{1024}=\dfrac{1}{2}$

also, $T_n=a\,.\,r^{n-1}$

$$32=1024\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \frac{32}{1024}=\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \frac{1}{32}=\frac{1}{2^{n-1}}=\frac{1}{2^5}$$

$$\Rightarrow n-1=5$$

$$n=6$$

[Note : $32=2^{n-1}$

$\Rightarrow \log 32=(n-1)\log 2$

$\Rightarrow \dfrac{\log 32}{\log 2}=n-1$

$5=n-1$]

Hence, $S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_6=\frac{1024\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$=2048\left(1-\left(\frac{1}{2}\right)^6\right)=2016$$

3. $1+2+4+8$

$\Rightarrow a=1, r=\dfrac{2}{1}=2$

$\therefore S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_8=\frac{1(1-2^8)}{1-2}=2^8-1=255$$

4. $32+16+8+\ldots\ldots$

$\Rightarrow a=32, r=\dfrac{16}{32}=\dfrac{1}{2}$

$\therefore S_n=\dfrac{a(1-r^n)}{1-r}$

$$\Rightarrow S_{10} = \frac{32\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$$

$$= 64\left(1 - \left(\frac{1}{2}\right)^{10}\right) = 63\frac{15}{16} = 63.94$$

5. $4 - 12 + 36 - 108 + \ldots$

$$\Rightarrow a = 4, r = \frac{-12}{4} = -3$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\Rightarrow S_6 = \frac{4(1 - (-3)^6)}{1 - (-3)}$$

$$= 1 - (-3)^6 = -728$$

6. $729 - 243 + 81 - \ldots - \frac{1}{3}$

$$\Rightarrow a = 729, r = \frac{-243}{729} = \frac{-1}{3}$$

also, $T_n = a \cdot r^{n-1}$

$$\therefore -\frac{1}{3} = 729\left(-\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow \frac{-1}{2187} = \frac{1}{(-3)^{n-1}} = \frac{-1}{3^7} = \frac{1}{(-3)^7} \quad \text{(or use logs)}$$

$$\Rightarrow \quad n - 1 = 7$$

$$n = 8$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_8 = \frac{729\left(1 - \left(-\frac{1}{3}\right)^8\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$= 546.75\left(1 - \left(-\frac{1}{3}\right)^8\right) = 546\,\frac{2}{3}$$

7. $\sum_{r=1}^{6} 4^r = 4^1 + 4^2 + 4^3 + \ldots$

$$= 4 + 16 + 64$$

$$\Rightarrow a = 4, r = \frac{16}{4} = 4, n = 6$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\Rightarrow S_6 = \frac{4(1 - 4^6)}{1 - 4}$$

$$= \left(\frac{-4}{3}\right)(1 - 4^6) = 5460$$

8. $\sum_{r=1}^{8} 2 \times 3^r = 2 \times 3^1 + 2 \times 3^2 + 2 \times 3^3 + \ldots$

$$\Rightarrow a = 6, r = \frac{18}{6} = 3, n = 8$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\Rightarrow S_8 = \frac{6(1 - 3^8)}{1 - 3}$$

$$= -3(1 - 3^8) = 19\,680$$

9. $\sum_{r=1}^{10} 6 \times \left(\frac{1}{2}\right)^r = 6 \times \frac{1}{2} + 6 \times \left(\frac{1}{2}\right)^2 + 6 \times \left(\frac{1}{2}\right)^3 + \ldots$

$= 3 + \frac{6}{4} + \frac{6}{8} + \ldots$

$\Rightarrow \quad a = 3, r = \frac{1}{2}, n = 10$

$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$

$= \frac{3\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$

$= 6\left(1 - \left(\frac{1}{2}\right)^{10}\right)$

$= 5.994145$

$= 5.994$

10. (i) $0.\dot{7} = 0.7777$

$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots..$

$\Rightarrow a = \frac{7}{10}, r = \frac{\frac{7}{100}}{\frac{7}{10}} = \frac{1}{10}$

$\therefore S_\infty = \frac{a}{1 - r}$

$= \frac{\frac{7}{10}}{1 - \frac{1}{10}}$

$= \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$

(ii) $0.\dot{3}\dot{5} = 0.353535$

$= \frac{35}{100} + \frac{35}{10\,000} + \frac{35}{1\,000\,000} + \ldots..$

$\Rightarrow a = \frac{35}{100}, r = \frac{35}{10\,000} \times \frac{100}{35} = \frac{1}{100}$

$\therefore S_\infty = \frac{a}{1 - r}$

$= \frac{\frac{35}{100}}{1 - \frac{1}{100}}$

$= \frac{35}{100} \times \frac{100}{99} = \frac{35}{99}$

(iii) $0.2\dot{3} = 0.23333$

$= \frac{2}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10\,000} + \ldots..$

$= \frac{2}{10} + \left[\frac{3}{100} + \frac{3}{1000} + \frac{3}{10\,000} + \ldots\right]$

Let $a = \frac{3}{100}, r = \frac{3}{1000} \times \frac{100}{3} = \frac{1}{10}$

$$\therefore S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{3}{100}}{1-\frac{1}{10}}$$
$$= \frac{3}{100} \times \frac{10}{9} = \frac{1}{30}$$
$$\therefore 0.2\dot{3} = \frac{2}{10} + \left[\frac{1}{30}\right]$$
$$= \frac{7}{30}$$

(iv) $0.\dot{3}7\dot{0} = 0.370370370$
$$= \frac{370}{1000} + \frac{370}{1\,000\,000}$$
$$\Rightarrow a = \frac{370}{1000},\ r = \frac{\cancel{370}}{1\,000\,000} \times \frac{1000}{\cancel{370}} = \frac{1}{1000}$$

$$\therefore S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{370}{1000}}{1-\frac{1}{1000}}$$
$$= \frac{370}{1000} \times \frac{1000}{999} = \frac{370}{999} = \frac{10}{27}$$

(v) $0.1\dot{6}\dot{2} = 0.1626262$
$$= \frac{1}{10} + \left[\frac{62}{1000} + \frac{62}{1\,00\,000} + \ldots\right]$$
$$\text{Let } a = \frac{62}{1000},\ r = \frac{\cancel{62}}{100\,\cancel{000}} \cdot \frac{1\cancel{000}}{\cancel{62}} = \frac{1}{100}$$

$$\therefore S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{62}{1000}}{1-\frac{1}{100}}$$
$$= \frac{62}{10\cancel{00}} \cdot \frac{1\cancel{00}}{99} = \frac{62}{990}$$
$$\therefore 0.1\dot{6}\dot{2} = \frac{1}{10} + \frac{62}{990}$$
$$= \frac{161}{990}$$

(vi) $0.3\dot{2}\dot{1} = 0.3212121$
$$= \frac{3}{10} + \left[\frac{21}{1000} + \frac{21}{100\,000} + \ldots\right]$$
$$\text{Let } a = \frac{21}{1000},\ r = \frac{\cancel{21}}{100\,\cancel{000}} \times \frac{1\cancel{000}}{\cancel{21}} = \frac{1}{100}$$

$$\therefore S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{21}{1000}}{1-\frac{1}{100}}$$
$$= \frac{\cancel{21}^{7}}{10\cancel{00}} \times \frac{1\cancel{00}}{\cancel{99}_{33}} = \frac{7}{330}$$
$$\therefore 0.3\dot{2}\dot{1} = \frac{3}{10} + \frac{7}{330}$$
$$= \frac{106}{330} = \frac{53}{165}$$

11. $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ... \left(\frac{1}{2}\right)^{n-1}$

$$a = 1, r = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_n = \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$= 2 - \frac{1}{2^{n-1}}$$

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[2 - \frac{1}{2^{n-1}}\right]$$

$$= 2 \quad \text{since } \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0.$$

$$S_\infty - S_n < 0.001$$

$$\Rightarrow 2 - \left[2 - \frac{1}{2^{n-1}}\right] < 0.001$$

$$\Rightarrow \quad \frac{1}{2^{n-1}} < 0.001$$

$$\Rightarrow \quad 2^{n-1} > \frac{1}{0.001}$$

$$2^{n-1} > 1000$$

Using logs: $\log 2^{n-1} > \log 1000$

$$(n - 1)\log 2 > \log 1000$$

$$n - 1 > \frac{\log 1000}{\log 2}$$

$$n - 1 > 9.97$$

$$n > 10.97$$

$$\therefore \quad n = 11$$

Exercise 9.7

1. $T_1 = 2$

$T_2 = 4T_1 = 4(2) = 8$

$T_3 = 4T_2 = 4(8) = 32$

$T_4 = 4T_3 = 4(32) = 128$

$T_5 = 4T_4 = 4(128) = 512$

2. (i) $G_1 = 2$

$G_2 = 2 + 3G_1 = 2 + 3(2) = 8$

$G_3 = 2 + 3G_2 = 2 + 3(8) = 26$

$G_4 = 2 + 3G_3 = 2 + 3(26) = 80$

$G_5 = 2 + 3G_4 = 2 + 3(80) = 242$

(ii) $G_1 = 1$

$G_2 = 3$

$G_3 = 2G_2 + G_1 = 2(3) + (1) = 7$

$G_4 = 2G_3 + G_2 = 2(7) + (3) = 17$

$G_5 = 2G_4 + G_3 = 2(17) + 7 = 41$

(iii) $G_1 = 0$

$G_2 = 1$

$G_3 = 3G_2 - G_1 = 3(1) - (0) = 3$

$G_4 = 3G_3 - G_2 = 3(3) - (1) = 8$

$G_5 = 3G_4 - G_3 = 3(8) - (3) = 21$

3. $S_0 = 6$
$S_1 = 2 + S_0 = 2 + (6) = 8$
$S_2 = 2 + S_1 = 2 + (8) = 10$
$S_3 = 2 + S_2 = 2 + (10) = 12$
$S_4 = 2 + S_3 = 2 + (12) = 14$
As the first differences are constant, S_n is a linear sequence.
Let $S_n = an + b$.
$S_0 = 6$: $\quad a(0) + b = 6$
$\qquad b = 6$
$S_1 = 8$: $\quad a(1) + 6 = 8$
$\qquad a = 2$
Thus $\quad S_n = 2n + 6$

4. Let T_n be any term in the sequence. Then
$T_n = T_{n-1} + T_{n-2}$
Then:
$T_1 = 0$
$T_2 = 0$
$T_3 = T_2 + T_1 = (1) + (0) = 1$
$T_4 = T_3 + T_2 = (1) + (1) = 2$
$T_5 = T_4 + T_3 = (2) + (1) = 3$
$T_6 = T_5 + T_4 = (3) + (2) = 5$
$T_7 = T_6 + T_5 = (5) + (3) = 8$
$T_8 = T_7 + T_6 = (8) + (5) = 13$

5. (i) $T_1 = 1$
$T_2 = 3$
$T_3 = 3T_2 - T_1 = 3(3) - 1 = 8$
$T_4 = 3T_3 - T_2 = 3(8) - 3 = 21$

(ii) $T_1 = 5$
$T_2 = T_1 - 2 = 5 - 2 = 3$
$T_3 = T_2 - 2 = 3 - 2 = 1$
$T_4 = T_3 - 2 = 1 - 2 = -1$

6. $G_1 = 2$
$G_2 = (G_1 - 2)^2 = (2 - 2)^2 = 0$
$G_3 = (G_2 - 2)^2 = (0 - 2)^2 = 4$
$G_4 = (G_3 - 2)^2 = (4 - 2)^2 = 4$
$G_5 = (G_4 - 2)^2 = (4 - 2)^2 = 4$

7. (a) (i) $T_1 = 1$
$T_2 = T_1 + 2 = 1 + 2 = 3$
$T_3 = T_2 + 2 = 3 + 2 = 5$
$T_4 = T_3 + 2 = 5 + 2 = 7$
$T_5 = T_4 + 2 = 7 + 2 = 9$
$T_6 = T_5 + 2 = 9 + 2 = 11$

(ii) $S_1 = 3$
$S_2 = S_1 + 4 = 3 + 4 = 7$
$S_3 = S_2 + 4 = 7 + 4 = 11$
$S_4 = S_3 + 4 = 11 + 4 = 15$
$S_5 = S_4 + 4 = 15 + 4 = 19$
$S_6 = S_5 + 4 = 19 + 4 = 23$

(b) T_n: The first differences are 2, 2, 2, 2, 2. As these are constant, T_n is arithmetic.
S_n: The first differences are 4, 4, 4, 4, 4. As these are constant, S_n is arithmetic.

(c) $S_n = an + b$
$S_1 = 3$: $\quad a(1) + b = 3$
$\qquad a + b = 3 \quad$ **... 1**
$S_2 = 7$: $\quad a(2) + b = 7$
$\qquad 2a + b = 7 \quad$ **... 2**

Then

1 × −1: $-a - b = -3$

2: $2a + b = 7$

$a = 4$

1: $4 + b = 3$

$b = -1$

(d) Thus $S_n = 4n - 1$

$S_{20} = 4(20) - 1 = 79.$

Exercise 9.8

1. (i) 5, 9, 13, 17, 21, ...

1st difference = 4, 4, 4, 4, etc

$\therefore \quad T_n = 4n + a$

Since $T_1 = 4(1) + a = 5$

$\Rightarrow \quad a = 1$

$\therefore \quad T_n = 4n + 1$

(ii) 1, 4, 7, 10, 13, ...

1st difference = 3, 3, 3, 3, etc

$\therefore \quad T_n = 3n + a$

Since $T_1 = 3(1) + a = 1$

$\Rightarrow \quad a = -2$

$\therefore \quad T_n = 3n - 2$

(iii) 11, 16, 21, 26, 31, ...

1st difference = 5, 5, 5, 5, etc

$\therefore \quad T_n = 5n + a$

Since $T_1 = 5(1) + a = 11$

$\Rightarrow \quad a = 6$

$\therefore \quad T_n = 5n + 6$

2. (i) 2, 1, 0, −1, −2, ...

First difference = −1 $\Rightarrow T_n = -n + a$

Since $T_1 = -(1) + a = 2$

$\Rightarrow \quad a = 3$

$\therefore \quad T_n = -n + 3$

(ii) 0, −2, −4, −6, −8, ...

First difference = −2 $\Rightarrow T_n = -2n + a$

Since $T_1 = -2(1) + a = 0$

$\Rightarrow \quad a = 2$

$\therefore \quad T_n = -2n + 2$

(iii) −6, −4, −2, 0, 2, ...

First difference = 2 $\Rightarrow T_n = -2n + a$

Since $T_1 = 2(1) + a = -6$

$\Rightarrow \quad a = -8$

$\therefore \quad T_n = 2n - 8$

3. (i) Sequence of squares = 1, 3, 6, 10

First difference = 2, 3, 4, ... etc

Second difference = 1, 1, 1, ... etc

$\therefore T_n = an^2 + bn + c$, where $2a = 1$

$\Rightarrow \quad a = \frac{1}{2}.$

$\therefore T_n = \frac{1}{2}n^2 + bn + c$

Since $T_1 = \frac{1}{2}(1)^2 + b(1) + c = 1$

$\Rightarrow \quad b + c = \frac{1}{2}$

$\Rightarrow \quad 2b + 2c = 1$ ..... A

Also, $T_2 = \frac{1}{2}(2)^2 + b(2) + c = 3$

$\Rightarrow \quad 2b + c = 1$ .... B

A: $\cancel{2b} + 2c = 1$

B: $\cancel{2b} + c = 1$

A − B: $c = 0$ and $b + 0 = \frac{1}{2}$

$\Rightarrow \quad b = \frac{1}{2}$

$\therefore \quad T_n = \frac{1}{2}n^2 + \frac{1}{2}n$

$\Rightarrow T_{28} = \frac{1}{2}(28)^2 + \frac{1}{2}(28)$

$= 406$

$\Rightarrow$ Area $= 406 \times 25 \text{ mm}^2$

$= 10\,150 \text{ mm}^2$

(ii) Sequence for perimeter $= 4, 8, 12, 16$

First difference $= 4, 4, 4$, etc

$\therefore T_n = an + b$, where $a = 4$

$= 4n + b$

Since $T_1 = 4(1) + b = 4$

$b = 0$

$\therefore T_n = 4n$

$\Rightarrow \quad T_{28} = 4 \times 28 = 112$

$\Rightarrow$ Perimeter $= 112 \times 5$ mm $= 560$ mm.

4. (i) 1, 4, 9, 16 ....

We recognise immediately that this is a sequence of squared numbers.

Alternatively, we could use differences.

$$\left.\begin{array}{ll} & 1, 4, 9, 16 \\ 1^{\text{st}} \text{ difference:} & 3, 5, 7 \\ 2^{\text{nd}} \text{ difference:} & 2, 2 \\ & \end{array}\right\} \begin{array}{l} \Rightarrow an^2 + bn + c = T_n \\ \text{where } 2a = 2 \\ \Rightarrow a = 1 \\ \therefore n^2 + bn + c \end{array}$$

We can then proceed that $b = c = 0$.

$\Rightarrow T_n = n^2$

$\therefore T_{30} = 30^2 = 900$ Triangles $= 900 \text{ cm}^2$

(ii) $T_n = n^2 = 441$

$n = \sqrt{441} = 21$

5. (i) Areas $= (1 \times 2), (2 \times 3), (3 \times 4)$

$= T_1, \quad T_2, \quad T_3$

By inspection: $T_n = n(n + 1) = n^2 + n$

$$\left.\begin{array}{ll} \textbf{or} & 2, 6, 12, 20, \ldots \\ \text{First difference:} & 4, 6, 8 \\ \text{Second difference:} & 2, 2 \\ & \end{array}\right\} \begin{array}{l} \Rightarrow T_n = an^2 + bn + c \\ \quad \text{where } 2a = 2 \\ \quad \Rightarrow a = 1 \\ \Rightarrow T_n = n^2 + bn + c \end{array}$$

Since $T_1 = (1)^2 + b(1) + c = 2$

$\Rightarrow b + c = 1$ ...... A

$$\left.\begin{array}{ll} \text{Also, } T_2 = (2)^2 + b(2) + c = 6 & \\ \quad \Rightarrow 2b + c = 2 \text{ .... B} & \\ \text{A:} \quad b + c = 1 & \\ \text{B:} \quad \underline{2b + c = 2} & \end{array}\right\} \Rightarrow T_n = n^2 + n \text{ (as before)}$$

A$-$B: $-b \quad = -1$

$b \quad = 1 \qquad \Rightarrow c = 0$

$\therefore T_{100} = 100^2 + 100$

$= 10\,100 \text{ cm}^2$

(ii) $T_n = n^2 + n = 240$

$\Rightarrow n^2 + n - 240 = 0$

$(n + 16)(n - 15) = 0$

$\Rightarrow n = -16 \quad$ or $\quad n = 15$

Since $n \in N$, $n = 15$.

6. (i) Sequence: 6, 27, 74, 159, 294
1st difference: 21, 47, 85, 135
2nd difference: 26, 38, 50
3rd difference: 12, 12

$\Rightarrow T_n = an^3 + bn^2 + cn + d$
where $6a = 12$
$\Rightarrow a = 2$

$\therefore \quad T_n = 2n^3 + bn^2 + cn + d$

Since $T_1 = 2(1)^3 + b(1)^2 + c(1) + d = 6$

$b + c + d = 4$ .... A

$T_2 = 2(2)^3 + b(2)^2 + c(2) + d = 27$

$4b + 2c + d = 11$ .... B

$T_3 = 2(3)^3 + b(3)^2 + c(3) + d = 74$

$9b + 3c + d = 20$ C

A: $b + c + d = 4$
B: $\underline{4b + 2c + d = 11}$
A − B: $-3b - c = -7$ .... D

B: $4b + 2c + d = 11$
C: $\underline{9b + 3c + d = 20}$
B − C: $-5b - c = -9$ .... E

D: $-3b - c = -7$
E: $\underline{-5b - c = -9}$
D − E: $+2b = 2$
$\Rightarrow \quad b = 1$
D: $\therefore \quad -3(1) - c = -7$
$\Rightarrow \quad c = 4$
A: $\therefore 1 + 4 + d = 4$
$\Rightarrow \quad d = -1$

$\Rightarrow T_n = 2n^3 + n^2 + 4n - 1$

[Note: We could have proceeded by subtracting the sequence $T_n = 2n^3$, i.e 2, 16, 54, 128, etc, and found quadratic element of pattern as before.]

(ii) Sequence: 3, −1, −1, 9, 35
1st difference: −4, 0, 10, 26
2nd difference: 4, 10, 16
3rd difference: 6, 6

$\Rightarrow T_n = an^3 + bn^2 + cn + d$
where $6a = 6$
$\Rightarrow a = 1$

$\therefore T_n = n^3 + bn^2 + cn + d$

Sequence: 3, −1, −1, 9, 35
n^3: 1, 8, 27, 64

Subtracting: 2, −9, −28, −55

$\Rightarrow bn^2 + cn + d$: 2, −9, −28, −55
1st difference: −11, −19, −27
2nd difference: −8, −8

$\Rightarrow 2b = -8$
$b = -4$

$\therefore \quad T_n = n^3 - 4n^2 + cn + d$

Since $T_1 = (1)^3 - 4(1)^2 + c(1) + d = 3$

$\Rightarrow \quad c + d = 6$ ...... A

$T_2 = (2)^3 - 4(2)^3 + c(2) + d = -1$

$2c + d = 7$ ..... B

A: $c + d = 6$
B: $2c + d = 7$
A − B: $-c = -1$
$\Rightarrow \quad c = 1 \therefore 1 + d = 6 \quad \Rightarrow d = 5$
$\therefore T_n = n^3 - 4n^2 + n + 5$

[Note: Simultaneous equations in b, c, d could be used as in part (i) to evaluate coefficients of the quadratic part of T_n.]

(iii) Sequence: 3, −1, 2, 17, 50, 107

$\Rightarrow T_n = an^3 - bn^2 + cn + d$

when $6a = 6$

$\Rightarrow a = 1$

1st difference: 3 15 33 57

2nd difference 12 18 24

3rd difference: 6 6

$\therefore T_n = n^3 + bn^2 + cn + d$

Sequence: −1, 2, 17, 50, 107

n^3: 1, 8, 27, 64

$bn^2 + cn + d$: −2, −6, −10, −14

1st difference: −4, −4, −4

$\Rightarrow b = 0$ and $c = -4$

$\therefore T_n = n^3 - 4n + d$

Since $T_1 = (1)^3 - 4(1) + d = -1$

$d = +2$

$\therefore T_n = n^3 - 4n + d$

[Note: Simultaneous equations in b, c, d could be used as in part (i) to evaluate coefficients of the quadratic part of T_n.]

7. (a) Sequence: 0, 3, 8, 15, ...

By inspection: $n^2 - 1$

$\Rightarrow T_n = n^2 - 1$

$\Rightarrow T_{24} = 24^2 - 1$

$= 575$ bright tiles

and 1 dark tile

Difference: 0, 3, 8, 15

1st difference: 3, 5, 7

2nd difference: 2, 2

$\Rightarrow T_n = an^2 + bn + c$

where $2a = 2 \Rightarrow a = 1$

$\therefore T_n = n^2 + bn + c$

$T_1 = (1)^2 + b(1) + c = 0$

$\Rightarrow \quad b + c = -1$ ...... A

$T_2 = (2)^2 + b(2) + c = 3$

$\Rightarrow \quad 2b + c = -1$ ...... B

$\therefore$ B − A : $b = 0$

A: $b + c = -1$

$\Rightarrow c = -1$

$\therefore T_n = n^2 - 1$

(b) Sequence: 0, 2, 7, 14, ...

By inspection $n^2 - 2$ [We note that $n = 1$ produces $1^2 - 2 = -1$ bright tiles! However, for $n = 2, 3, ...$ the formula fits exactly.]

$\Rightarrow T_n = n^2 - 2$

$\Rightarrow T_{24} = 24^2 - 2$

$= 574$ bright tiles

and 2 dark tiles.

(c) Sequence: 0, 2, 6, 12, ...

By inspection: $n^2 - n$

$\Rightarrow T_n = n^2 - n$

$\Rightarrow T_{24} = 24^2 - 24$

$= 552$ bright tiles

and 24 dark tiles.

8. (i) Sequence: 7, 16, 31, 52, 79 ...

1st difference: 9, 15, 21, 27 ...

2nd difference: 6, 6, 6 ...

$\Rightarrow T_n = an^2 + bn + c$

where $2a = 6$

$\Rightarrow a = 3$

$\therefore T_n = 3n^2 + bn + c$

Since $T_1 = 3(1)^2 + b(1) + c = 7$

$\Rightarrow b + c = 4$ ..... A

Also, $T_2 = 3(2)^2 + b(2) + c = 16$

$2b + c = 14$ ..... B

$\Rightarrow$ B − A: $b = 0$

$\therefore$ A: $0 + c = 4$

$\Rightarrow c = 4$

$\therefore T_n = 3n^2 + 4.$

(ii) Sequence: $1, 0, -3, -8, -15, \ldots$
1st difference: $-1, -3, -5, -7, \ldots$
2nd difference: $-2, -2, -2, \ldots$

$\Rightarrow T_n = an^2 + bn + c$ where $2a = -2$ $\Rightarrow a = -1$

$\therefore T_n = -n^2 + bn + c$

Since $T_1 = -(1)^2 + b(1) + c = 1$

$b + c = 2$ ..... A

Also, $T_2 = -(2)^2 + b(2) + c = 0$

$2b + c = 4$ ..... B

$\Rightarrow$ B − A: $b = 2$

$\therefore$ A: $2 + c = 2$

$\Rightarrow c = 0$

$\therefore T_n = -n^2 + 2n.$

(iii) Sequence: $-1, 14, 53, 128, 251, \ldots$
1st difference: $15, 39, 75, 123, \ldots$
2nd difference: $24, 36, 48, \ldots$
3rd difference: $12, 12, \ldots$

$\Rightarrow T_n = an^3 + bn^2 + cn + d$ where $6a = 12$ $\Rightarrow a = 2$

$\therefore T_n = 2n^3 + bn^2 + cn + d$

Sequence:	$-1, 14, 53, 128, 251$
$2n^3$:	$2, 16, 54, 128, 250$
$bn^2 + cn + d$:	$-3, -2, -1, 0, 1$
1st difference:	$1, 1, 1, 1$

$\Rightarrow b = 0$, $c = 1$

$\therefore T_n = 2n^3 + n + d$

Since $T_1 = 2(1)^3 + (1) + d = -1$

$d = -4$

$\therefore T_n = 2n^3 + n - 4.$

(iv) Sequence: $-2, 2, 6, 10, 14, \ldots$
1st difference: $4, 4, 4, 4, \ldots$

$\Rightarrow T_n = an + b$ where $a = 4$

$\therefore T_n = 4n + b$

Since $T_1 = 4(1) + b = -2$

$b = -6$

$\therefore T_n = 4n - 6.$

(v) Sequence: $4, 31, 98, 223, 424, \ldots$
1st difference: $27, 67, 125, 201, \ldots$
2nd difference: $40, 58, 76, \ldots$
3rd difference: $18, 18, \ldots$

$T_n = an^3 + bn^2 + cn + d$ where $6a = 18$ $\Rightarrow a = 3$

$\therefore T_n = 3n^3 + bn^2 + cn + d$

$$\begin{array}{rl} \text{Sequence:} & 4, 31, 98, 223, 424 \\ 3n^3: & 3, 24, 81, 192, 375 \\ \hline bn^2 + cn + d: & 1, \ 7, 17, \ 31, \ 49 \\ 1^{st} \text{ difference:} & 6, 10, 14, 18 \\ 2^{nd} \text{ difference:} & 4, 4, 4 \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \begin{array}{l} \Rightarrow 2b = 4 \\ b = 2 \end{array}$$

$\therefore T_n = 2n^2 + cn + d$

Since $T_1 = 2(1)^2 + c(1) + d = 1$

$c + d = -1$ ..... A

Also, $T_2 = 2(2)^2 + c(2) + d = 7$

$2c + d = -1$ ..... B

$\therefore$ B − A: $c = 0$ $\quad\quad$ $\therefore$ A: $0 + d = -1$

$\Rightarrow \quad d = -1$

$\therefore T_n = 3n^3 + 2n^2 - 1.$

Revision Exercise 9 (Core)

1. (i) $T_n = 3n + 4$

$$\left.\begin{array}{l} \Rightarrow T_1 = 3(1) + 4 = 7 \\ T_2 = 3(2) + 4 = 10 \\ T_3 = 3(3) + 4 = 13 \\ T_4 = 3(4) + 4 = 16 \end{array}\right\} 7, 10, 13, 16$$

(ii) $T_n = 6n - 1$

$$\left.\begin{array}{l} \Rightarrow T_1 = 6(1) - 1 = 5 \\ T_2 = 6(2) - 1 = 11 \\ T_3 = 6(3) - 1 = 17 \\ T_4 = 6(4) - 1 = 23 \end{array}\right\} 5, 11, 17, 23$$

(iii) $T_n = 2^{n-1}$

$$\left.\begin{array}{l} \Rightarrow T_1 = 2^{1-1} = 2^0 = 1 \\ T_2 = 2^{2-1} = 2^1 = 2 \\ T_3 = 2^{3-1} = 2^2 = 4 \\ T_4 = 2^{4-1} = 2^3 = 8 \end{array}\right\} 1, 2, 4, 8$$

(iv) $T_n = (n + 3)(n + 4)$

$$\left.\begin{array}{l} \Rightarrow T_1 = (1 + 3)(1 + 4) = 20 \\ T_2 = (2 + 3)(2 + 4) = 30 \\ T_3 = (3 + 3)(3 + 4) = 42 \\ T_4 = (4 + 3)(4 + 4) = 56 \end{array}\right\} 20, 30, 42, 56$$

(v) $T_n = n^3 + 1$

$$\left.\begin{array}{l} \Rightarrow T_1 = 1^3 + 1 = 2 \\ T_2 = 2^3 + 1 = 9 \\ T_3 = 3^3 + 1 = 28 \\ T_4 = 4^3 + 1 = 65 \end{array}\right\} 2, 9, 28, 65$$

2. $T_3 = 71, \quad T_7 = 55$

$T_n = a + (n - 1)d, \quad\quad a$ = first term, d = common difference

$\Rightarrow T_3 = a + 2d = 71$ .... A

$\Rightarrow T_7 = a + 6d = 55$ .... B

$\Rightarrow$ A − B: $\quad -4d = 16$

$\Rightarrow \quad d = -4$

$\Rightarrow \quad$ A: $a + 2(-4) = 71$

$\Rightarrow \quad a = 79$

$\Rightarrow$ first term $= 79$, common difference $= -4$.

3. $T_1 = 12, \quad S_\infty = 36$

$T_1 = a = 12$

$$S_\infty = \frac{a}{1-r} = 36$$

$$\Rightarrow \frac{12}{1-r} = 36$$

$\Rightarrow 12 = 36 - 36r$

$$r = \frac{24}{36} = \frac{2}{3}$$

4. (i) $-2, 4, -8, \ldots \Rightarrow r = \dfrac{4}{-2} = -2$

$\therefore T_n = a \,.\, r^{n-1}$

$= -2(-2)^{n-1}$

(ii) $1, \dfrac{1}{2}, \dfrac{1}{4}, \ldots. \quad \Rightarrow r = \dfrac{\frac{1}{2}}{1} = \dfrac{1}{2}$

$\therefore T_n = a \,.\, r^{n-1}$

$= 1 \,.\, \left(\dfrac{1}{2}\right)^{n-1} \quad \left[= \dfrac{1}{2^{n-1}} = 2^{1-n}\right]$

(iii) $2, -6, 18, \ldots \quad \Rightarrow r = \dfrac{-6}{2} = -3$

$\therefore T_n = a \,.\, r^{n-1}$

$T_n = 2(-3)^{n-1}$

5. (i) Sequence : 12, 20, 28

1^{st} difference: 8, 8 $\quad \Rightarrow T_n = an + b$

$T_n = 8n + b$

$\therefore T_n = 8n + b$

Since $T_1 = 8(1) + b = 12$,

$b = 4$.

$\therefore T_n = 8n + 4$.

[Note: We observe an arithmetic sequence since there is a common difference $\Rightarrow T_n = a + (n-1)d$ where $a = 12$ and $d = 8$. $\quad \therefore T_n = 12 + (n-1)8$

$= 8n + 4$]

(ii) $T_n = 8n + 4 = 2006$

$8n = 2002$

$$n = \frac{2002}{8} = 250.25.$$

$\therefore$ The maximum number of complete cubes $= 250$.

6. (i) $T_2 = 21, \quad T_3 = -63$

$$r = \frac{T_{n+1}}{T_n} = \frac{T_3}{T_2} = \frac{-63}{21} = -3$$

(ii) $T_n = a \,.\, r^{n-1}$

$T_2 = a \,.\, (-3)^{2-1} = 21$

$\Rightarrow \quad -3a = 21$

$a = -7$.

$T_n = a \,.\, r^{n-1}$
$T_1 = (-7)\,.\,(-3)^{1-1}$
$= (-7)\,.\,(-3)^0$
$= (-7)\,.\,(1)$
$T_1 = -7$

7. €2000 is invested at 2.5% compound interest.
$\Rightarrow$ After 1 year, amount on deposit, $A = 2000 + 2000(0.025)$
$= 2000(1 + 0.025)^1$
$= 2000(1.025)^1$
After 2 years, $A = 2000(1.025) + 2000(1.025)(0.025)$
$= 2000(1.025)[1 + 0.025]$
$= 2000(1.025)^2$
$\therefore$ After 5 years, $A = 2000(1.025)^5$.

8. $1 + 2 + 3 + 4 + \ldots\ldots 200.$
$\Rightarrow\; a = 1, d = 1, n = 200$

$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{200} = \frac{200}{2}(2(1) + (200-1)1)$

$= 100(201) = 20{,}100$

9. $T_5 = 2T_2$
$T_5 - T_2 = 9$

$T_n = a + (n-1)d$
$\Rightarrow T_2 = a + (2-1)d = a + d$
$T_5 = a + (5-1)d = a + 4d$

$T_5 = 2T_2 \Rightarrow a + 4d = 2(a + d)$
$\Rightarrow a - 2d = 0 \ldots\ldots A$
$T_5 - T_2 = 9 \Rightarrow a + 4d - (a + d) = 9$
$\Rightarrow \quad 3d = 9$
$d = 3$
$\Rightarrow \quad A: a - 2(3) = 0$
$a = 6$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$\Rightarrow S_{10} = \frac{10}{2}(2(6) + (10-1)3) = 195$

10. $\sum_{r=3}^{16}(2r + 1) = 2(3) + 1, 2(4) + 1, 2(5) + 1 \ldots\ldots 2(16) + 1$

$= 7, 9, 11, \ldots\ldots 33$
To find the number of terms n; $\quad r = 3, 4, 5, \ldots\ldots 16$
$\Rightarrow n = 14\,[= 16 - 3 + 1]$

or $T_n = a + (n-1)d$
$\Rightarrow 7 + (n-1)2 = 33$
$\Rightarrow 2n + 5 = 33$
$n = 14$

$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{14}{2}(2(7) + (14-1)2)$

$= 7(14 + 26) = 280$

Revision Exercise 9 (Advanced)

1. (i) $a = 2000$ lumens

$r = \frac{3}{5}$

$T_n = a \cdot r^n$

$\Rightarrow T_{10} = 2000\left(\frac{3}{5}\right)^{10}$

$= 20$ lumens

(ii) $T_n = 2000(0.6)^n$

(iii) $\frac{1}{10}^{\text{th}}$ of original value $= 200$

$\therefore 2000(0.6)^n = 200$

$(0.6)^n = 0.1$

$\Rightarrow n\log(0.6) = \log(0.1)$

$n = \left(\frac{\log 0.1}{\log 0.6}\right) = 4.51$

$\Rightarrow$ After the 5^{th} mirror.

2. $A = P(1 + i)^t$

(i) Original investment $= €P$

$\Rightarrow$ double value $= €2P$

$\therefore 2\cancel{P} = \cancel{P}(1 + i)^t$

$\therefore 2 = (1 + i)^t$

$\Rightarrow t$ for doubling depends only on i.

$\therefore \log 2 = \log(1 + i)^t$

$\therefore \log 2 = t\log(1 + i)$

$t = \frac{\log 2}{\log(1 + i)}$

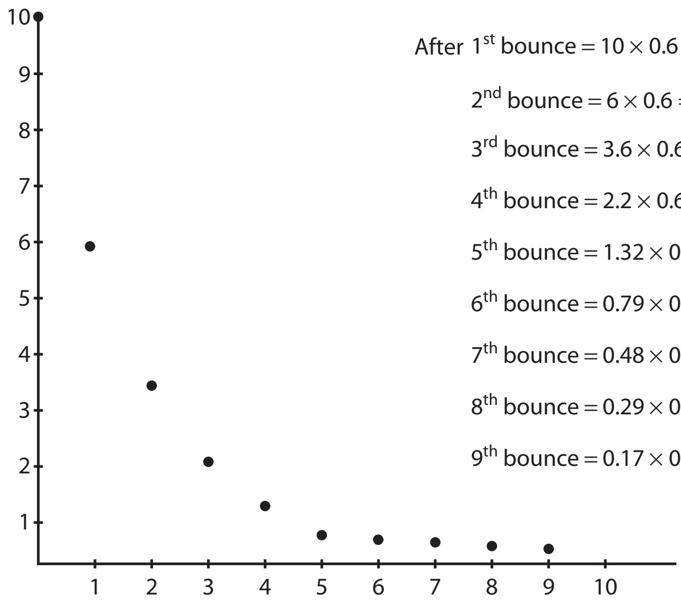
(ii) (a) $i = 2\% \Rightarrow t = \frac{\log 2}{\log(1 + 0.02)} = 35$ years

(b) $i = 5\% \Rightarrow t = \frac{\log 2}{\log(1 + 0.05)} = 14.2$ years

(c) $i = 10\% \Rightarrow t = \frac{\log 2}{\log(1 + 0.1)} = 7.3$ years

3. (i) $10 + 6 + 6 + 3.6 + 3.6 + 2.2 + 2.2 + \ldots$

$10 \rightarrow 6 \Rightarrow r = \frac{6}{10} = 0.6$



After 1^{st} bounce $= 10 \times 0.6 = 6$

2^{nd} bounce $= 6 \times 0.6 = 3.6$

3^{rd} bounce $= 3.6 \times 0.6 = 2.2$

4^{th} bounce $= 2.2 \times 0.6 = 1.32$

5^{th} bounce $= 1.32 \times 0.6 = 0.79$

6^{th} bounce $= 0.79 \times 0.6 = 0.48$

7^{th} bounce $= 0.48 \times 0.6 = 0.29$

8^{th} bounce $= 0.29 \times 0.6 = 0.17$

9^{th} bounce $= 0.17 \times 0.6 = 0.1$

(ii) $10 + 2(6 + 3.6 + 2.2 + \ldots)$

An infinite geometric series.

(iii) Sequence: 6, 3.6, 2.2, ...

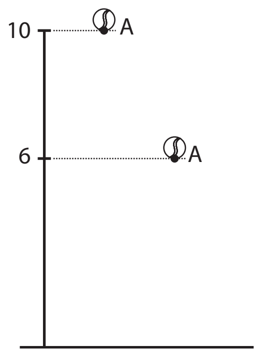
$\Rightarrow a = 6, \quad r = \dfrac{3.6}{6} = 0.6$

$S_\infty = \dfrac{a}{1-r} = \dfrac{6}{1-0.6} = 15$

$\Rightarrow$ Distance travelled $= 10 + 2(15)$

$= 40$m.

(iv) Considering a point A on the bottom of the ball as the point to measure to, then the size of the ball has no effect on the answer.



4. (i) Sequence: 3, 6, 12, 24, 48

$\Rightarrow a = 3, \; r = \dfrac{6}{3} = \dfrac{12}{6} = 2.$

$\therefore T_n = a \,.\, r^{n-1}$

$= 3 \,.\, 2^{n-1}$

(ii) $T_n = 3 \,.\, 2^{n-1} > 100\,000$

$\Rightarrow 2^{n-1} > \dfrac{100\,000}{3}$

$\therefore (n-1)\log 2 > \log 333\,333.3$

$n - 1 > \dfrac{\log 333\,333.3}{\log 2}$

$n > 1 + 18.35$

$n > 19.35$

$\Rightarrow$ The 20$^{\text{th}}$ term will exceed 1 000 000.

5. Sequence: 1, 2, 4, 8, 16 .....

$\Rightarrow a = 1, \; r = \dfrac{2}{1} = \dfrac{4}{2} = 2.$

$\therefore T_n = a \cdot r^{n-1}$

(i) $T_{32} = 1 \,.\, 2^{32-1} = 2^{31} = 2\,147\,483\,648$ cent

$= €21\,474\,836$

(ii) $T_{64} = 1 \,.\, 2^{64-1} = 2^{63} = 9.22 \times 10^{18}$ cent

$= €9.22 \times 10^{16}$

6. Let sequence be $a - d, a, a + d$

$\Rightarrow (a-d) + a + (a+d) = 33$

$\Rightarrow \quad 3a = 33$

$a = 11$

$(a-d)(a)(a+d) = 935$

$\Rightarrow (11-d)(11)(11+d) = 935$

$\Rightarrow (11-d)(11+d) = 8$

$\Rightarrow \quad 121 - d^2 = 8$

$d^2 = 36$

$d = \pm 6.$

When $d = +6, \quad \Rightarrow \quad (11-6), 11, (11+6) = 5, 11, 17$

When $d = -6, \quad \Rightarrow \quad (11+6), 11, (11-6) = 17, 11, 5$

7. (i) Value of car $= €30\,000$

After year 1 $= €30\,000 - 30\,000(0.13)$ $\quad [13\% = 0.13]$

$= 30\,000(1 - 0.13)$

After year 2 $= 30\,000(1 - 0.13) - 30\,000(1 - 0.13)(0.13)$

$= 30\,000(1 - 0.13)[1 - 0.13]$

$= 30\,000(1 - 0.13)^2$

After year "a" $= 30\,000(1 - 0.13)^a$

(ii) $a = 5$: Value $= 30\,000(1 - 0.13)^5$

$= €14\,953$

(iii) $€30\,000(1 - 0.13)^a < €6000$

$$(0.87)^a < \frac{\cancel{6\,000}^{\,1}}{\cancel{30\,000}_{\,5}} = 0.2$$

$$a\log(0.87) < \log 0.2$$

$$a < \frac{\log 0.2}{\log 0.87}$$

$$a < 11.56$$

$\therefore$ During the 12th year.

8. $T_n = 3\left(\frac{2}{3}\right)^n - 1$

(i) $T_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 1$

$T_2 = 3\left(\frac{2}{3}\right)^2 - 1 = \frac{1}{3}$

$T_3 = 3\left(\frac{2}{3}\right)^3 - 1 = -\frac{1}{9}$

(ii) $T_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1$

$= \cancel{3}\left(\frac{2}{3}\right)^n \cdot \frac{2}{\cancel{3}} - 1$

$= 2\,.\left(\frac{2}{3}\right)^n - 1$

(iii) $3T_{n+1} = 3\left(2\left(\frac{2}{3}\right)^n - 1\right)$

$2T_n = 2\left(3\left(\frac{2}{3}\right)^n - 1\right)$

$\therefore 3T_{n+1} - 2T_n = 3\left(2\left(\frac{2}{3}\right)^n - 1\right) - 2\left(3\left(\frac{2}{3}\right)^n - 1\right)$

$= \cancel{6\left(\frac{2}{3}\right)^n} - 3 - \cancel{6\left(\frac{2}{3}\right)^n} + 2$

$= -1$

$\Rightarrow \quad k = -1$

(iv) $\sum_{n=1}^{15}\left[3\left(\frac{2}{3}\right)^n - 1\right] = \sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - 15$

Sequence $= 3\left(\frac{2}{3}\right)^1, 3\left(\frac{2}{3}\right)^2, 3\left(\frac{2}{3}\right)^3, \,.....$

$\Rightarrow a = 3\left(\frac{2}{3}\right), r = \dfrac{3\cdot\left(\frac{2}{3}\right)^2}{3\cdot\left(\frac{2}{3}\right)} = \left(\frac{2}{3}\right)$

$= 2$

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{2\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}} = 6\left(1-\left(\frac{2}{3}\right)^{15}\right)$$

$$= 5.98629$$

$$\therefore \sum_{n=1}^{15}\left[3\left(\frac{2}{3}\right)^n - 1\right] = 5.98629 - 15$$

$$= -9.014$$

9. (i) $S_n = T_1 + T_2 + T_3 + \dots T_{n-1} + T_n$

$\underline{S_{n-1} = T_1 + T_2 + T_3 + \dots T_{n-1}}$

$\therefore S_n - S_{n-1} = T_n$

(ii) $S_n = 3n^2 + n$

$\Rightarrow S_{n-1} = 3(n-1)^2 + (n-1)$

$$\therefore T_n = S_n - S_{n-1} = 3n^2 + n - [3(n-1)^2 + (n-1)]$$
$$= 3n^2 + n - [3(n^2 - 2n + 1) + n - 1]$$
$$= \cancel{3n^2} + \cancel{n} - \cancel{3n^2} + 6n - 3 - \cancel{n} + 1$$
$$= 6n - 2$$

(iii) $$\sum_{r=1}^{n}(T_r)^2 = T_1^2 + T_2^2 + T_3^2 + \dots + T_n^2$$
$$= (6-2)^2 + (6(2)-2)^2 + (6(3)-2)^2 + \dots + (6n-2)^2$$
$$= \sum(6n-2)^2$$
$$= \sum 36n^2 - 24n + 4$$
$$= 36\sum_{n=1}^{n} n^2 - 24\sum_{n=1}^{n} n + \sum_{n=1}^{n} 4$$

but $\sum n = \frac{n}{2}(n+1)$, $\qquad \sum_{n=1}^{n} 4 = 4n.$

and $\sum n^2 = \frac{n}{6}(2n+1)(n+1)$

$$\sum(T_r)^2 = 36\left[\frac{n}{6}(2n+1)(n+1)\right] - 24\left[\frac{n}{2}(n+1)\right] + 4n$$
$$= n(n+1)[12n + 6 - 12] + 4n$$
$$= n(n+1)(12n-6) + 4n$$
$$= n[(n+1)(12n-6) + 4]$$
$$= n[12n^2 + 6n - 2]$$
$$= 2n[6n^2 + 3n - 1]$$

10. $\log_4 x = \dfrac{\log_2 x}{\log_2 4}$

$$\left(\begin{array}{l} \text{let } y = \log_2 4 \\ \Rightarrow 2^y = 4 = 2^2 \\ \Rightarrow y = 2 \end{array}\right)$$

$\therefore \log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{\log_2 x}{2}$

Also, $\qquad \log_{16} x = \dfrac{\log_2 x}{\log_2 16}$

$$\left(\begin{array}{l} \text{let } y = \log_2 16 \\ \Rightarrow 2^y = 16 = 2^4 \\ \Rightarrow y = 4 \end{array}\right)$$

$$\therefore \log_{16} x = \frac{\log_2 x}{\log_2 16} = \frac{\log_2 x}{4}$$

$$\therefore \log_2 x, \log_4 x, \log_{10} x = \log_2 x, \frac{\log_2 x}{2}, \frac{\log_2 x}{4}$$

$$\Rightarrow \frac{\frac{\log_2 x}{2}}{\log_2 x} = \frac{\frac{\log_2 x}{4}}{\frac{\log_2 x}{2}} = \frac{1}{2}$$

$$\therefore r = \frac{1}{2}.$$

$$S_\infty = \frac{a}{1-r} = \frac{\log_2 x}{1 - \frac{1}{2}} = 2 \log_2 x$$

$$= k \log_2 x$$

$$\Rightarrow \quad k = 2.$$

Revision Exercise 9 (Extended-Response Questions)

1. $T_n = an^3 + bn^2 + cn + d.$

(i) $\Rightarrow$

$$\begin{array}{ccccc} T_1 & T_2 & T_3 & T_4 & T_5 \\ a + b + c + d, & 8a + 4b + 2c + d, & 27a + 9b + 3c + d, & 64a + 16b + 4c + d, & 125a + 25b + 5c + d \end{array}$$

1st difference: $7a + 3b + c, 19a + 5b + c, 37a + 7b + c, 61a + 9b + c$

2nd difference: $12a + 2b, 18a + 2b, 24a + 2b$

3rd difference: $6a, 6a$

(ii) The 3rd difference for all cubic sequences is always $6a$.

(iii) (a) The 2nd difference for all quadratic sequences is always $2a$.

(b) The 1st difference ($T_2 - T_1$) for all quadratic sequences is always $3a + b$.

[Note: $T_n = an^2 + bn + c$

$$\begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \\ a + b + c, & 4a + 2b + c, & 9a + 3b + c, & 16a + 4b + c \end{array}$$

1st difference: $3a + b, \quad 5a + b, \quad 7a + b$

2nd difference: $2a, \quad 2a$]

(iv)

$$\begin{array}{ccccccc} T_1 & & T_2 & & T_3 & & T_4 \\ 5, & & 12, & & 25, & & 44 \\ & 7, & & 13, & & 19 & \text{1st difference} \\ & & 6, & & 6 & & \text{2nd difference} \end{array}$$

(v) $\therefore T_n = an^2 + bn + c$

where $2a = 6$

$a = 3$

$\therefore T_n = 3n^2 + bn + c$

Also, $3a + b = 7$

$\therefore \quad 9 + b = 7 \quad \Rightarrow b = -2$

$\therefore \quad T_n = 3n^2 - 2n + c$

Since $\quad T_1 = 3(1)^2 - 2(1) + c = 5$

$\Rightarrow \quad c = 4$

$\therefore \quad T_n = 3n^2 - 2n + 4.$

(vi) $\therefore \quad T_{20} = 3(20)^2 - 2(20) + 4.$

$= 1164$

2. (i) Initial height $= 40\,\text{m}$;

After 1 bounce $= 40 \,.\, r^1$

After 2 bounces $= 40 \,.\, r^2$

After n bounces $= 40 \,.\, r^n$, Where r is a fraction representing how far the ball bounces back up.

(ii) $T_n = 40r^n$

$T_{10} = 40r^{10} = 1\text{m}$

$\Rightarrow \quad r^{10} = \dfrac{1}{40}$

$\Rightarrow \quad 10 \log r = \log \dfrac{1}{40}$

$\log r = \dfrac{\log \frac{1}{40}}{10} = -0.1602$

$r = 10^{-0.1602} = 0.69 = 69\%$

(iii) $T_1 = 40(0.69)^1 = 27.6\text{m}$

$T_2 = 40(0.69)^2 = 19.044\text{m}$

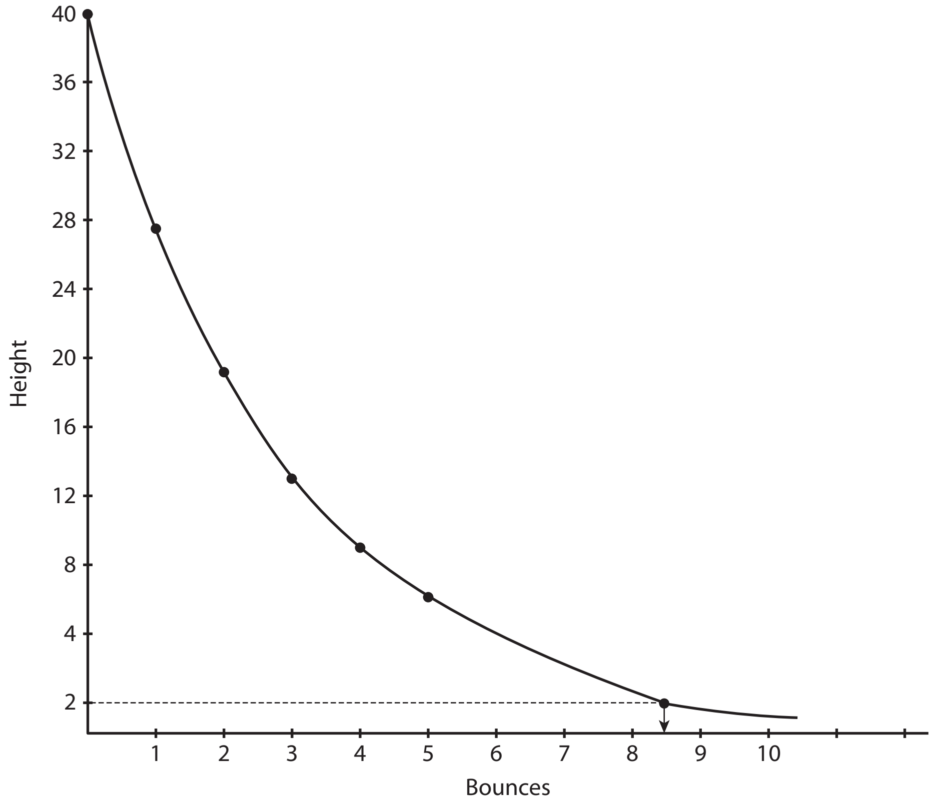
$T_3 = 40(0.69)^3 = 13.14\text{m}$

$T_4 = 40(0.69)^4 = 9.067\text{m}$

$T_5 = 40(0.69)^5 = 6.256\text{m}.$

Bounce	1st	2nd	3rd	4th	5th
Height	27.6	19.04	13.14	9.07	6.26

(iv)



(v) At least 9 bounces

(vi) $T_n = 40(0.69)^n < 2$

$(0.69)^n < \dfrac{2}{40} = 0.05$

$n \log 0.69 < \log 0.05$

$n < \dfrac{\log 0.05}{\log 0.69} = 8.07$

$\therefore$ 9 bounces are needed.

(vii) If conditions stayed exactly the same as on the first bounce, then the ball would continue to bounce for ever. But in a real-life situation, the conditions do not same the same; the ball loses energy. The ball may heat up and transfer energy to the ground. Also, sound energy may be dissipated.

3. (i) Scheme 1: €20, €22, €24, €26, ...

$$\Rightarrow a = €20, d = 2.$$
$$T_n = a + (n-1)d$$
$$= 20 + (n-1)2$$
$$= 2n + 18 = \text{amount of money in week } n.$$
$$S_n = \frac{n}{2}(2a + (n-1)\,d)$$
$$= \frac{n}{2}(40 + (n-1)\,2)$$
$$= \frac{n}{2}(2n + 38)$$
$$= n(n+19) = \text{Total amount of money after week } n.$$

Scheme 2: $€20, €20\left(\frac{21}{20}\right), €20\left(\frac{21}{20}\right)^2, €20\left(\frac{21}{20}\right)^3, ...$

$$\Rightarrow a = €20, r = \frac{21}{20}.$$
$$T_n = ar^{n-1}$$
$$= 20\left(\frac{21}{20}\right)^{n-1} = \text{amount of money in week } n.$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$= 20\left(\frac{1-\left(\frac{21}{20}\right)^n}{1-\frac{21}{20}}\right)$$
$$= -400\left(1-\left(\frac{21}{20}\right)^n\right)$$
$$S_n = 400\left(\left(\frac{21}{20}\right)^n - 1\right)$$

(ii) Scheme 1: $S_{36} = 36(36+19) = €1980$

Scheme 2: $S_{36} = 400\left(\left(\frac{21}{20}\right)^{36} - 1\right) = €1916.73$

$\Rightarrow$ Scheme 1 is better.

(iii) Assuming scheme 1: $€n(n+19)$ = total amount after n weeks

€7.50 spent per week $\Rightarrow$ €$7.5n$ spent after n weeks.

$\therefore n(n+19) - 7.5n$ is saved after n weeks.

$\therefore n^2 + 19n - 7.5n = €400$

$\therefore n^2 + 11.5n - 400 = 0.$

$a = 1,\ b = 11.5,\ c = -400$

$$\Rightarrow n = \frac{-11.5 \pm \sqrt{(11.5)^2 - 4(1)(-400)}}{2}$$
$$= \frac{30.12}{2} \text{ or } \frac{-53.12}{2}$$
$$= 15.06 \text{ or } -26.5$$

$\therefore$ Ronan needs to save for 15 weeks and can buy the console in the 16th week.

4. (i) $160l$ at the start.

15% lost = 85% left at the end of the year.

$\Rightarrow 160 \times 85\% = 136l$ left.

(ii) $2010 \rightarrow 2020 = 10$ years (end of year)

end of year 1 $= 160(0.85)^1$

end of year 2 $= 160(0.85)^2$

$\Rightarrow$ end of year 10 $= 160(0.85)^{10}$

$= 31.499$

$= 31.5l$

(iii) Last barrel left for 1 year $= 160(0.85) = 136l$ left

2nd last barrel left for 2 years $= 160(0.85)(0.85)$

$= 136(0.85)^2 = 115.6$ left

$\Rightarrow$ Total amount of liquid after 20 years

$= 160(0.85) + 160(0.85)^2 + 160(0.85)^3 + \ldots.\ 160(0.85)^{20}$

$\Rightarrow a = 160(0.85) = 136$

$r = 0.85$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{136(1-(0.85)^{20})}{1-0.85}$$

$= 871.52l$

$= 872l$

5. (i) Cost = €15 000

Depreciation = 20% $\Rightarrow$ 80% of value left at the end of each year

$2005 \rightarrow 2007 = 2$ years.

$\Rightarrow$ Value = €15 000$(0.8)^2$

= €9,600

(ii) €15 000 (0.8) < €500

$$\Rightarrow (0.8)^n < \frac{\cancel{500}}{\cancel{15000}} = \frac{1}{30}$$

$$\Rightarrow n\log(0.8) < \log\left(\frac{1}{30}\right)$$

$$n < \frac{\log \frac{1}{30}}{\log 0.8}$$

$n < 15.242$ years

$\Rightarrow 2005 + 15 = 2020$

$\Rightarrow$ In year 2020.

(iii) Deposit = €1000, $r = 5\% = 0.05$

$\Rightarrow$ Savings $= 1000\,(1.05) + 1000\,(1.05)^2 + 1000\,(1.05)^3 + \ldots..$

$\ldots\ 1000\,(1.05)^{15}$

$\Rightarrow a = 1000\,(1.05),\ r = 1.05.$

$$\therefore \text{Savings} = €1000(1.05)\frac{(1-(1.05)^{15})}{(1-1.05)}$$

= €22 657.49

$\simeq$ €22 657

(iv) Amount saved $= €22\,657$

Let inflation $= r\% = 0.0r$

$\Rightarrow$ Cost of new machine after 15 years $= 15\,000\,(1.05)^{15}$

$$\Rightarrow \quad 15\,000\,(1.0r)^{15} = €22\,657$$

$$(1.0r)^{15} = \frac{22\,657}{15\,000}$$

$$(1.0r)^{15} = 1.51$$

$$(1.0r) = (1.51)^{\frac{1}{15}}$$

$$1.0r = 1.028$$

$$\Rightarrow 0.0r = 0.028$$

$$r\% = 2.8\%$$

Chapter 10

Exercise 10.1

1. (i) Numerical (ii) Categorical (iii) Numerical (iv) Categorical

2. (i) Discrete
(ii) Discrete
(iii) Continuous
(iv) Discrete
(v) Continuous
(vi) Discrete
(vii) Discrete
(viii) Discrete

3. (i) Categorical
(ii) Numerical
(iii) Numerical
Part (ii) is discrete

4. Race time is continuous
Number on bib is discrete

5. (i) No (ii) Yes (iii) Yes (iv) No

6. (i) Contains two pieces of information
(ii) Number of eggs
(iii) Amount of flour

7. (i) Categorical
(ii) Numerical
(iii) Numerical
(iv) Categorical
Part (iii) is discrete
Part (ii) is bivariate continuous numerical

8. (i) True
(ii) True
(iii) False
(iv) False
(v) True
(vi) True
(vii) True
(viii) True

9. Small, medium, large;
1-bedroom house, 2-bedroom house, 3-bedroom house;
Poor, fair, good, very good

10. (i) Primary (ii) Secondary (iii) Primary (iv) Secondary

11. (i) Secondary
(ii) Roy's data; It is more recent.

12. (i) Number of bedrooms in family home and the number of children in the family
(ii) An athlete's height and his distance in a long-jump competition.

Exercise 10.2

1. (i) Too personal (it identifies respondent)
(ii) Too vague/subjective

2. (i) Too personal
(ii) Too leading
(iii) (a) Overlapping
(b) "Roughly how many times per annum do you visit your doctor?"

3. **Q A**: Judgmental and subjective
   Q B: Leading and biased

4. Not suitable; too vague, not specific enough

5. Where did you go on holidays last year?
   ☐ Ireland
   ☐ Europe, excluding Ireland
   ☐ Rest of the world

   What type of accommodation did you use?
   ☐ Self-catering
   ☐ Guesthouses/Hotels
   ☐ Camping

6. B and D are biased:
   B gives an opinion: D is a leading question.

7. Do you have a part-time job?
   Are you male or female?

8. Explanatory variable: Length of legs.
   Response variable: Time recorded in sprint race.

9. (i) Explanatory variable: Number of operating theatres.
   (ii) Response variable: Number of operations per day.

10. (i) Group B
    (ii) Explanatory variable: The new drug.
    (iii) Response variable: Blood pressure.
    (iv) (a) a designed experiment = carry out some controlled activity and record the results.

Exercise 10.3

1. Census — all members of the population surveyed.
   Sample — only part of the population surveyed.

2. Any sample of size *n* which has an equal chance of being selected.

3. (i) Likely biased (ii) Random (iii) Random (iv) Random (v) Random

4. Selecting a sample in the easiest way
   (i) Convenience sample.
   (ii) (a) High level of bias likely.
   (b) Unrepresentative of the population.

5. (i) Convenience sampling (ii) Systematic sampling (iii) Stratified sampling

6. (i) Very small sample; not random and therefore not representative
   (ii) Each member of the local population should have an equal chance of being asked. The sample should not be too small. The sample should be stratified to ensure all age and class groups are represented.

7. (i) Convenience sampling.
   (ii) Her street may not be representative of the whole population.
   (iii) Systematic random sampling from a directory <u>or</u> cluster sampling of travel agent's clients, ie. pick one travel agent at random and survey them about all their clients.

8. (i) Assign a number to each student and then use a random number generator to pick n numbers.

(ii) (a) $\frac{230}{1000} \times 100 = 23$ students

(b) $\frac{80}{1000} \times 100 = 8$ boys

9. (i) Quota sampling.

(ii) Advantage: Convenient as no sampling frame required.
Disadvantage: Left to the discretion of the interviewer so possible bias.

10. (i) Cost and time, without a great loss in accuracy.

(ii) Sampling frame: a list of all the items that could be included in the survey.

11. (i) Junior Cycle: $\frac{460}{880} \times 100 = 52.27 = 52$ pupils

Senior Cycle: $\frac{420}{880} \times 100 = 47.72 = 48$ pupils

(ii) Stratified sampling is better if there are different identifiable groups with different views in the population.

12. (i) Cluster sampling (ii) Convenience sampling (iii) Systematic sampling

Exercise 10.4

1. (a) 2, 2, 5, 5, 7, 8, 8, 8, 11

$\Rightarrow$(i) Mode = 8 (ii) Median = 7

(b) 3, 3, 5, 7, 7, 7, 8, 8, 9, 11, 12

$\Rightarrow$ (i) Mode = 7 (ii) Median = 7

2. 31, 34, 36, 37, 41, 41, 42, 42, 42, 43, 45

(i) Median speed = 41 km/hr

(ii) Mean speed $= \frac{31 + 34 + 36 + 37 + 41 + 41 + 42 + 42 + 42 + 43 + 45}{11}$

$= \frac{434}{11} = 39.45$ km/hr

3. 7, 11, 12, 14, 14, 14, 18, 22, 22, 36

(i) Mode = 14 points

(ii) Median $= \frac{14 + 14}{2} = 14$ points

(iii) Mean $= \frac{7 + 11 + 12 + 14 + 14 + 14 + 18 + 22 + 22 + 36}{10}$

$= \frac{170}{10} = 17$ points

4. The four numbers are 21, 25, 16 and x.

$\Rightarrow \frac{21 + 25 + 16 + x}{4} = 19$

$\Rightarrow 62 + x = 76$

$\Rightarrow x = 76 - 62 = 14$, the fourth number.

5. Results for six tests were: 8, 4, 5, 3, x and y.

Modal mark $= 4 \Rightarrow x = 4$

Mean $= 5 \Rightarrow \dfrac{8 + 4 + 5 + 3 + 4 + y}{6} = 5$

$\Rightarrow \quad 24 + y = 30$

$\Rightarrow \quad y = 30 - 24 = 6$

6. Numbers: 9, 11, 11, 15, 17, 18, 100

(i) Mean $= \dfrac{9 + 11 + 11 + 15 + 17 + 18 + 100}{7}$

$= \dfrac{181}{7} = 25.877$

(ii) Median $= 15$

$\Rightarrow$ Median is the best

7. Numbers: 103, 35, x, x, x.

Mean $= 39 \Rightarrow \dfrac{103 + 35 + x + x + x}{5} = 39$

$\Rightarrow 138 + 3x = 195$

(i) Total of the five numbers $= 195$

(ii) $3x = 195 - 138$

$\Rightarrow 3x = 57$

$\Rightarrow \quad x = \dfrac{57}{3} = 19$

8. Mean for 12 children $= 76\%$

$\Rightarrow$ Total for 12 children $= 76\% \times 12 = 912\%$

Mean for 8 children $= 84\%$

$\Rightarrow$ Total for 8 children $= 84\% \times 8 = 672\%$

$\Rightarrow$ Total for 20 children $= 912\% + 672\% = 1584\%$

$\Rightarrow$ Overall mean $= \dfrac{1584\%}{20} = 79.2\%$

9. Median, since 50% of the marks will be above the median mark.

10. (i) Mean for 20 boys $= 17.4$

$\Rightarrow$ Total for 20 boys $= 17.4 \times 20$

$= 348$ marks

Mean for 10 girls $= 13.8$

$\Rightarrow$ Total for 10 girls $= 13.8 \times 10$

$= 138$ marks

Total for 30 students $= 348 + 138$

$= 486$

$\Rightarrow$ Mean for whole class $= \dfrac{486}{30}$

$= 16.2$

(ii) Median for 12, 18, 20, 25 and $x = 20$

Mean $= \dfrac{12 + 18 + 20 + 25 + x}{5} = 22$ [i.e. 20 (the median) + 2]

$\Rightarrow \quad 75 + x = 110$

$\Rightarrow \quad x = 110 - 75 = 35$

11.

x = Marks	3	4	5	6	7	8	9	
f = No. of students	3	2	6	10	0	3	1	= 25
$f \cdot x$ =	9	8	30	60	0	24	9	= 140

(i) Number of students = 3 + 2 + 6 + 10 + 0 + 3 + 1 = 25

(ii) Mode = 6 marks

(iii) Mean $= \frac{\Sigma fx}{\Sigma f} = \frac{140}{25} = 5.6$ marks

(iv) 10 + 0 + 3 + 1 = 14 students

(v) 25 students $\Rightarrow$ median is the mark of 13th student = 6 marks

12.

x = No. in family	2	3	4	5	6	7	8	
f = frequency	2	4	6	5	2	0	1	= 20
$f \cdot x$ =	4	12	24	25	12	0	8	= 85

(i) Mode = 4 people

(ii) Median $= \frac{4+4}{2} = 4$ people

(iii) Mean $= \frac{\Sigma fx}{\Sigma f} = \frac{85}{20} = 4.25$

13.

Age	10–20	20–30	30–40	40–50	
f = No. of people	4	15	11	10	= 40
x = mid-interval	15	25	35	45	
$f \cdot x$	60	375	385	450	= 1270

(i) Mean $= \frac{\Sigma fx}{\Sigma f} = \frac{1270}{40} = 31.75 = 32$ years

(ii) (30–40) years

14. (i) Mean $= \frac{\Sigma x}{N} = \frac{256.2}{6} = 42.7$

(ii) Mean will increase

15. (i) (a) Mode = B

(b) Median = C

(ii) Categorical data is not numerical

16.

Rainfall (mm)	0	1	2	3	3	26	3	2	3	0
Sunshine (hours)	70	15	10	15	18	0	15	21	21	80
Rainfall (mm)	0	0	1	2	2	3	3	3	3	26
Sunshine (hours)	0	10	15	15	15	18	21	21	70	80

(i) Mean $= \frac{0+0+1+2+2+3+3+3+3+26}{10} = \frac{43}{10} = 4.3$ mm of rainfall

(ii) Mean $= \frac{0+10+15+15+15+18+21+21+70+80}{10}$

$= \frac{265}{10} = 26.5$ hours of sunshine

(iii) Rainfall mode = 3 mm

Sunshine mode = 15 hours

(iv) Rainfall median $= \frac{2+3}{2} = 2.5$ mm

Sunshine median $= \frac{15+18}{2} = 16.5$ hours

(v) Median rainfall and mean sunshine
(least rainfall and highest sunshine).

17. Dice was thrown 50 times, mean score = 3.42.
$\Rightarrow$ Total scores $= 50 \times 3.42 = 171$

outcomes	1	2
frequency	9	12

outcomes	1	2
frequency	12	9

scores $= 9 + 24 = 33$ scores $= 12 + 18 = 30$
Hence, there is an increase of 3 (or a decrease of 3) in the total scores when the frequencies had been swapped.

Mean $= \frac{171+3}{50} = \frac{174}{50} = 3.48$

or Mean $= \frac{171-3}{50} = \frac{168}{50} = 3.36$

Exercise 10.5

1. (i) Range $= 10 - 2 = 8$
(ii) Range $= 73 - 16 = 57$

2. Marks in order: 4, 10, 27, 27, 29, 34, 34, 34, 37
(i) Range $= 37 - 4 = 33$
(ii) Median $= 29$
(iii) (a) Lower quartile $= 27$
(b) Upper quartile $= 34$
(c) Interquartile range $= 34 - 27 = 7$

3. Times in order: 6, 7, 8, 9, 9, 9, 11, 12, 15, 16, 19
(i) Range $= 19 - 6 = 13$
(ii) Lower quartile $= 8$
(iii) Upper quartile $= 15$

(iv) Interquartile range $= 15 - 8 = 7$ 4. Marks in order: 12, 13, 14, 14, 14, 14, 14, 15, 15, 16, 16, 17
(i) Range $= 17 - 12 = 5$ marks
(ii) Mean $= \frac{12 + 13 + 5(14) + 2(15) + 2(16) + 17}{12} = \frac{174}{12} = 14.5$ marks
(iii) On average, the girls didn't do as well as the boys. The girls' marks were more dispersed.

5. Scores in order: 41, 50, 50, 51, 53, 59, 64, 65, 66
(i) Range $= 66 - 41 = 25$
(ii) Lower quartile $= 50$
(iii) Upper quartile $= 65$
(iv) Interquartile range $= 65 - 50 = 15$

6. Results in order: 2.2, 2.2, 2.3, 2.3, 2.5, 2.7, 3.1, 3.2, 3.6, 3.7, 3.7, 3.8, 3.8, 3.8, 3.8, 3.9, 3.9, 3.9, 4.0, 4.0, 4.0, 4.0, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 5.1, 5.5
(i) Lower Quartile, $Q_1 = 3.2$
Upper Quartile, $Q_3 = 4.0$
Interquartile range $= 4.0 - 3.2 = 0.8$

(ii) $(1.5) \times (0.8) = 1.2$

$\Rightarrow$ Outlier $= 5.5$ as it is more than $1\frac{1}{2}$ times the interquartile range above Q_3.

7. (i) Mean $= \mu = \dfrac{\Sigma x}{n} = \dfrac{1 + 3 + 7 + 9 + 10}{5} = \dfrac{30}{5} = 6$

$$\sigma = \sqrt{\frac{\Sigma(x-\mu)^2}{n}} = \sqrt{\frac{(1-6)^2 + (3-6)^2 + (7-6)^2 + (9-6)^2 + (10-6)^2}{5}}$$

$$= \sqrt{\frac{(-5)^2 + (-3)^2 + (1)^2 + (3)^2 + (4)^2}{5}}$$

$$= \sqrt{\frac{25 + 9 + 1 + 9 + 16}{5}}$$

$$= \sqrt{\frac{60}{5}} = \sqrt{12} = 3.464 = 3.5$$

(ii) Mean $= \mu = \dfrac{8 + 12 + 15 + 9}{4} = \dfrac{44}{4} = 11$

$$\sigma = \sqrt{\frac{(8-11)^2 + (12-11)^2 + (15-11)^2 + (9-11)^2}{4}}$$

$$= \sqrt{\frac{(-3)^2 + (1)^2 + (4)^2 + (-2)^2}{4}}$$

$$= \sqrt{\frac{9 + 1 + 16 + 4}{4}} = \sqrt{\frac{30}{4}} = \sqrt{7.5} = 2.73 = 2.7$$

(iii) Mean $= \mu = \dfrac{1 + 3 + 4 + 6 + 10 + 12}{6} = \dfrac{36}{6} = 6$

$$\sigma = \sqrt{\frac{(1-6)^2 + (3-6)^2 + (4-6)^2 + (6-6)^2 + (10-6)^2 + (12-6)^2}{6}}$$

$$= \sqrt{\frac{(-5)^2 + (-3)^2 + (-2)^2 + (0)^2 + (4)^2 + (6)^2}{6}}$$

$$= \sqrt{\frac{25 + 9 + 4 + 0 + 16 + 36}{6}} = \sqrt{\frac{90}{6}} = \sqrt{15} = 3.87 = 3.9$$

8. Mean $= \mu = \dfrac{2 + 3 + 4 + 5 + 6}{5} = \dfrac{20}{5} = 4$

$$\sigma = \sqrt{\frac{(2-4)^2 + (3-4)^2 + (4-4)^2 + (5-4)^2 + (6-4)^2}{5}}$$

$$= \sqrt{\frac{(-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2}{5}}$$

$$= \sqrt{\frac{4 + 1 + 0 + 1 + 4}{5}} = \sqrt{\frac{10}{5}} = \sqrt{2} = 1.414$$

Mean $= \mu = \dfrac{12 + 13 + 14 + 15 + 16}{5} = \dfrac{70}{5} = 14$

$$\sigma = \sqrt{\frac{(12-14)^2 + (13-14)^2 + (14-14)^2 + (15-14)^2 + (16-14)^2}{5}}$$

$$= \sqrt{\frac{(-2)^2(-1)^2 + (0)^2 + (1)^2 + (2)^2}{5}}$$

$$= \sqrt{\frac{4 + 1 + 0 + 1 + 4}{5}} = \sqrt{\frac{10}{5}} = \sqrt{2} = 1.414$$

(i) New set is $x + 10$.

(ii) Both the same.

(iii) If all the numbers are increased by the same amount, the standard deviation does not change.

9. Mean $= \mu = \dfrac{2+3+4+5+6+8+8}{7} = \dfrac{36}{7}$

$$\sigma = \sqrt{\frac{\left(2-\frac{36}{7}\right)^2+\left(3-\frac{36}{7}\right)^2+\left(4-\frac{36}{7}\right)^2+\left(5-\frac{36}{7}\right)^2+\left(6-\frac{36}{7}\right)^2+\left(8-\frac{36}{7}\right)^2+\left(8-\frac{36}{7}\right)^2}{7}}$$

$$= \sqrt{\frac{\left(\frac{-22}{7}\right)^2+\left(\frac{-15}{7}\right)^2+\left(\frac{-8}{7}\right)^2+\left(\frac{-1}{7}\right)^2+\left(\frac{6}{7}\right)^2+\left(\frac{20}{7}\right)^2+\left(\frac{20}{7}\right)^2}{7}}$$

$$= \sqrt{\frac{\frac{484}{49}+\frac{225}{49}+\frac{64}{49}+\frac{1}{49}+\frac{36}{49}+\frac{400}{49}+\frac{400}{49}}{7}}$$

$$= \sqrt{\frac{1610}{343}} = \sqrt{4.69388} = 2.166 = 2.17$$

10. (i) Route 1 mean $= \dfrac{15+15+11+17+14+12}{6} = \dfrac{84}{6} = 14$

Route 2 mean $= \dfrac{11+14+17+15+16+11}{6} = \dfrac{84}{6} = 14$

(ii) Route 1 $\Rightarrow$ $\sigma = \sqrt{\dfrac{(15-14)^2+(15-14)^2+(11-14)^2+(17-14)^2+(14-14)^2+(12-14)^2}{6}}$

$$= \sqrt{\frac{(1)^2+(1)^2+(-3)^2+(3)^2+(0)^2+(-2)^2}{6}}$$

$$= \sqrt{\frac{1+1+9+9+0+4}{6}} = \sqrt{\frac{24}{6}} = \sqrt{4} = 2$$

Route 2 $\Rightarrow$ $\sigma = \sqrt{\dfrac{(11-14)^2+(14-14)^2+(17-14)^2+(15-14)^2+(16-14)^2+(11-14)^2}{6}}$

$$= \sqrt{\frac{(-3)^2+(0)^2+(3)^2+(1)^2+(2)^2+(-3)^2}{6}}$$

$$= \sqrt{\frac{9+0+9+1+4+9}{6}}$$

$$= \sqrt{\frac{32}{6}} = \sqrt{5\tfrac{1}{3}} = 2.309 = 2.3$$

(iii) Route 1, as times are less dispersed.

11.

Variable = x	0	2	3	4	
Frequency = f	4	3	2	3	= 12
$f \cdot x$	0	6	6	12	= 24

Mean $= \mu = \dfrac{\Sigma fx}{\Sigma f} = \dfrac{24}{12} = 2$

x	f	$(x-\mu)$	$(x-\mu)^2$	$f(x-\mu)^2$
0	4	−2	4	16
2	3	0	0	0
3	2	1	1	2
4	3	2	4	12

$\Sigma f = 12$ $\qquad$ $\Sigma f(x-\mu)^2 = 30$

$$\sigma = \sqrt{\frac{\Sigma f(x-\mu)^2}{\Sigma f}} = \sqrt{\frac{30}{12}} = \sqrt{2.5} = 1.58 = 1.6$$

12. Mean $= \dfrac{(1 \times 1) + (4 \times 2) + (9 \times 3) + (6 \times 4)}{1 + 4 + 9 + 6}$

$\Rightarrow \mu = \dfrac{1 + 8 + 27 + 24}{20} = \dfrac{60}{20} = 3$

x	f	$x - \mu$	$(x - \mu)^2$	$f(x - \mu)^2$
1	1	−2	4	4
2	4	−1	1	4
3	9	0	0	0
4	6	1	1	6
	↓			↓
	$\Sigma f = 20$			$\Sigma f(x - \mu)^2 = 14$

$\sigma = \sqrt{\dfrac{\Sigma f(x - \mu)^2}{\Sigma f}} = \sqrt{\dfrac{14}{20}} = 0.83666 = 0.84$

13.

Class	Mid-interval = x	f	fx	$(x - \mu)$	$(x - \mu)^2$	$f(x - \mu)^2$
1–3	2	4	8	−3	9	36
3–5	4	3	12	−1	1	3
5–7	6	9	54	1	1	9
7–9	8	2	16	3	9	18
		18				66

Mean $= \mu = \dfrac{\Sigma fx}{\Sigma f} = \dfrac{90}{18} = 5$

$\sigma = \sqrt{\dfrac{\Sigma f(x - \mu)^2}{\Sigma f}} = \sqrt{\dfrac{66}{18}} = 1.91 = 1.9$

14.

Class	Mid-interval = x	f	fx	$x - \mu$	$(x - \mu)^2$	$f(x - \mu)^2$
0–4	2	2	4	−9	81	162
4–8	6	3	18	−5	25	75
8–12	10	9	90	−1	1	9
12–16	14	7	98	3	9	63
16–20	18	3	54	7	49	147
		24	264			456

Mean $= \dfrac{264}{24} = 11$

$\sigma = \sqrt{\dfrac{456}{24}} = \sqrt{19} = 4.358 = 4.36$

15. (i) Mean $(\overline{x}) = \dfrac{18 + 26 + 22 + 34 + 25}{5} = \dfrac{125}{5} = 25$ letters

(ii) $\sigma = \sqrt{\dfrac{(18 - 25)^2 + (26 - 25)^2 + (22 - 25)^2 + (34 - 25)^2 + (25 - 25)^2}{5}}$

$= \sqrt{\dfrac{(-7)^2 + (1)^2 + (-3)^2 + (9)^2 + (0)^2}{5}}$

$= \sqrt{\dfrac{49 + 1 + 9 + 81 + 0}{5}} = \sqrt{\dfrac{140}{5}} = \sqrt{28} = 5.29 = 5.3$

(iii) $\overline{x} + \sigma = 25 + 5.3 = 30.3$

$\overline{x} - \sigma = 25 - 5.3 = 19.7$

(iv) 3 days

16. (i) Mean $(\bar{x}) = \dfrac{1 + 9 + a + 3a - 2}{4}$

$= \dfrac{4a + 8}{4} = a + 2$

(ii) $\sigma = \sqrt{20} \quad \Rightarrow \quad \sigma = \sqrt{\dfrac{[1 - (a + 2)]^2 + [9 - (a + 2)]^2 + [a - (a + 2)]^2 + [3a - 2 - (a + 2)]^2}{4}}$

$= \sqrt{\dfrac{(-a - 1)^2 + (7 - a)^2 + (-2)^2 + (2a - 4)^2}{4}}$

$= \sqrt{\dfrac{a^2 + 2a + 1 + 49 - 14a + a^2 + 4 + 4a^2 - 16a + 16}{4}}$

$= \sqrt{\dfrac{6a^2 - 28a + 70}{4}} = \sqrt{20}$

$\Rightarrow \quad \dfrac{3a^2 - 14a + 35}{2} = 20$

$\Rightarrow \quad 3a^2 - 14a + 35 = 40$

$\Rightarrow \quad 3a^2 - 14a - 5 = 0$

$\Rightarrow \quad (3a + 1)(a - 5) = 0$

$\Rightarrow \quad a = -\dfrac{1}{3}, a = 5$

$\Rightarrow \quad a = 5 \quad \text{as} \quad a \in Z$

17. (i) 80% (ii) 20%

18. (i) No, as it does not tell you what percentage did worse than Elaine.

(ii) $P_{40} = \dfrac{40}{100} \times \dfrac{800}{1} = 320$

$\Rightarrow$ 800 − 320 = 480 people did better than Tanya.

19. (i) $P_{25} = \dfrac{52 + 55}{2} = 53.5$

(ii) $P_{75} = \dfrac{72 + 77}{2} = 74.5$

(iii) $P_{40} = \dfrac{63 + 65}{2} = 64$

$\Rightarrow P_{75} - P_{40} = 74.5 - 64 = 10.5$

(iv) $P_{80} = \dfrac{77 + 79}{2} = 78$

$\Rightarrow$ 4 people have scores $\geq P_{80}$

(v) $\dfrac{9}{20} \times \dfrac{100}{1} = 45 \quad \Rightarrow$ Eoins mark is at the 45th percentile

20. (i) $\dfrac{70}{100} \times 36 = 25.2 \quad \Rightarrow$ Next whole number = 26

$\Rightarrow \quad P_{70}$ = 26th number in the set = €55

(ii) $\dfrac{40}{100} \times 36 = 14.4 \quad \Rightarrow$ Next whole number = 15

$\Rightarrow \quad P_{40}$ = 15th number in the set = €32

(iii) 14

(iv) $\dfrac{80}{100} \times 36 = 28.8 \quad \Rightarrow$ Next whole number = 29

$\Rightarrow \quad P_{80}$ = 29th number in the set = €59 and 7 are more expensive.

(v) Price = €40 $\Rightarrow$ 19 t-shirts are lower than this

$\Rightarrow \quad \dfrac{19}{36} \times \dfrac{100}{1} = 52.77 \quad \Rightarrow$ 53rd to 56th percentile

21. Mean $= \frac{a + b + 8 + 5 + 7}{5} = 6$

$$\Rightarrow \quad a + b + 20 = 30$$
$$\Rightarrow \quad a + b = 10$$
$$\Rightarrow \quad a = 10 - b$$

Find σ for $10 - b, b, 8, 5, 7$.

$$\Rightarrow \quad \sigma = \sqrt{\frac{(10 - b - 6)^2 + (b - 6)^2 + (8 - 6)^2 + (5 - 6)^2 + (7 - 6)^2}{5}}$$
$$\Rightarrow \quad \sqrt{\frac{(4 - b)^2 + (b - 6)^2 + (2)^2 + (-1)^2 + (1)^2}{5}} = \sqrt{2}$$
$$\Rightarrow \quad \sqrt{\frac{16 - 8b + b^2 + b^2 - 12b + 36 + 4 + 1 + 1}{5}} = \sqrt{2}$$
$$\Rightarrow \quad \sqrt{\frac{2b^2 - 20b + 58}{5}} = \sqrt{2}$$
$$\Rightarrow \quad \frac{2b^2 - 20b + 58}{5} = 2$$
$$\Rightarrow \quad 2b^2 - 20b + 58 = 10$$
$$\Rightarrow \quad 2b^2 - 20b + 48 = 0$$
$$\Rightarrow \quad b^2 - 10b + 24 = 0$$
$$\Rightarrow \quad (b - 4)(b - 6) = 0$$
$$\Rightarrow \quad b = 4, b = 6$$
$$\Rightarrow \quad a = 10 - 4 = 6, \quad a = 10 - 6 = 4$$

Since $a > b$, hence $a = 6, b = 4$.

Exercise 10.6

1. (i) 4 people
(ii) 27 years
(iii) 8 people
(iv) Median age is the age of the 13th person = 36 years

2. (i)

stem	leaf							
0	0	1	2	4	6	6	7	8
1	2	4	4	5	7	8	8	9
2	1	1	3	5	6			
3	1	1	2					

Key: 2|3 = 23 CDs

(ii) 8 pupils

(iii) Median $= \frac{15 + 17}{2} = 16$ CDs

3.

stem	leaf								
1	5								
2	4	4	5	6	7	8	8	9	
3	0	1	2	3	5	5	5	6	7
4	2	2	3						
5	6	6	8						

Key: 3|2 means 3.2 seconds

(i) 6 calls
(ii) $5.8 - 1.5 = 4.3$ seconds
(iii) Median $= \frac{3.2 + 3.3}{2} = 3.25$ seconds
(iv) Mode = 3.5 seconds

4. (i) Range = 84 − 22 = 62 marks

(ii) Q_1? $\Rightarrow$ $\frac{1}{4}(19) = 4.75$ $\Rightarrow$ Q_1 is the 5th value = 47 marks

(iii) Q_3? $\Rightarrow$ $\frac{3}{4}(19) = 14.25$ $\Rightarrow$ Q_3 is the 15th value = 67 marks

(iv) 67 − 47 = 20 marks

5. (i) Median $= \frac{38 + 44}{2} = \frac{82}{2} = 41$ laptops

(ii) Q_1? $\Rightarrow$ $\frac{1}{4}(26) = 6.5$ $\Rightarrow$ Q_1 is the 7th value = 32 laptops

(iii) Q_3? $\Rightarrow$ $\frac{3}{4}(26) = 19.5$ $\Rightarrow$ Q_3 is the 20th value = 47 laptops

(iv) Interquartile range = 47 − 32 = 15 laptops

(v) Mode = 47 laptops

6. (i) 19 students took both Science and French

(ii) (a) Science range = 91 − 25 = 66 marks

(b) French range = 85 − 36 = 49 marks

(iii) Median for Science = 55 marks

(iv) Q_1? $\Rightarrow$ $\frac{1}{4}(19) = 4.75$ $\Rightarrow$ Q_1 is the 5th value = 48 marks

Q_3? $\Rightarrow$ $\frac{3}{4}(19) = 14.25$ $\Rightarrow$ Q_3 is the 15th value = 74 marks

$\Rightarrow$ Interquartile range of the French marks = 74 − 48 = 26 marks

7. (i) Median = 76 bpm

Range = 92 − 65 = 27 bpm

(ii) Median $= \frac{68 + 68}{2} = 68$ bpm

Range = 88 − 50 = 38 bpm

(iii) Those who did not smoke; significantly lower median.

8. (i)

French							English					
				2	1	3	8					
	7	6	5	4	4	4	3	4	4			
8	8	7	3	3	0	5	2	6	8			
		9	6	1	1	6	3	5	5	8	9	
				8	5	7	1	2	2	7	9	9
					1	8	4	5				

Key: 7|5 = 57 marks Key: 6|9 = 69 marks

(ii) Median for French $= \frac{53 + 57}{2} = 55$ marks

(iii) Median for English $= \frac{65 + 68}{2} = 66.5$ marks

(iv) English; higher median.

9. (i) Range = 33 − 2 = 31 minutes

(ii) Median for Matrix 1 $= \frac{17 + 18}{2} = 17.5$ minutes

(iii) 15 minutes (i.e the digit 5 is missing)

(iv) Prob (person waited > 10 mins) $= \frac{15}{20} = 0.75$

(v) Median for Matrix 1 $= 17.5$ minutes

Median for Matrix 2 $= \dfrac{17 + 18}{2} = 17.5$ minutes

$\Rightarrow$ Both have median 17.5, similar ranges (one minute in the difference); hence no significant difference.

10.

Men				Women		
2	1	0	4	0	1	
2	2	2	5	1		
5	5	4	6	2	3	
		1	7	5		
			8	7	8	
Key: 5|6 = 65 mins			9	3	5	Key: 8|7 = 87 mins

(i) Modal time for men $= 52$ minutes

(ii) (a) Median for men $= \dfrac{52 + 52}{2} = 52$ minutes

(b) Median for women $= \dfrac{63 + 75}{2} = 69$ minutes

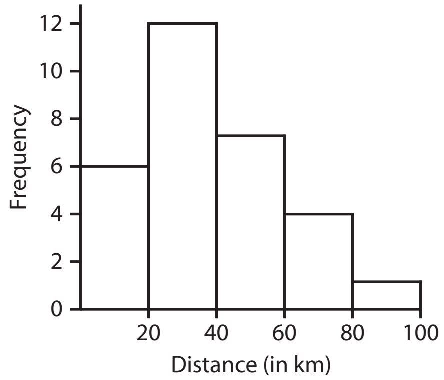
(iii) (a) Range for men $= 71 - 40 = 31$ minutes

(b) Range for women $= 95 - 40 = 55$ minutes

(iv) Women in the survey have a higher median and a wider range.
The far wider range is significant here as both the men's and women's shortest time spent watching t.v. was the same (i.e. 40 mins). Accordingly, the women dominated the longer tv-watching times in this survey (87, 95 mins etc), especially with there being no outliers.

Exercise 10.7

1. (i)



(ii) 12 motorists

(iii) (20–40) km

(iv) Percentage $= \dfrac{12}{30} \times \dfrac{100}{1} = 40\%$

2. (i) 10 people

(ii) Modal class $=$ (40–50) years

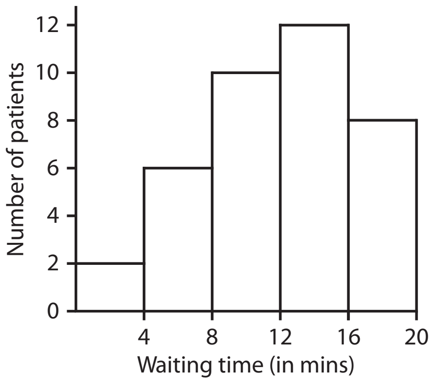
(iii) How many < 30 years? $= 2 + 4 + 6 = 12$ people

(iv) Total $= 2 + 4 + 6 + 10 + 17 + 12 + 6 + 3 = 60$ people

(v) (50–60) years

(vi) Median lies in the (40–50) years interval.

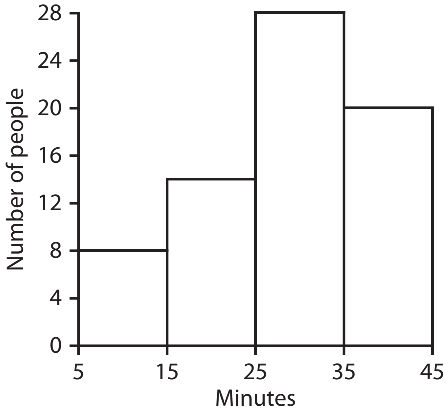
3. (i)



(ii) Number of patients $= 2 + 6 + 10 + 12 + 8 = 38$ patients
(iii) Modal class = (12–16) mins
(iv) Median lies in the (12–16) mins interval
(v) Greatest number $>$ 10 minutes $= 10 + 12 + 8 = 30$ patients
(vi) Least number $>$ 14 minutes $= 8$ patients

4. (i) Number of pupils $\geq$ 15 secs $= 10 + 9 = 19$ pupils
(ii) Total $= 8 + 12 + 15 + 10 + 9 = 54$ pupils
(iii) Modal class = (10–15) secs
(iv) Median lies in the (10–15) secs interval
(v) Greatest number $<$ 8 secs $= 8 + 12 = 20$ pupils
(vi) Least number $<$ 12 secs $= 8 + 12 = 20$ pupils

5. (i)



(ii) Modal class = (25–35) mins
(iii) Median lies in the (25–35) mins interval
(iv) (15–25) mins because $\frac{14}{70} \times \frac{100}{1} = 20\%$
(v) Greatest number $>$ 30 minutes $= 28 + 20 = 48$ people
(vi) Mean $= \frac{(10)(8) + (20)(14) + (30)(28) + (40)(20)}{8 + 14 + 28 + 20}$

$= \frac{80 + 280 + 840 + 800}{70}$

$= \frac{2000}{70} = 28.57 = 29$ minutes

Exercise 10.8

1. Symmetrical;
(i) Normal distribution
(ii) People's heights

2. Positively skewed; age at which people start third-level education.

3. (i) c (ii) a (iii) b (iv) b (v) c

4. Negatively skewed
(i) Mean (ii) Mode

5. More of the data is closer to the mean in distribution Ⓐ

6. (i) Ⓑ (ii) Ⓑ

7. (i) Ⓐ (ii) Equal

8. (i) Ⓑ (ii) Ⓐ

9. (i) Ⓐ (ii) Ⓑ

10. (i)

A	B	C	D
×	×	✓	×
✓	×	×	×
×	✓	×	✓
✓	×	×	×
✓	✓	✓	×

(ii) D has the largest standard deviatioin as more of the data is located further from the mean.

Revision Exercise 10 (Core)

1. (i) Primary (ii) Secondary (iii) Primary (iv) Secondary (v) Secondary

2. Times arranged in order: 6, 7, 8, 9, 9, 9, 11, 12, 15, 16, 19
(i) Range $= 19 - 6 = 13$ minutes
(ii) Q_1? $\Rightarrow$ $\frac{1}{4}(11) = 2.75$ $\Rightarrow$ Q_1 is the 3rd value $= 8$ minutes
(iii) Q_3? $\Rightarrow$ $\frac{3}{4}(11) = 8.25$ $\Rightarrow$ Q_3 is the 9th value $= 15$ minutes
(iv) Interquartile range $= 15 - 8 = 7$ minutes

3. Mean $= \dfrac{3 + 6 + 7 + x + 14}{5} = 8$

$\Rightarrow \quad 30 + x = 40$

$\Rightarrow \quad x = 40 - 30 = 10$

Standard Deviation $(\sigma) = \sqrt{\dfrac{(3-8)^2 + (6-8)^2 + (7-8)^2 + (10-8)^2 + (14-8)^2}{5}}$

$= \sqrt{\dfrac{(-5)^2 + (-2)^2 + (-1)^2 + (2)^2 + (6)^2}{5}}$

$= \sqrt{\dfrac{25 + 4 + 1 + 4 + 36}{5}} = \sqrt{\dfrac{70}{5}} = \sqrt{14} = 3.74 = 3.7$

4. (i) Census surveys the entire population; sample surveys only part of the population.
(ii) $P_{72} \Rightarrow$ 28% higher than his mark

$= \dfrac{28}{100} \times 90 = 25.2$

$= 25$ students

5. (i) 32 students
(ii) €48
(iii) Median = amount spent by the 8th female = €25
(iv) Median = amount spent by the 9th male = €29
(v) Males; higher median.

6. (i) (b); because it has the greater spread.

(ii) Mean $(\mu) = \dfrac{(2 \times 0) + (5 \times 1) + (6 \times 2) + (5 \times 3) + (2 \times 4)}{2 + 5 + 6 + 5 + 2}$

$= \dfrac{0 + 5 + 12 + 15 + 8}{20}$

$= \dfrac{40}{20} = 2$

x	f	$x - \mu$	$(x - \mu)^2$	$f(x - \mu)^2$
0	2	-2	4	8
1	5	-1	1	5
2	6	0	0	0
3	5	1	1	5
4	2	2	4	8
	20			26

$\sigma = \sqrt{\dfrac{\Sigma f(x - \mu)^2}{\Sigma f}} = \sqrt{\dfrac{26}{20}} = \sqrt{1.3} = 1.140$

7. (i) Stratified, then simple random sampling.

(ii) $\dfrac{100}{5} = 20$ students

(iii) Give each student a number and then select 10, using random number key on calculator.

8. (i) Yes; it may not be representative as there is no random element to the survey.

(ii) Use stratified sampling based on gender, age, marital status, income level, etc. and then use simple random sampling.

9. Stratified sampling is used when the population can be split into separate groups or strata that are quite different from each other. The number selected from each group is proportional to the size of the group. Separate random samples are then taken from each group.

In cluster sampling, the population is divided into groups or clusters. Then, some of these clusters are randomly selected and all items from these clusters are chosen. A large number of small clusters is best as this minimises the chances of the sample being unrepresentative. Cluster sampling is very popular with scientists.

10. (i)

stem	leaf							
3	1	2	3	4	5	6	8	
4	0	1	2	6	6	7	7	7
5	9							

Key: 4|2 = 4.2 mins

(ii) Mode = 4.7 minutes

(iii) Median $= \dfrac{4.0 + 4.1}{2} = 4.05$ minutes

Q_1? $\Rightarrow \dfrac{1}{4}(16) = 4 \Rightarrow$ Q_1 is the 4th value = 3.4 minutes

Q_3? $\Rightarrow \dfrac{3}{4}(16) = 12 \Rightarrow$ Q_3 is the 12th value = 4.6 minutes

$\Rightarrow$ Interquartile range = 4.6 − 3.4 = 1.2 minutes

Revision Exercise 10 (Advanced)

1. David; as the standard deviation of his marks is smaller.

2. Marks in order: 37, 38, 42, 46, 46, 46, 48, 54, 55, 57, 59, 63, 64, 65, 66, 68, 68, 68, 71, 73, 74, 76, 78, 82.

(i) $\dfrac{40}{100} \times 24 = 9.6$

$\Rightarrow$ P_{40} = 10th number in the set = 57% (i.e. 57 marks out of 100)

(ii) Score = 71 marks $\Rightarrow$ 18 students are lower than this.

$\Rightarrow \dfrac{18}{24} \times \dfrac{100}{1} = 75$

$\Rightarrow$ Gillian's score is P_{75}, the 75^{th} percentile.

3. (i) A, D (iii) B (v) A
(ii) C, A (iv) A

4.

Class	Mid-interval = x	f	fx	$x - \mu$	$(x - \mu)^2$	$f(x - \mu)^2$
1–3	2	4	8	−2	4	16
3–5	4	3	12	0	0	0
5–7	6	0	0	2	4	0
7–9	8	2	16	4	16	32
		9	36			48

Mean $(\mu) = \dfrac{36}{9} = 4$

Standard deviation $(\sigma) = \sqrt{\dfrac{48}{9}} = \sqrt{5\frac{1}{3}} = 2.309 = 2.3$

5. (i) Negatively skewed as most of the data occurs at the higher values.
(ii) A = mode, B = median, C = mean.
(iii) Age when people retire

6. $300 + 500 + 400 = 1200$ cans

Large $\Rightarrow \dfrac{300}{1200} \times 60 = 15$ large cans

Medium $\Rightarrow \dfrac{500}{1200} \times 60 = 25$ medium cans

Small $\Rightarrow \dfrac{400}{1200} \times 60 = 20$ small cans

7. (i) Mean $(\mu) = \dfrac{(26 \times 0) + (90 \times 1) + (57 \times 2) + (19 \times 3) + (5 \times 4) + (3 \times 5) + (200 \times 6)}{26 + 90 + 57 + 19 + 5 + 3 + 200}$

$= \dfrac{0 + 90 + 114 + 57 + 20 + 15 + 1200}{400}$

$= \dfrac{1496}{400} = 3.74$

(ii)

x	f	$(x - \mu)$	$(x - \mu)^2$	$f(x - \mu)^2$
0	26	−3.74	13.9876	363.6776
1	90	−2.74	7.5076	675.684
2	57	−1.74	3.0276	172.5732
3	19	−0.74	0.5476	10.4044
4	5	0.26	0.0676	0.338
5	3	1.26	1.5876	4.7628
6	200	2.26	5.1076	1021.52
	400			2248.96

$\sigma = \sqrt{\dfrac{2248.96}{400}} = \sqrt{5.6224} = 2.371 = 2.37$

(iii) In the earlier study of the same junction, there were less crashes, on average, each day (mean 0.54 lower). The far lower standard deviation also tells us that there used to be far fewer days where there were a high number (i.e. 5 or 6) of road accidents at the junction.

8. (i) Explanatory: Fertilizer used
Response: Wheat yield
(ii) Explanatory: Suitable habitat
Response: Number of species
(iii) Explanatory: Amount of water
Response: Time taken to cool
(iv) Explanatory: Size of engine
Response: Petrol consumption

9. A: Systematic
B: Convenience
C: Simple random
D: Stratified
E: Quota

10. Runs in order: 0, 0, 13, 28, 35, 40, 47, 51, 63, 77, a
(i) Median = 40

Q_1? $\Rightarrow$ $\frac{1}{4}(11) = 2.75$ $\Rightarrow$ Lower Q. is the 3rd number = 13

Q_3? $\Rightarrow$ $\frac{3}{4}(11) = 8.25$ $\Rightarrow$ Upper Q. is the 9th amount = 63

$\Rightarrow$ Interquartile range = 63 − 13 = 50

(ii) Mode = 0 $\Rightarrow$ not an appropriate average because zero would not be a typical value. (In fact, it is the lowest value.)
Range is between 0 and a > 100 so it would be distorted by the two zeros and the one very high value.

Revision Exercise 10 (Extended-Response Questions)

1. (i) (a) Median is between 190th and 191st matches

$= \frac{2+2}{2} = 2$ goals

Q_1? $\Rightarrow$ $\frac{1}{4}(380) = 95$ $\Rightarrow$ 95th match had 1 goal scored in it.

Q_3? $\Rightarrow$ $\frac{3}{4}(380) = 285$ $\Rightarrow$ 285th match had 4 goal scored in it

$\Rightarrow$ Interquartile range = 4 − 1 = 3 goals

(b)

Goals = x	Matches = f	$f \cdot x$	$x - \mu$	$(x - \mu)^2$	$f(x - \mu)^2$
0	30	0	−2.56	6.5536	196.608
1	79	79	−1.56	2.4336	192.2544
2	99	198	−0.56	0.3136	31.0464
3	68	204	0.44	0.1936	13.1648
4	60	240	1.44	2.0736	124.416
5	24	120	2.44	5.9536	142.8864
6	11	66	3.44	11.8336	130.1696
7	6	42	4.44	19.7136	118.2816
8	2	16	5.44	29.5936	59.1872
9	1	9	6.44	41.4736	41.4736
	380	974			1049.488

Mean $(\mu) = \frac{974}{380} = 2.563 = 2.56$

$\sigma = \sqrt{\frac{1049.488}{380}} = \sqrt{2.76181} = 1.661 = 1.66$

(ii) The mean is slightly higher in the 2008/09 season and the standard deviation is also marginally higher. The wider spread in the 2008/09 season suggests a few more open, high-scoring games. However, the median number of goals per game is the same for both seasons. Overall, there is little significant difference between the two seasons.

2. (i) Mode = 22

$Q_1 = X \quad \Rightarrow \quad \frac{1}{4}(21) = 5.25 \quad \Rightarrow \quad Q_1$ is the 6^{th} number $= 11 = X$

(ii) Median $= Y \quad \Rightarrow \quad 11^{th}$ number for Jack $= 27 = Y$

$Q_3 = Z \quad \Rightarrow \quad \frac{3}{4}(21) = 15.75 \quad \Rightarrow \quad Q_3$ is the 16^{th} number $= 22 = Z$

(iii) Strand road; median is more than double that of market street.

3. (i) Driver: Positively skewed as a lot of the data is clustered to the left, especially in the (20–30) year age-group.
Passenger: From the ages (0–40) years, it is a symmetrical distribution with a mean of approximately 20 years. The values then fall away as you move away from the centre.

(ii) (a) Driver: 20 years old
(b) Passenger: 18 years old

(iii) A uniform distribution suggesting casualities equally likely at all ages, with a moderate peak from (15–25) years.

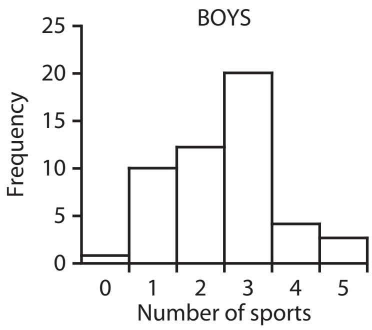
(iv) The (17–30) years age-group. Most of the casualities among both drivers and passengers occur in this group.

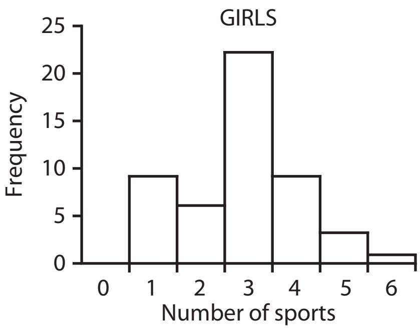
4. (i) Boys

Numbers of sports	0	1	2	3	4	5
Frequency	1	10	12	20	4	3

Girls

Numbers of sports	1	2	3	4	5	6
Frequency	9	6	22	9	3	1





(i) **Similarity**: Both have the same mode (3).
Difference: Girls' distribution resembles a normal distribution. For the boys, most of the data is concentrated at the lower values (1–3).

(ii) Though the medians are the same, the girls' distribution has a greater spread. The samples' findings are sufficiently different to suggest that this could not happen by chance, i.e. that there is evidence that there are differences between the two populations.

(iv) They could include more boys and girls who are not in G.A.A. clubs. Include both urban and rural children from different parts of the country so the sample would be less biased. Also, be more precise about what "playing sports" means.

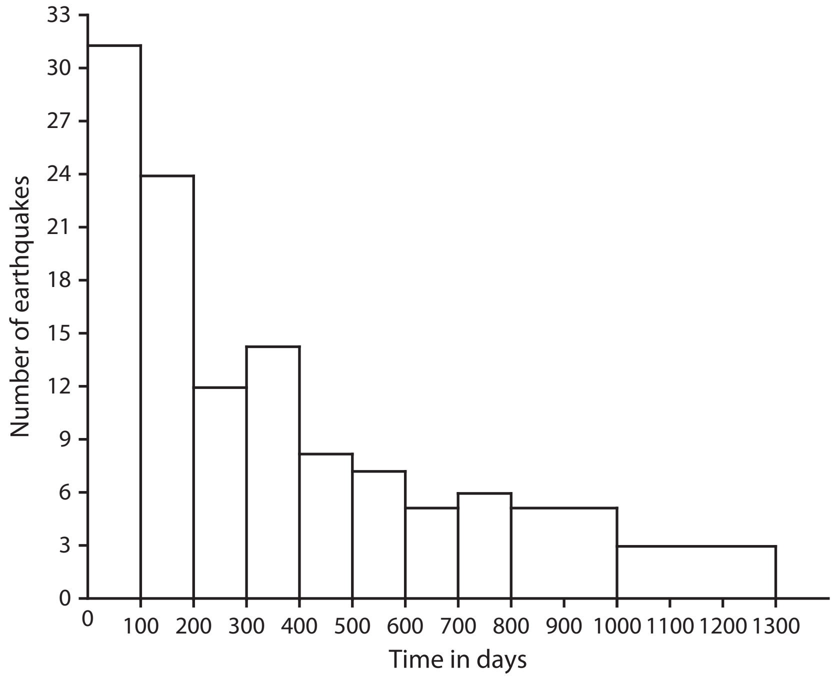
5. (i) A – run; B – cycle; C – swim

(ii) 25 minutes

(iii) Standard deviation for the swim times lies between 4.533 and 3.409 $\Rightarrow$ Approx: 3 minutes

(iv) It would be very unusual for two (or more) athletes to have the same time as it is continuous numerical data and time were to the nearest 100^{th} of a second.

6. (i)



(ii) The distribution has a positive skew (tail to the right).

Median = 58th earthquake = $31 + 24 + \frac{1}{3}(12)$

$\Rightarrow \frac{1}{4}$ into 3rd interval [200 − 300] = 225 days

(iii) It is not a normal distribution and so z-scores are not appropriate. The distribution has a positive skew and hence, it is not a normal distribution.

(iv) $\frac{24}{100} = 0.2087 = 0.21$. This is the relative frequency of the next earthquake occurring between 100 and 200 days later.

(v) They could have looked at the number of earthquakes each year, or some other interval of time (eg. distribution of earthquakes per decade, per year, etc)
They could have redefined serious earthquakes as earthquakes greater than a certain magnitude; earthquakes in less-populated areas are not included.
The data set could have been broadened to include less serious earthquakes. This could result in a different pattern.

Chapter 11

Exercise 11.1

1. (i) $x^2 + y^2 = (2)^2 \Rightarrow x^2 + y^2 = 4$

(ii) $x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$

(iii) $x^2 + y^2 = (\sqrt{2})^2 \Rightarrow x^2 + y^2 = 2$

(iv) $x^2 + y^2 = (3\sqrt{2})^2 \Rightarrow x^2 + y^2 = 18$

(v) $x^2 + y^2 = \left(\dfrac{3}{4}\right)^2 \Rightarrow x^2 + y^2 = \dfrac{9}{16}$

$\Rightarrow 16x^2 + 16y^2 = 9$

(vi) $x^2 + y^2 = \left(2\dfrac{1}{2}\right)^2 \Rightarrow x^2 + y^2 = \dfrac{25}{4}$

$\Rightarrow 4x^2 + 4y^2 = 25$

2. $(0, 0), (3, 4) \Rightarrow$ radius $= \sqrt{(3 - 0)^2 + (4 - 0)^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25} = 5$

Hence, $x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$

3. $(0, 0), (-4, 1) \Rightarrow$ radius $= \sqrt{(-4 - 0)^2 + (1 - 0)^2}$

$= \sqrt{16 + 1}$

$= \sqrt{17}$

Hence, $x^2 + y^2 = (\sqrt{17})^2 \Rightarrow x^2 + y^2 = 17$

4. (i) $(-4, -3), (4, 3) \Rightarrow$ midpoint $= \left(\dfrac{-4 + 4}{2}, \dfrac{-3 + 3}{2}\right)$

$= (0, 0)$ [the midpoint is the centre]

(ii) $(0, 0)\ (4, 3) \Rightarrow$ radius $= \sqrt{(4 - 0)^2 + (3 - 0)^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25} = 5$

(iii) $x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$

5. $(4, -1)(-4, 1) \Rightarrow$ midpoint $= \left(\dfrac{4 - 4}{2}, \dfrac{-1 + 1}{2}\right)$

$= (0, 0)$

$(0, 0)\ (4, -1) \Rightarrow$ radius $= \sqrt{(4 - 0)^2 + (-1 - 0)^2}$

$= \sqrt{16 + 1} = \sqrt{17}$

Hence, $x^2 + y^2 = (\sqrt{17})^2 \Rightarrow x^2 + y^2 = 17$ is the equation of the circle.

6. (i) $r^2 = 9 \Rightarrow r = \sqrt{9} = 3$

(ii) $r^2 = 1 \Rightarrow r = \sqrt{1} = 1$

(iii) $r^2 = 27 \Rightarrow r = \sqrt{27} = 3\sqrt{3}$

(iv) $4x^2 + 4y^2 = 25$

$\Rightarrow x^2 + y^2 = \dfrac{25}{4}$

$\Rightarrow r^2 = \dfrac{25}{4} \Rightarrow r = \sqrt{\dfrac{25}{4}} = \dfrac{5}{2}$

(v) $9x^2 + 9y^2 = 4$

$\Rightarrow x^2 + y^2 = \dfrac{4}{9}$

$\Rightarrow r^2 = \dfrac{4}{9} \Rightarrow r = \sqrt{\dfrac{4}{9}} = \dfrac{2}{3}$

(vi) $16x^2 + 16y^2 = 49$

$\Rightarrow x^2 + y^2 = \dfrac{49}{16}$

$\Rightarrow r^2 = \dfrac{49}{16} \Rightarrow r = \sqrt{\dfrac{49}{16}} = \dfrac{7}{4}$

7. (i) Radius = perpendicular distance from (0, 0) to $2x + y - 5 = 0$.

$$\Rightarrow \text{radius} = \frac{|2(0) + 1(0) - 5|}{\sqrt{(2)^2 + (1)^2}} = \frac{5}{\sqrt{5}} = \frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \sqrt{5}$$

(ii) Equation of circle: $x^2 + y^2 = (\sqrt{5})^2 \Rightarrow x^2 + y^2 = 5$

8. Point (0, 0); tangent: $4x - 3y - 25 = 0$

$$\Rightarrow \text{radius} = \frac{|4(0) - 3(0) - 25|}{\sqrt{(4)^2 + (-3)^2}} = \frac{25}{\sqrt{25}} = \frac{25}{5} = 5$$

Hence, equation of circle: $x^2 + y^2 = 5^2 \Rightarrow x^2 + y^2 = 25$

9. Centre = (0, 0); tangent: $3x - y + 10 = 0$

$$\Rightarrow \text{radius} = \frac{|3(0) - 1(0) + 10|}{\sqrt{(3)^2 + (-1)^2}} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Hence, equation of circle: $x^2 + y^2 = (\sqrt{10})^2$

$\Rightarrow x^2 + y^2 = 10$

10. Centre (0, 0), radius = $2\sqrt{5}$

$\Rightarrow$ Equation of circle: $x^2 + y^2 = (2\sqrt{5})^2$

$\Rightarrow x^2 + y^2 = 20$

Centre (0, 0); line t: $x - 2y + 10 = 0$

$$\Rightarrow \text{Perpendicular distance} = \frac{|1(0) - 2(0) + 10|}{\sqrt{(1)^2 + (-2)^2}}$$

$$= \frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{\sqrt{5}\,.\,\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5} = \text{radius}$$

Hence, t is a tangent.

Exercise 11.2

1. (i) Centre (3, 1), radius = 2

Equation of circle: $(x - 3)^2 + (y - 1)^2 = (2)^2 = 4$

(ii) Centre (1, −4), radius = $\sqrt{8}$

Equation of circle: $(x - 1)^2 + (y + 4)^2 = (\sqrt{8})^2 = 8$

(iii) Centre (4, 0), radius = $2\sqrt{3}$

Equation of circle: $(x - 4)^2 + (y - 0)^2 = (2\sqrt{3})^2 = 12$

(iv) Centre (0, −5), radius = $3\sqrt{2}$

Equation of circle: $(x - 0)^2 + (y + 5)^2 = (3\sqrt{2})^2 = 18$

2. Centre (2, 2), point (5, 1) $\Rightarrow$ radius $= \sqrt{(5 - 2)^2 + (1 - 2)^2}$

$= \sqrt{3^2 + (-1)^2}$

$= \sqrt{9 + 1} = \sqrt{10}$

Equation of circle: $(x - 2)^2 + (y - 2)^2 = (\sqrt{10})^2 = 10$

3. (i) (3, 5), (−1, 1) $\Rightarrow$ midpoint $= \left(\frac{3 - 1}{2}, \frac{5 + 1}{2}\right) = (1, 3)$

(ii) Centre (1, 3), point (3, 5) $\Rightarrow$ radius $= \sqrt{(3 - 1)^2 + (5 - 3)^2}$

$= \sqrt{2^2 + 2^2} = \sqrt{8}$

Equation of circle: $(x - 1)^2 + (y - 3)^2 = (\sqrt{8})^2 = 8$

4. (i) Centre = (3, 2), radius = $\sqrt{16} = 4$

(ii) Centre = (−2, 6), radius = $\sqrt{8} = 2\sqrt{2}$

(iii) Centre = (3, 0), radius = $\sqrt{5}$

(iv) Centre = (0, −2), radius = $\sqrt{10}$

5. Radius $= \sqrt{18} = 3\sqrt{2}$

$\Rightarrow$ diameter $= 2(3\sqrt{2}) = 6\sqrt{2}$

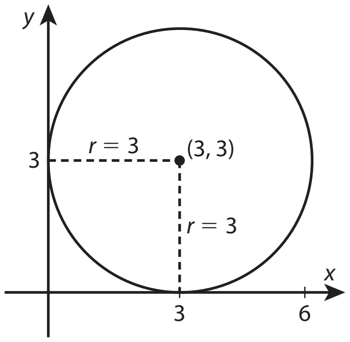
Hence, centre $= (-2, 5)$, radius $= 6\sqrt{2}$

$\Rightarrow$ Equation of circle: $(x + 2)^2 + (y - 5)^2 = (6\sqrt{2})^2 = 72$

6. Centre $= (3, 3)$, radius $= 3$

Equation of circle: $(x - 3)^2 + (y - 3)^2 = (3)^2$

$\Rightarrow \quad (x - 3)^2 + (y - 3)^2 = 9$



7. (i) $2g = -4 \quad \Rightarrow \quad g = -2$

$2f = 8 \quad \Rightarrow \quad f = 4$

$\Rightarrow$ centre $= (-g, -f) = (2, -4)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (4)^2 + 5}$

$= \sqrt{4 + 16 + 5} = \sqrt{25} = 5$

(ii) $2g = -2 \quad \Rightarrow \quad g = -1$

$2f = -6 \quad \Rightarrow \quad f = -3$

$\Rightarrow$ centre $= (-g, -f) = (1, 3)$

and radius $= \sqrt{(-1)^2 + (-3)^2 + 15}$

$= \sqrt{1 + 9 + 15} = \sqrt{25} = 5$

(iii) $2g = -8 \Rightarrow g = -4$

$2f = 0 \quad \Rightarrow f = 0$

$\Rightarrow$ centre $= (4, 0)$

and radius $= \sqrt{(-4)^2 + (0)^2 + 8} = \sqrt{16 + 8} = \sqrt{24} = 2\sqrt{6}$

(iv) $2g = 5 \quad \Rightarrow \quad g = \frac{5}{2} = 2\frac{1}{2}$

$2f = -6 \quad \Rightarrow \quad f = -3$

$\Rightarrow$ centre $= \left(-2\frac{1}{2}, 3\right)$

and radius $= \sqrt{\left(2\frac{1}{2}\right)^2 + (-3)^2 + 5} = \sqrt{20\frac{1}{4}} = \frac{9}{2}$

(v) $x^2 + y^2 - 2x + \frac{3}{2}y = 0$

$\Rightarrow \quad 2g = -2 \quad \Rightarrow \quad g = -1$

and $2f = \frac{3}{2} \quad \Rightarrow \quad f = \frac{3}{4}$

$\Rightarrow$ centre $= \left(1, -\frac{3}{4}\right)$

and radius $= \sqrt{(-1)^2 + \left(\frac{3}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

(vi) $x^2 + y^2 - 7y + \frac{33}{4} = 0$

$\Rightarrow \quad 2g = 0 \quad \Rightarrow \quad g = 0$

and $2f = -7 \quad \Rightarrow \quad f = -\frac{7}{2} = -3\frac{1}{2}$

$\Rightarrow \quad$ centre $= \left(0, 3\frac{1}{2}\right)$

and radius $= \sqrt{(0)^2 + \left(-3\frac{1}{2}\right)^2 - \frac{33}{4}} = \sqrt{\frac{49}{4} - \frac{33}{4}} = \sqrt{4} = 2$

8. Circle: $x^2 + y^2 - 4x + 2y - 20 = 0$

Point $(5, -5) \Rightarrow (5)^2 + (-5)^2 - 4(5) + 2(-5) - 20$

$= 25 + 25 - 20 - 10 - 20 = 50 - 50 = 0$, true

9. Circle: $x^2 + y^2 + 2x - 4y - 20 = 0$

Point $(3, 6) \Rightarrow (3)^2 + (6)^2 + 2(3) - 4(6) - 20$

$= 9 + 36 + 6 - 24 - 20$

$= 51 - 44 = 7 > 0 \Rightarrow$ outside circle

10. Circle: $x^2 + y^2 - 2x + 4y - 15 = 0$

Point $(3, 1) \Rightarrow (3)^2 + (1)^2 - 2(3) + 4(1) - 15$

$= 9 + 1 - 6 + 4 - 15$

$= 14 - 21 = -7 < 0 \Rightarrow$ inside circle

11. Circle: $x^2 + y^2 - 6x + 4y + 4 = 0$

Point $(1, 1) \Rightarrow (1)^2 + (1)^2 - 6(1) + 4(1) + 4$

$= 1 + 1 - 6 + 4 + 4$

$= 10 - 6 = 4 > 0 \Rightarrow$ outside circle

12. $2g = -8 \Rightarrow g = -4$

$2f = 10 \Rightarrow f = 5$

Radius $= 7 \quad \Rightarrow \sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + (5)^2 - k} = 7$

$\Rightarrow 16 + 25 - k = 49$

$\Rightarrow \quad -k = 49 - 41 = 8$

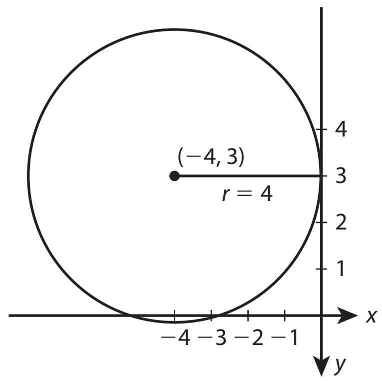
$\Rightarrow \quad k = -8$

13. (i) Sketch circle k

(ii) Radius = 4

(iii) Centre = $(-4, 3)$

Equation of circle: $(x + 4)^2 + (y - 3)^2 = (4)^2 = 16$



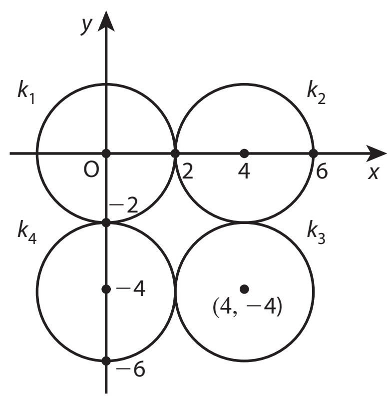
14. (i) $r^2 = 4 \quad \Rightarrow \quad r = 2$

(ii) Centre $k_3 = (4, -4)$

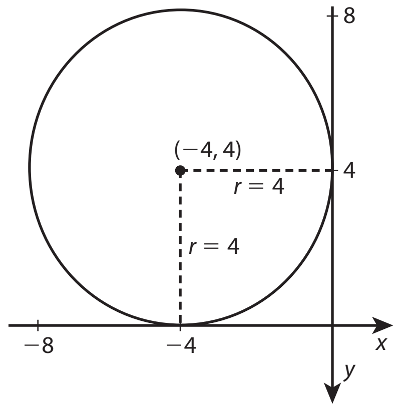
(iii) k_3 has centre $(4 - 4)$, radius = 2

$\Rightarrow \quad$ Equation k_3: $(x - 4)^2 + (y + 4)^2 = (2)^2 = 4$

(iv) k_4 … its centre is $(0, -4)$.



15. Centre $= (-4, 4)$
Radius $= 4$
Equation of circle: $(x + 4)^2 + (y - 4)^2 = (4)^2$
$\Rightarrow (x + 4)^2 + (y - 4)^2 = 16$



16. (i) $(x - 2)^2 + (y - 6)^2 = 100$
Centre $= (2, 6) = C$

(ii) $C = (2, 6)$, $P = (10, 0)$ slope $CP = \dfrac{0 - 6}{10 - 2} = \dfrac{-6}{8} = \dfrac{-3}{4}$

Line $4x - 3y - 40 = 0 \Rightarrow$ slope $= \dfrac{-a}{b} = \dfrac{-4}{-3} = \dfrac{4}{3}$

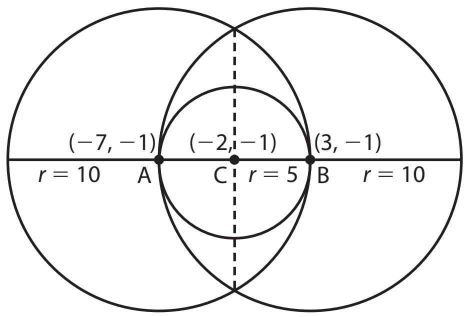
Product of slopes $w_1 \cdot w_2 = \left(\dfrac{-3}{4}\right)\left(\dfrac{4}{3}\right) = \dfrac{-12}{12} = -1$

Hence, CP is perpendicular to the given line.

17. Centre $\Rightarrow y = x \cap y = -x + 4$
$\Rightarrow x = -x + 4$
$\Rightarrow 2x = 4$
$\Rightarrow x = 2 \Rightarrow y = 2$
Centre $= (2, 2)$ and diameter $= 2 \Rightarrow$ radius $= 1$
Equation of circle: $(x - 2)^2 + (y - 2)^2 = (1)^2 = 1$

18. (i) $(x + 2)^2 + (y + 1)^2 = 25$
Centre $C = (-2, -1)$, radius $= 5$
Centre $B = (3, -1)$, radius $= 10$
Centre $A = (-7, -1)$, radius $= 10$

(ii) $(x + 7)^2 + (y + 1)^2 = (10)^2 = 100$



Exercise 11.3

1. Circle: $x^2 + y^2 = 10$
$\Rightarrow$ centre $= (0, 0)$, radius $= \sqrt{10}$
l: $3x + y + 10 = 0$

Perpendicular distance $= \dfrac{|3(0) + 1(0) + 10|}{\sqrt{(3)^2 + (1)^2}} = \dfrac{10}{\sqrt{10}}$

$= \dfrac{10\sqrt{10}}{\sqrt{10}\sqrt{10}} = \sqrt{10} =$ radius

Hence, l is a tangent to the circle.

2. $(x-3)^2+(y+4)^2=50$

$\Rightarrow$ centre $=(3,-4)$, radius $=\sqrt{50}=5\sqrt{2}$

Line: $x-y+3=0$

$$\text{Perpendicular distance} = \frac{|1(3)-1(-4)+3|}{\sqrt{(1)^2+(-1)^2}}$$

$$= \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} = \text{radius}$$

Hence, the line is a tangent to the circle.

3. $(x+2)^2+(y-1)^2=16$

$\Rightarrow$ centre $=(-2,1)$, radius $=\sqrt{16}=4$

Line: $3x-4y-12=0$

$$\text{Perpendicular distance} = \frac{|3(-2)-4(1)-12|}{\sqrt{(3)^2+(-4)^2}}$$

$$= \frac{|-6-4-12|}{\sqrt{25}}$$

$$= \frac{22}{5} = 4\frac{2}{5} \neq 4$$

Hence, the line is not a tangent to the circle.

4. Centre $=(-1,2)$, tangent: $2x-3y-5=0$

$$\text{Radius} = \frac{|2(-1)-3(2)-5|}{\sqrt{(2)^2+(-3)^2}} = \frac{|-2-6-5|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Equation of circle: $(x+1)^2+(y-2)^2=(\sqrt{13})^2=13$

5. Centre $=(2,1)$, tangent: $x-y+5=0$

$$\text{Radius} = \frac{|1(2)-1(1)+5|}{\sqrt{(1)^2+(-1)^2}} = \frac{|2-1+5|}{\sqrt{2}} = \frac{6}{\sqrt{2}}$$

$$= \frac{6}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Equation of circle: $(x-2)^2+(y-1)^2=(3\sqrt{2})^2=18$

6. $x^2+y^2-2x-2y+1=10$

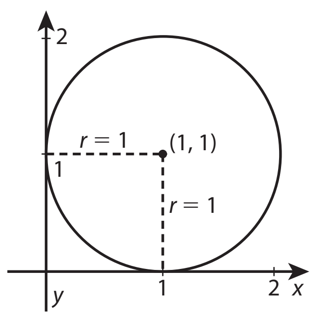
(i) $2g=-2 \Rightarrow g=-1$

$2f=-2 \Rightarrow f=-1$

$\Rightarrow$ centre $=(1,1)$

(ii) Radius $=\sqrt{g^2+f^2-c}=\sqrt{(-1)^2+(-1)^2-1}=\sqrt{1+1-1}=\sqrt{1}=1$

(iii) Sketch of circle



(iv) $|-g|=|-1|=1=$ radius

$|-f|=|-1|=1=$ radius

7. Centre $=(2,2)=(-g,-f)$

Radius $=|-g|=|-f|=|-2|=2$

Equation of circle: $(x-2)^2+(y-2)^2=(2)^2=4$

8. Centre $=(2,3)=(-g,-f)$

Touches y-axis $\Rightarrow$ radius $=|-g|=|2|=2$

Equation of circle: $(x-2)^2+(y-3)^2=(2)^2=4$

9. $x^2 + y^2 - 4x + 6y - 12 = 0$

(i) $2g = -4 \quad \Rightarrow \quad g = -2$

$2f = 6 \quad \Rightarrow \quad f = 3$

centre $= (-g, -f) = (2, -3)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 + 12}$

$= \sqrt{4 + 9 + 12} = \sqrt{25} = 5$

(ii) Tangent: $3x + 4y - k = 0$

$$\text{Radius} = \frac{|3(2) + 4(-3) - k|}{\sqrt{(3)^2 + (4)^2}} = 5$$

$$\Rightarrow \quad \frac{|6 - 12 - k|}{\sqrt{25}} = 5$$

$$\Rightarrow \quad |-6 - k| = 25$$

$\Rightarrow -6 - k = 25 \quad$ or $\quad 6 + k = 25$

$\Rightarrow \quad -k = 31 \quad$ or $\quad k = 25 - 6$

$\Rightarrow \quad k = -31 \quad$ or $\quad k = 19$

10. (i) Midpoint [AB] $= \left[\frac{0 + 1}{2}, \frac{-2 + 2}{2}\right] = \left(\frac{1}{2}, 0\right)$

Slope AB $= \frac{2 + 2}{1 - 0} = \frac{4}{1} = 4$

Perpendicular slope $= -\frac{1}{4}$

$\Rightarrow$ Equation: $y - 0 = -\frac{1}{4}\left(x - \frac{1}{2}\right)$

$$\Rightarrow \quad 4y = -1\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \quad 4y = -x + \frac{1}{2}$$

$$\Rightarrow \quad 8y = -2x + 1$$

$$\Rightarrow \quad 2x + 8y - 1 = 0$$

(ii) Midpoint [BC] $= \left[\frac{0 + 4}{2}, \frac{-2 - 3}{2}\right] = \left(2, -2\frac{1}{2}\right)$

Slope BC $= \frac{-3 + 2}{4 - 0} = -\frac{1}{4}$

Perpendicular slope $= 4$

$\Rightarrow$ Equation: $y + 2\frac{1}{2} = 4(x - 2)$

$$\Rightarrow \quad y + 2\frac{1}{2} = 4x - 8$$

$$\Rightarrow \quad 2y + 5 = 8x - 16$$

$$\Rightarrow \quad 8x - 2y - 21 = 0$$

(iii)

$$\begin{aligned} 8x - 2y &= 21 \quad (\times 4) \\ 2x + 8y &= 1 \quad (\times 1) \\ \hline \Rightarrow \quad 32x - 8y &= 84 \\ 2x + 8y &= 1 \\ \hline 34x \quad &= 85 \quad \text{(adding)} \end{aligned}$$

$$\Rightarrow \quad x = \frac{85}{34} = \frac{5}{2}$$

$$\Rightarrow \quad 8\left(\frac{5}{2}\right) - 2y = 21$$

$$\Rightarrow \quad 20 - 2y = 21$$

$$\Rightarrow \quad -2y = 1$$

$$\Rightarrow \quad y = -\frac{1}{2} \Rightarrow \text{point} = \left(\frac{5}{2}, -\frac{1}{2}\right) = \text{centre}$$

(iv) $(1, 2), \left(\frac{5}{2}, -\frac{1}{2}\right) \Rightarrow$ radius $= \sqrt{\left(\frac{5}{2}-1\right)^2 + \left(-\frac{1}{2}-2\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{17}{2}}$

(v) Equation of circle: $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\sqrt{\frac{17}{2}}\right)^2 = \frac{17}{2}$

11. General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

Point (0, 0) $\Rightarrow (0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$

$\Rightarrow 0 + 0 + 0 + 0 + c = 0 \quad \Rightarrow \quad c = 0$

Point (2, 0) $\Rightarrow (2)^2 + (0)^2 + 2g(2) + 2f(0) + c = 0$

$\Rightarrow 4 + 0 + 4g + 0 + c = 0$

$\Rightarrow 4g + c = -4$

Point (3, −1) $\Rightarrow (3)^2 + (-1)^2 + 2g(3) + 2f(-1) + c = 0$

$\Rightarrow 9 + 1 + 6g - 2f + c = 0$

$\Rightarrow 6g - 2f + c = -10$

$c = 0 \Rightarrow 4g + 0 = -4$

$4g = -4 \quad \Rightarrow \quad g = -1$

Hence, $6(-1) - 2f + 0 = -10$

$-2f = -4$

$f = 2$

$\Rightarrow x^2 + y^2 + 2(-1)x + 2(2)y + 0 = 0$

$\Rightarrow x^2 + y^2 - 2x + 4y = 0$

12. General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

Point (0, 0) $\Rightarrow (0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$

$\Rightarrow 0 + 0 + 0 + 0 + c = 0 \quad \Rightarrow \quad c = 0$

Point (−2, 4) $\Rightarrow (-2)^2 + (4)^2 + 2g(-2) + 2f(4) + c = 0$

$\Rightarrow 4 + 16 - 4g + 8f + c = 0$

$\Rightarrow -4g + 8f + c = -20$

$c = 0 \Rightarrow -4g + 8f = -20$

$\Rightarrow -g + 2f = -5$

Point (−1, 7) $\Rightarrow (-1)^2 + (7)^2 + 2g(-1) + 2f(7) + c = 0$

$\Rightarrow 1 + 49 - 2g + 14f + c = 0$

$\Rightarrow -2g + 14f + c = -50$

$c = 0 \Rightarrow -2g + 14f = -50$

$\Rightarrow -g + 7f = -25$

$-g + 2f = -5$

$\underline{-g + 7f = -25}$ (subtracting)

$-5f = +20$

$\Rightarrow f = -4$

$\Rightarrow -g + 2(-4) = -5$

$\Rightarrow -g - 8 = -5$

$\Rightarrow -g = 3 \quad \Rightarrow \quad g = -3$

$\Rightarrow x^2 + y^2 + 2(-3)x + 2(-4)y + 0 = 0$

$\Rightarrow x^2 + y^2 - 6x - 8y = 0$

13. Centre $(-g, -f)$ on line $x + 2y - 6 = 0$

$$\Rightarrow -g - 2f - 6 = 0 \quad \Rightarrow \quad -g - 2f = 6$$

Point (3, 5) $\Rightarrow (3)^2 + (5)^2 + 2g(3) + 2f(5) + c = 0$

$$\Rightarrow 9 + 25 + 6g + 10f + c = 0$$

$$\Rightarrow 6g + 10f + c = -34$$

Point $(-1, 3)$ $\Rightarrow (-1)^2 + (3)^2 + 2g(-1) + 2f(3) + c = 0$

$$\Rightarrow 1 + 9 - 2g + 6f + c = 0$$

$$\Rightarrow -2g + 6f + c = -10$$

$$\begin{array}{r} -2g + 6f + c = -10 \\ \underline{6g + 10f + c = -34} \\ -8g - 4f = 24 \end{array} \quad \text{(subtracting)}$$

$$\Rightarrow -2g - f = 6$$

$$\begin{array}{r} -2g - f = 6 \\ \underline{-2g - 4f = 12} \\ 3f = -6 \end{array}$$

$$\Rightarrow f = -2$$

$$\Rightarrow -g - 2(-2) = 6$$

$$\Rightarrow -g + 4 = 6$$

$$\Rightarrow -g = 2 \Rightarrow g = -2$$

Hence, $6(-2) + 10(-2) + c = -34$

$$\Rightarrow -12 - 20 + c = -34$$

$$\Rightarrow c = -2$$

$$\Rightarrow x^2 + y^2 + 2(-2)x + 2(-2)y - 2 = 0$$

$$\Rightarrow x^2 + y^2 - 4x - 4y - 2 = 0$$

14. Centre $(-g, -f)$ is on x-axis $\Rightarrow -f = 0$

$$\Rightarrow f = 0$$

Point (4, 5) $\Rightarrow (4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$

$$\Rightarrow 16 + 25 + 8g + 10f + c = 0$$

$$\Rightarrow 8g + 10f + c = -41$$

$f = 0 \Rightarrow 8g + 10(0) + c = -41 \Rightarrow 8g + c = -41$

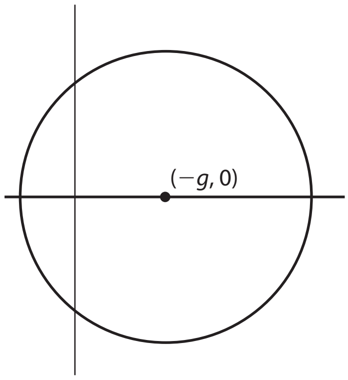
Point $(-2, 3)$ $\Rightarrow (-2)^2 + (3)^2 + 2g(-2) + 2f(3) + c = 0$

$$\Rightarrow 4 + 9 - 4g + 6f + c = 0$$

$$\Rightarrow -4g + 6f + c = -13$$

$f = 0 \Rightarrow -4g + 6(0) + c = -13 \Rightarrow -4g + c = -13$

$$\begin{array}{r} 8g + c = -41 \\ \underline{-4g + c = -13} \\ 12g \quad = -28 \end{array} \quad \text{(subtracting)}$$



$$\Rightarrow g = \frac{-28}{12} = -\frac{7}{3}$$

$$\Rightarrow -4\left(-\frac{7}{3}\right) + c = -13$$

$$\Rightarrow \frac{28}{3} + c = -13$$

$$\Rightarrow c = -13 - \frac{28}{3} = -\frac{67}{3}$$

Hence, $x^2 + y^2 + 2\left(-\frac{7}{3}\right)x + 2(0)y - \frac{67}{3} = 0$

$$\Rightarrow 3x^2 + 3y^2 - 14x - 67 = 0$$

15. (i) $x^2 + y^2 - 4x - 6y + k = 0$

$2g = -4 \quad \Rightarrow \quad g = -2$

$2f = -6 \quad \Rightarrow \quad f = -3$

$\Rightarrow$ centre $(-g, -f) = (2, 3)$

circle touches x-axis $\Rightarrow r = |-3| = 3$

(ii) Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-3)^2 - k} = 3$

$\Rightarrow \quad 4 + 9 - k = 9$

$\Rightarrow \quad -k = -4 \Rightarrow k = 4$

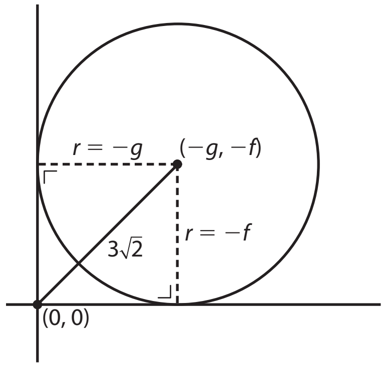
and point T = (2, 0).

16. (i) Centre $= (-g, -f)$

$\Rightarrow \quad r = -g \quad$ and $\quad r = -f$

$\Rightarrow -g = -f \quad \Rightarrow \quad g = f$

Centre in the first quadrant $\Rightarrow g, f < 0$



(ii) $(0, 0)\ (-g, -f) \quad \Rightarrow \quad$ distance $= 3\sqrt{2}$

$\Rightarrow \quad \sqrt{(-g - 0)^2 + (-f - 0)^2} = 3\sqrt{2}$

$\Rightarrow \quad g^2 + f^2 = (3\sqrt{2})^2 = 18$

$g = f \quad \Rightarrow \quad g^2 + g^2 = 18$

$\Rightarrow \quad 2g^2 = 18$

$\Rightarrow \quad g^2 = 9$

$\Rightarrow \quad g = -3$ or $g = 3$ (Not valid)

$\Rightarrow \quad -g = 3, -f = 3$

Centre (3, 3) radius = 3

Equation of circle: $(x - 3)^2 + (y - 3)^2 = 3^2$

$\Rightarrow \quad x^2 - 6x + 9 + y^2 - 6y + 9 = 9$

$\Rightarrow \quad x^2 + y^2 - 6x - 6y + 9 = 0$

17. (i) The perpendicular to a tangent at the point of contact passes through the centre of the circle.

(ii) Tangent: $3x + 2y - 12 = 0$

$\Rightarrow$ slope $= -\frac{a}{b} = -\frac{3}{2} \Rightarrow$ perpendicular slope $= \frac{2}{3}$

point B = (4, 0)

$\Rightarrow$ equation: $y - 0 = \frac{2}{3}(x - 4)$

$\Rightarrow$ equation: $y - 0 = \frac{2}{3}(x - 4)$

$\Rightarrow \quad 3y = 2x - 8$

$\Rightarrow \quad 2x - 3y - 8 = 0$

(iii) A = (3, −5), B = (4, 0) $\quad \Rightarrow \quad$ midpoint $= \left(\frac{3+4}{2}, \frac{-5+0}{2}\right) = \left(\frac{7}{2}, \frac{-5}{2}\right)$

Slope AB $= \frac{0+5}{4-3} = \frac{5}{1} \quad \Rightarrow \quad$ perpendicular slope $= -\frac{1}{5}$

$\Rightarrow \quad$ equation: $y + \frac{5}{2} = -\left(x - \frac{7}{2}\right)$

$\Rightarrow \quad 5y + \frac{25}{2} = -x + \frac{7}{2}$

$\Rightarrow \quad x + 5y + 9 = 0$

(iv) $2x - 3y = 8$

$x + 5y = -9$ $\times 2$

$2x - 3y = 8$

$2x + 10y = -18$ (subtracting)

$-13y = 26 \Rightarrow y = -2$

$\Rightarrow 2x - 3(-2) = 8$

$\Rightarrow 2x + 6 = 8$

$\Rightarrow 2x = 2 \Rightarrow x = 1$

$\Rightarrow$ centre $= (1, -2)$

point A $= (3, -5)$

$\Rightarrow$ radius $= \sqrt{(3-1)^2 + (-5+2)^2}$

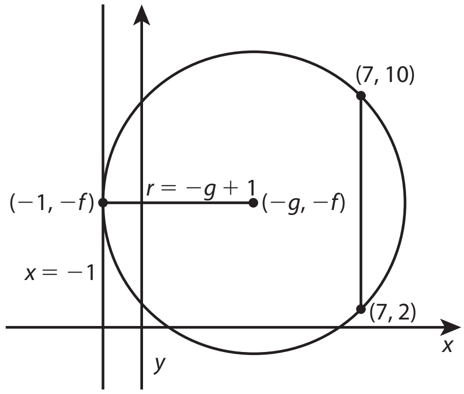
$= \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

(v) Equation of circle: $(x-1)^2 + (y+2)^2 = (\sqrt{13})^2 = 13$

$\Rightarrow x^2 - 2x + 1 + y^2 + 4y + 4 = 13$

$\Rightarrow x^2 + y^2 - 2x + 4y - 8 = 0$

18. (7, 2) (7, 10) Equation of chord: $x = 7$



Midpoint $= (7, 6)$

$\Rightarrow$ Equation of perpendicular bisector: $y = 6$

$\Rightarrow -f = 6 \Rightarrow f = -6$

Radius = distance between $(-g, -f)$ and $(-1, -f)$

Radius $= -1 + g$

$\Rightarrow \sqrt{g^2 + f^2 - c} = -1 + g$

$\Rightarrow g^2 + f^2 - c = (-1+g)^2 = 1 - 2g + g^2$

$\Rightarrow f^2 - c = 1 - 2g$

$f = -6 \Rightarrow (-6)^2 - c = 1 - 2g$

$\Rightarrow 36 - c = 1 - 2g$

$\Rightarrow 2g - c = -35$

Point (7, 2) $\Rightarrow (7)^2 + (2)^2 + 2g(7) + 2f(2) + c = 0$

$\Rightarrow 14g + 4f + c = -53$

$f = -6 \Rightarrow 14g + 4(-6) + c = -53$

$14g + c = -53 + 24 = -29$

$2g - c = -35$

$14g + c = -29$ (adding)

$16g = -64$

$\Rightarrow g = -4$

$14(-4) + c = -29$

$\Rightarrow -56 + c = -29$

$\Rightarrow c = -29 + 56 = 27$

Hence, equation of circle: $x^2 + y^2 + 2(-4)x + 2(-6)y + 27 = 0$

$\Rightarrow x^2 + y^2 - 8x - 12y + 27 = 0$

19. Radius $= \sqrt{20} \Rightarrow \sqrt{g^2 + f^2 - c} = \sqrt{20}$

$\Rightarrow g^2 + f^2 - c = 20$

Point $(-1, 3)$ $\Rightarrow (-1)^2 + (3)^2 + 2g(-1) + 2f(3) + c = 0$

$\Rightarrow 1 + 9 - 2g + 6f + c = 0$

$\Rightarrow -2g + 6f + c = -10$

and $g^2 + f^2 - c = 20$ (adding)

$\Rightarrow g^2 - 2g + f^2 + 6f = 10$

Centre $(-g, -f)$ on line $x + y = 0$

$\Rightarrow \quad -g - f = 0$

$\Rightarrow \quad g = -f$

Hence, $(-f)^2 - 2(-f) + f^2 + 6f = 10$

$\Rightarrow \quad f^2 + 2f + f^2 + 6f - 10 = 0$

$\Rightarrow \quad 2f^2 + 8f - 10 = 0$

$\Rightarrow \quad f^2 + 4f - 5 = 0$

$\Rightarrow \quad (f + 5)(f - 1) = 0$

$\Rightarrow \quad f = -5 \quad \text{or} \quad f = 1$

$\Rightarrow \quad g = -(-5) = 5 \quad \text{or} \quad g = -(1) = -1$

$\Rightarrow (5)^2 + (-5)^2 - c = 20$ or $(-1)^2 + (1)^2 - c = 20$

$\Rightarrow \quad 25 + 25 - c = 20 \quad \Rightarrow \quad 1 + 1 - c = 20$

$\Rightarrow \quad c = 30 \quad \Rightarrow \quad c = -18$

C_1: $x^2 + y^2 + 10x - 10y + 30 = 0$

C_2: $x^2 + y^2 - 2x + 2y - 18 = 0$

20. (i) $(x - 5)^2 + (y - 3)^2 = 4$

Centre $= (5, 3)$

$r^2 = 4 \quad \Rightarrow \quad r = 2$

Line is parallel to x-axis and goes through (5,1)

$\Rightarrow$ equation: $y = 1$

(ii) Rear wheel radius $= 3(2) = 6$

10 units from $(5,1) = (15, 1)$

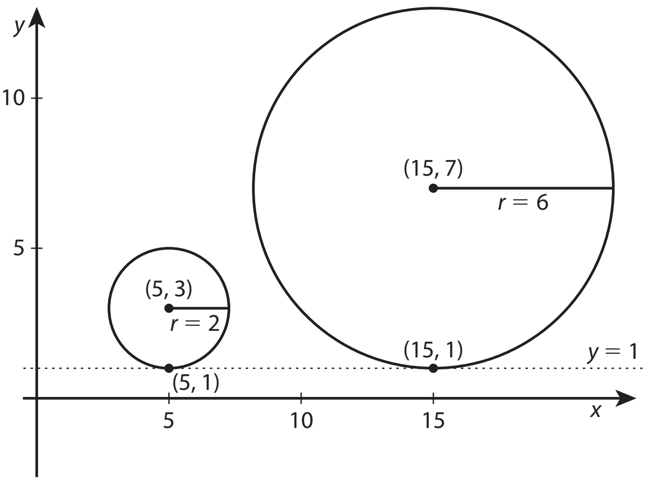
$\Rightarrow$ centre of rear wheel's rim $= (15, 7)$

$\Rightarrow$ equation: $(x - 15)^2 + (y - 7)^2$

$= (6)^2 = 36$

(iii) 2 units to the left $\Rightarrow$ centre $= (13, 7)$ and radius $= 6$

$\Rightarrow$ equation: $(x - 13)^2 + (y - 7)^2 = (6)^2 = 36$



Exercise 11.4

1. Circle: $x^2 + y^2 = 8 \quad \Rightarrow \quad$ centre $= (0, 0)$, point P(2, 2)

$\Rightarrow \quad$ slope of the radius [op] $= \dfrac{2 - 0}{2 - 0} = \dfrac{2}{2} = 1$

$\Rightarrow \quad$ perpendicular slope $= -1$

Equation of tangent: $y - 2 = -1(x - 2)$

$\Rightarrow \quad y - 2 = -x + 2$

$\Rightarrow \quad x + y = 4$

2. Circle: $x^2 + y^2 = 10 \quad \Rightarrow \quad$ centre $= (0, 0)$, point $(-3, 1)$

$\Rightarrow \quad$ slope of the radius $= \dfrac{1 - 0}{-3 - 0} = -\dfrac{1}{3}$

$\Rightarrow \quad$ perpendicular slope $= 3$

Equation of tangent: $\quad y - 1 = 3(x + 3)$

$\Rightarrow \quad y - 1 = 3x + 9$

$\Rightarrow \quad 3x - y + 10 = 0$

3. Circle: $x^2 + y^2 = 17 \quad \Rightarrow \quad$ centre $= (0, 0)$, point $(4, -1)$

$\Rightarrow \quad$ slope of radius $= \dfrac{-1-0}{4-0} = -\dfrac{1}{4}$

$\Rightarrow \quad$ perpendicular slope $= 4$

Equation of tangent: $y + 1 = 4(x - 4)$

$\Rightarrow \quad y + 1 = 4x - 16$

$\Rightarrow \quad 4x - y - 17 = 0$

4. Circle: $(x - 1)^2 + (y + 2)^2 = 20$

(i) Point P $(3, 2)$ $\Rightarrow \quad (3-1)^2 + (2+2)^2 = 20$

$\Rightarrow \quad (2)^2 + (4)^2 = 20$

$\Rightarrow \quad 4 + 16 = 20$

$\Rightarrow \quad 20 = 20 \quad \Rightarrow$ True

(ii) Centre $= (1, -2)$

(iii) $C(1, -2)$ $P(3, 2)$ $\Rightarrow \quad$ slope CP $= \dfrac{2+2}{3-1} = \dfrac{4}{2} = 2$

$\Rightarrow \quad$ perpendicular slope $= -\dfrac{1}{2}$

Equation of tangent: $\quad y - 2 = -\dfrac{1}{2}(x - 3)$

$\Rightarrow \quad 2y - 4 = -x + 3$

$\Rightarrow \quad x + 2y - 7 = 0$

5. Circle: $(x + 4)^2 + (y - 3)^2 = 17$

Centre $= (-4, 3)$

Point $(0, 2)$

Slope radius $= \dfrac{3-2}{-4-0} = -\dfrac{1}{4}$

Perpendicular slope $= 4$

Equation of tangent: $\quad y - 2 = 4(x-0)$

$\Rightarrow \quad y - 2 = 4x$

$\Rightarrow \quad 4x - y + 2 = 0$

6. Circle: $\quad x^2 + y^2 - 4x + 10y - 8 = 0$

$2g = -4 \quad \Rightarrow \quad g = -2$

$2f = 10 \quad \Rightarrow \quad f = 5$

Centre $= (-g, -f) = (2, -5)$

Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (5)^2 + 8}$

$= \sqrt{4 + 25 + 8} = \sqrt{37}$

Point $(3,1)$

Slope radius $= \dfrac{-5-1}{2-3} = \dfrac{-6}{-1} = 6$

Perpendicular slope $= -\dfrac{1}{6}$

Equation of tangent: $\quad y - 1 = -\dfrac{1}{6}(x - 3)$

$\Rightarrow \quad 6y - 6 = -x + 3$

$\Rightarrow \quad x + 6y - 9 = 0$

7. Point (0, 0) line: $3x - 4y - 25 = 0$

Perpendicular distance $= \dfrac{|3(0) - 4(0) - 25|}{\sqrt{(3)^2 + (-4)^2}} = \dfrac{|-25|}{\sqrt{25}}$

$= \dfrac{25}{5} = 5$

Circle $x^2 + y^2 = 25$

$\Rightarrow$ centre $= (0, 0)$ radius $= \sqrt{25} = 5$

Perpendicular distance from (0, 0) to line $= 5$

Hence, line $3x - 4y - 25 = 0$ is a tangent to the circle.

8. Circle: $x^2 + y^2 - 6x - 4y + 8 = 0$

$2g = -6 \quad \Rightarrow \quad g = -3$

$2f = -4 \quad \Rightarrow \quad f = -2$

$\Rightarrow$ centre $= (-g, -f) = (3, 2)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-3)^2 + (-2)^2 - 8} = \sqrt{5}$

line: $x + 2y - 12 = 0$

Perpendicular distance $= \dfrac{|1(3) + 2(2) - 12|}{\sqrt{(1)^2 + (2)^2}} = \dfrac{|-5|}{\sqrt{5}} = \sqrt{5}$

Hence, line $x + 2y - 12 = 0$ is a tangent to the circle because the perpendicular distance from the centre of the circle to the line = the radius length.

9. Circle: $x^2 + y^2 - 6x - 2y - 15 = 0$

$2g = -6 \quad \Rightarrow \quad g = -3$

$2f = -2 \quad \Rightarrow \quad f = -1$

$\Rightarrow$ centre $= (-g, -f) = (3, 1)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-3)^2 + (-1)^2 + 15} = \sqrt{25} = 5$

Tangent: $3x + 4y + c = 0 \quad \Rightarrow \quad$ perpendicular distance $= \dfrac{|3(3) + 4(1) + c|}{\sqrt{(3)^2 + (4)^2}} = 5$

$\Rightarrow \dfrac{|13 + c|}{\sqrt{25}} = \dfrac{|13 + c|}{5} = 5$

$\Rightarrow |13 + c| = 25 \quad \Rightarrow \quad 13 + c = -25 \quad \Rightarrow \quad c = -38$

OR $\quad 13 + c = 25 \quad \Rightarrow \quad c = 12$

10. Circle: $x^2 + y^2 + 4x - 4y - 5 = 0$

$\Rightarrow 2g = 4 \quad \Rightarrow \quad g = 2$

and $2f = -4 \quad \Rightarrow \quad f = -2$

$\Rightarrow$ centre $= (-g, -f) = (-2, 2)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (-2)^2 + 5} = \sqrt{4 + 4 + 5} = \sqrt{13}$

Tangent: $2x - ky - 3 = 0$

$\Rightarrow$ perpendicular distance $= \dfrac{|2(-2) - k(2) - 3|}{\sqrt{(2)^2 + (-k)^2}} = \sqrt{13}$

$\Rightarrow \quad \dfrac{|-4 - 2k - 3|}{\sqrt{4 + k^2}} = \sqrt{13}$

$\Rightarrow \quad |-2k - 7| = \sqrt{13}\sqrt{4 + k^2}$

$\Rightarrow \quad |-2k - 7| = \sqrt{52 + 13k^2}$

$\Rightarrow \quad (-2k - 7)^2 = 52 + 13k^2$

$\Rightarrow \quad 4k^2 + 28k + 49 - 52 - 13k^2 = 0$

$\Rightarrow \quad -9k^2 + 28k - 3 = 0$

$\Rightarrow \quad 9k^2 - 28k + 3 = 0$

$\Rightarrow \quad (k - 3)(9k - 1) = 0$

$\Rightarrow \quad k = 3, k = \dfrac{1}{9}$

11. $x^2 + y^2 = 13 \quad \Rightarrow \quad \text{centre} = (0, 0), \text{radius} = \sqrt{13}$

Point $(-2, 3) \quad \Rightarrow \quad \text{slope radius} = \dfrac{3-0}{-2-0} = -\dfrac{3}{2}$

$\Rightarrow \quad \text{perpendicular slope} = \dfrac{2}{3}$

$\Rightarrow$ Equation of tangent: $\quad y - 3 = \dfrac{2}{3}(x + 2)$

$\Rightarrow \quad 3y - 9 = 2x + 4$

$\Rightarrow \quad 2x - 3y + 13 = 0$

Circle: $x^2 + y^2 - 10x + 2y - 26 = 0$

$\Rightarrow$ centre $= (5, -1)$ and radius $= \sqrt{(-5)^2 + (1)^2 + 26} = \sqrt{52}$

$= 2\sqrt{13}$

Line: $2x - 3y + 13 = 0$

$$\Rightarrow \quad \text{perpendicular distance} = \frac{|2(5) - 3(-1) + 13|}{\sqrt{(2)^2 + (-3)^2}}$$

$$= \frac{|10 + 3 + 13|}{\sqrt{13}}$$

$$= \frac{26}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}}$$

$$= \frac{26\sqrt{13}}{13}$$

$$= 2\sqrt{13}$$

Hence, $2x - 3y + 13 = 0$ is a tangent to both circles.

12. Centre $(2, -1)$, tangent: $3x + y = 0$

$$\text{Perpendicular distance} = \frac{|3(2) + 1(-1)|}{\sqrt{(3)^2 + (1)^2}} = \frac{5}{\sqrt{10}} = \frac{5\sqrt{10}}{\sqrt{10}\sqrt{10}}$$

$$= \frac{5\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{2} = \text{radius}$$

Equation of circle: $(x - 2)^2 + (y + 1)^2 = \left(\dfrac{\sqrt{10}}{2}\right)^2 = \dfrac{10}{4} = \dfrac{5}{2}$

13. Point $= (0, 0)$, slope $= m$

Equation of line: $y - 0 = m(x - 0)$

$\Rightarrow \quad y = mx \quad \Rightarrow \quad mx - y = 0$

Circle: $x^2 + y^2 - 4x - 2y + 4 = 0$

$2g = -4 \quad \Rightarrow \quad g = -2$

$2f = -2 \quad \Rightarrow \quad f = -1$

$\Rightarrow$ centre $= (-g, -f) = (2, 1)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 - 4} = \sqrt{4 + 1 - 4}$

$= \sqrt{1} = 1$

Hence, perpendicular distance from centre $(2, 1)$ to the tangent $mx - y = 0$ equals radius $= 1$.

$$\Rightarrow \quad \frac{|m(2) - 1(1)|}{\sqrt{(m)^2 + (-1)^2}} = 1$$

$\Rightarrow \quad 2m - 1 = 1\sqrt{m^2 + 1}$

$\Rightarrow \quad (2m - 1)^2 = m^2 + 1$

$\Rightarrow \quad 4m^2 - 4m + 1 = m^2 + 1$

$\Rightarrow \quad 3m^2 - 4m = 0$

$\Rightarrow \quad m(3m - 4) = 0$

$\Rightarrow \quad m = 0 \quad \text{or} \quad 3m = 4$

$\Rightarrow \quad m = \frac{4}{3}$

Hence, $m = 0 \quad \Rightarrow \quad$ tangent: $(0)x - y = 0$

$\Rightarrow \quad y = 0$

and $m = \frac{4}{3} \quad \Rightarrow \quad$ tangent: $\frac{4}{3}(x) - y = 0$

$\Rightarrow \quad 4x - 3y = 0$

14. Point (3, 5), slope $= m$

$\Rightarrow$ Equation of line: $\quad y - 5 = m(x - 3)$

$\Rightarrow \quad y - 5 = mx - 3m$

$\Rightarrow \quad mx - y - 3m + 5 = 0$

Circle: $x^2 + y^2 + 2x - 4y - 4 = 0$

$\Rightarrow \quad$ centre $= (-1, 2)$, radius $= \sqrt{(1)^2 + (-2)^2 + 4} = \sqrt{9} = 3$

Perpendicular distance $= \dfrac{|m(-1) - 1(2) - 3m + 5|}{\sqrt{(m)^2 + (-1)^2}} = 3$

$\Rightarrow \quad \dfrac{|-m - 2 - 3m + 5|}{\sqrt{m^2 + 1}} = 3$

$\Rightarrow \quad |-4m + 3| = 3\sqrt{m^2 + 1}$

$\Rightarrow \quad (-4m + 3)^2 = (3\sqrt{m^2 + 1})^2$

$\Rightarrow \quad 16m^2 - 24m + 9 = 9(m^2 + 1) = 9m^2 + 9$

$\Rightarrow \quad 7m^2 - 24m = 0$

$\Rightarrow \quad m(7m - 24) = 0$

$\Rightarrow \quad m = 0, 7m = 24$

$\Rightarrow \quad m = \frac{24}{7}$

$m = 0 \quad \Rightarrow \quad$ tangent: $(0)x - y - 3(0) + 5 = 0$

$\Rightarrow \quad y - 5 = 0$

$m = \frac{24}{7} \quad \Rightarrow \quad$ tangent: $\left(\frac{24}{7}\right)x - y - 3\left(\frac{24}{7}\right) + 5 = 0$

$\Rightarrow \quad 24x - 7y - 72 + 35 = 0$

$\Rightarrow \quad 24x - 7y - 37 = 0$

15. Line parallel to $3x + 4y - 6 = 0$

is $3x + 4y + c = 0$

Circle: $x^2 + y^2 - 2x - 2y - 7 = 0$

$\Rightarrow \quad$ centre $= (1, 1)$, radius $= \sqrt{(-1)^2 + (-1)^2 + 7} = \sqrt{9} = 3$

Perpendicular distance: $\dfrac{|3(1) + 4(1) + c|}{\sqrt{(3)^2 + (4)^2}} = 3$

$\Rightarrow \quad \dfrac{|7 + c|}{\sqrt{25}} = \dfrac{|7 + c|}{5 = 3}$

$\Rightarrow \quad |7 + c| = 15$

$\Rightarrow \quad 7 + c = 15 \quad \text{or} \quad 7 + c = -15$

$\Rightarrow \quad c = 8 \quad \text{or} \quad c = -22$

Hence, tangents are $3x + 4y + 8 = 0$

and $3x + 4y - 22 = 0$.

16. (i) Centre $= (3, 5)$, tangent: $2x - y + 4 = 0$

Radius = perpendicular distance $= \dfrac{|2(3) - 1(5) + 4|}{\sqrt{(2)^2 + (-1)^2}}$

$$= \frac{|6 - 5 + 4|}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

(ii) Equation of circle: $(x - 3)^2 + (y - 5)^2 = (\sqrt{5})^2 = 5$

(iii) Point $(1, 4)$, centre $= (3, 5)$

Slope radius $= \dfrac{5 - 4}{3 - 1} = \dfrac{1}{2}$

$\Rightarrow$ perpendicular slope $= -2$

Equation of tangent: $y - 4 = -2(x - 1)$

$\Rightarrow \quad y - 4 = -2x + 2$

$\Rightarrow \quad 2x + y - 6 = 0$

17. (i) Circle: $x^2 + y^2 - 10kx + 6y + 60 = 0$

$2g = -10k \quad \Rightarrow \quad g = -5k$

$2f = 6 \quad \Rightarrow \quad f = 3$

$\Rightarrow$ Centre $= (-g, -f) = (5k, -3)$

(ii) Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-5k)^2 + (3)^2 - 60} = 7$

$\Rightarrow \quad 25k^2 + 9 - 60 = 49$

$\Rightarrow \quad 25k^2 = 100$

$\Rightarrow \quad k^2 = 4$

$\Rightarrow \quad k = 2$ or $k = -2$ (Not valid)

(iii) Tangent: $3x + 4y + d = 0$, centre $= (10, -3)$

Perpendicular distance $= \dfrac{|3(10) + 4(-3) + d|}{\sqrt{(3)^2 + (4)^2}} = 7$

$\Rightarrow \quad \dfrac{|30 - 12 + d|}{\sqrt{25}} = \dfrac{|18 + d|}{5} = 7$

$\Rightarrow \quad |18 + d| = 35$

$\Rightarrow \quad 18 + d = 35 \quad$ or $\quad 18 + d = -35$

$\Rightarrow \quad d = 17 \quad$ or $\quad d = -53$

18. Circle: $x^2 + y^2 + 4x - 2y - 4 = 0$

(i) $2g = 4 \quad \Rightarrow \quad g = 2$

$2f = -2 \quad \Rightarrow \quad f = -1$

Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (-1)^2 + 4} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

(ii) The perpendicular to a tangent at the point of contact passes through the centre of the circle.

(iii) $C(-2, 1)$, $P(3, 1) \quad \Rightarrow \quad |CP| = \sqrt{(3 + 2)^2 + (1 - 1)^2}$

$= \sqrt{25 + 0} = 5$

Radius $= |CT| = 3$

Pythagoras' theorem $\Rightarrow \quad |PT|^2 + |CT|^2 = |CP|^2$

$\Rightarrow \quad |PT|^2 + (3)^2 = (5)^2$

$\Rightarrow \quad |PT|^2 + 9 = 25$

$\Rightarrow \quad |PT|^2 = 25 - 9 = 16$

$\Rightarrow \quad |PT| = \sqrt{16} = 4$

19. Circle: $x^2 + y^2 - 14x - 2y + 34 = 0$

$2g = -14 \quad \Rightarrow \quad g = -7$

$2f = -2 \quad \Rightarrow \quad f = -1$

$\Rightarrow$ Centre $= (-g, -f) = (7, 1)$ and radius $= \sqrt{(-7)^2 + (-1)^2 - 34}$

$= \sqrt{16} = 4$

Point $(2, 5) \quad \Rightarrow \quad$ hypotenuse $= \sqrt{(7-2)^2 + (1-5)^2} = \sqrt{(5)^2 + (-4)^2} = \sqrt{41}$

Radius $= 4$

$\Rightarrow$ Length of tangent $= \sqrt{(\text{hypotenuse})^2 - (\text{radius})^2}$

$= \sqrt{41 - 16}$

$= \sqrt{25}$

$= 5$

20. Centre $= (4, 2)$, radius $= \sqrt{(-4)^2 + (-2)^2 - 10}$

$= \sqrt{16 + 4 - 10} = \sqrt{10}$

Point $(0, 0) \quad \Rightarrow \quad$ hypotenuse $= \sqrt{(4-0)^2 + (2-0)^2}$

$= \sqrt{4^2 + 2^2} = \sqrt{20}$

$\Rightarrow$ Length of tangent $= \sqrt{(\text{hypotenuse})^2 - (\text{radius})^2}$

$= \sqrt{20 - 10} = \sqrt{10}$

21. Centre $= (2, 5)$, radius $= \sqrt{16} = 4$

Point $(7, 8) \quad \Rightarrow \quad$ hypotenuse $= \sqrt{(7-2)^2 + (8-5)^2}$

$= \sqrt{5^2 + 3^2} = \sqrt{34}$

$\Rightarrow$ Length of tangent $= \sqrt{(\text{hypotenuse})^2 - (\text{radius})^2}$

$= \sqrt{34 - 16} = \sqrt{18} = 3\sqrt{2}$

22. Centre $= (2, 3)$, radius $= \sqrt{(-2)^2 + (-3)^2 - c}$

$= \sqrt{4 + 9 - c} = \sqrt{13 - c}$

Point $(1, 1) \quad \Rightarrow \quad$ hypotenuse $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$

Length of tangent $= 2$

$\Rightarrow$ $(\text{radius})^2 + (2)^2 = (\sqrt{5})^2$

$\Rightarrow$ $(\text{radius})^2 + 4 = 5$

$\Rightarrow$ $(\text{radius})^2 = 5 - 4 = 1$

$\Rightarrow$ radius $= 1$

Hence, $\sqrt{13 - c} = 1$

$\Rightarrow \quad 13 - c = (1)^2 = 1$

$\Rightarrow \quad c = 12$

Exercise 11.5

1. Line: $3x - y + 5 = 0 \quad \Rightarrow \quad y = 3x + 5$

Circle: $\quad x^2 + y^2 = 5 \quad \Rightarrow \quad x^2 + (3x + 5)^2 = 5$

$\Rightarrow \quad x^2 + 9x^2 + 30x + 25 - 5 = 0$

$\Rightarrow \quad 10x^2 + 30x + 20 = 0$

$\Rightarrow \quad x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x + 1)(x + 2) = 0$

$\Rightarrow \quad x = -1, \qquad x = -2$

$\Rightarrow \quad y = 3(-1) + 5 \qquad y = 3(-2) + 5$

$= 2 \qquad\qquad = -1$

Points $= (-1, 2)$ and $(-2, -1)$

2. Circle: $x^2 + y^2 = 10 \Rightarrow$ centre $= (0, 0)$, radius $= \sqrt{10}$

Line: $x - 3y - 10 = 0$

Perpendicular distance $= \dfrac{|1(0) - 3(0) - 10|}{\sqrt{(1)^2 + (-3)^2}} = \dfrac{|-10|}{\sqrt{10}} = \sqrt{10}$ = radius

Hence, the line is a tangent to the circle.

$$\begin{aligned}
\text{Line: } x = 3y + 10 \quad &\Rightarrow & \text{Circle: } (3y + 10)^2 + y^2 &= 10 \\
&\Rightarrow & 9y^2 + 60y + 100 + y^2 - 10 &= 0 \\
&\Rightarrow & 10y^2 + 60y + 90 &= 0 \\
&\Rightarrow & y^2 + 6y + 9 &= 0 \\
&\Rightarrow & (y + 3)(y + 3) &= 0 \\
&\Rightarrow & y &= -3 \\
&\Rightarrow & x &= 3(-3) + 10 \\
&\Rightarrow & x &= -9 + 10 = 1
\end{aligned}$$

$\Rightarrow$ Point of contact $= (1, -3)$

3. Line: $2x - y - 5 = 0 \quad \cap \quad$ Circle: $x^2 + y^2 = 5$

$$\begin{aligned}
\Rightarrow \quad y = 2x - 5 \quad &\Rightarrow & x^2 + (2x - 5)^2 &= 5 \\
&\Rightarrow & x^2 + 4x^2 - 20x + 25 - 5 &= 0 \\
&\Rightarrow & 5x^2 - 20x + 20 &= 0 \\
&\Rightarrow & x^2 - 4x + 4 &= 0 \\
&\Rightarrow & (x - 2)(x - 2) &= 0 \\
&\Rightarrow & x &= 2 \\
&\Rightarrow & y &= 2(2) - 5 = -1
\end{aligned}$$

$\Rightarrow$ Point of intersection $= (2, -1)$

4. (i) $x + y = 6 \quad \cap \quad x^2 + y^2 + 2x - 4y - 20 = 0$

$$\begin{aligned}
\Rightarrow \quad y = -x + 6 \quad &\Rightarrow & x^2 + (-x + 6)^2 + 2x - 4(-x + 6) - 20 &= 0 \\
&\Rightarrow & x^2 + x^2 - 12x + 36 + 2x + 4x - 24 - 20 &= 0 \\
&\Rightarrow & 2x^2 - 6x - 8 &= 0 \\
&\Rightarrow & x^2 - 3x - 4 &= 0 \\
&\Rightarrow & (x - 4)(x + 1) &= 0
\end{aligned}$$

$\Rightarrow x = 4 \quad$ or $\quad x = -1$

$\Rightarrow y = -4 + 6 = 2 \quad$ or $\quad y = -(-1) + 6 = 7$

$\Rightarrow$ Points $= (4, 2)$ and $(-1, 7)$

(ii) $2x + y - 2 = 0 \quad \cap \quad x^2 + y^2 - 10x - 4y - 11 = 0$

$$\begin{aligned}
\Rightarrow \quad y = -2x + 2 \quad &\Rightarrow & x^2 + (-2x + 2)^2 - 10x - 4(-2x + 2) - 11 &= 0 \\
&\Rightarrow & x^2 + 4x^2 - 8x + 4 - 10x + 8x - 8 - 11 &= 0 \\
&\Rightarrow & 5x^2 - 10x - 15 &= 0 \\
&\Rightarrow & x^2 - 2x - 3 &= 0 \\
&\Rightarrow & (x + 1)(x - 3) &= 0
\end{aligned}$$

$\Rightarrow x = -1 \quad$ or $\quad x = 3$

$\Rightarrow y = -2(-1) + 2 = 4 \quad$ or $\quad y = -2(3) + 2 = -4$

$\Rightarrow$ Points $= (-1, 4)$ and $(3, -4)$

(iii) $3x - y - 5 = 0 \quad \cap \quad x^2 + y^2 - 2x + 4y - 5 = 0$

$$
\begin{aligned}
\Rightarrow \quad y = 3x - 5 \quad &\Rightarrow \quad x^2 + (3x - 5)^2 - 2x + 4(3x - 5) - 5 = 0 \\
&\Rightarrow \quad x^2 + 9x^2 - 30x + 25 - 2x + 12x - 20 - 5 = 0 \\
&\Rightarrow \quad 10x^2 - 20x = 0 \\
&\Rightarrow \quad x^2 - 2x = 0 \\
&\Rightarrow \quad x(x - 2) = 0 \\
&\Rightarrow \quad x = 0 \quad \text{or} \quad x = 2 \\
&\Rightarrow \quad y = 3(0) - 5 \quad \text{or} \quad y = 3(2) - 5 \\
&\qquad\;\; = -5 \qquad\qquad\quad = 1
\end{aligned}
$$

Points $= (0, -5)$ and $(2, 1)$

5. $x - 2y + 12 = 0 \quad \cap \quad x^2 + y^2 - x - 31 = 0$

$$
\begin{aligned}
\Rightarrow \quad x = 2y - 12 \quad &\Rightarrow \quad (2y - 12)^2 + y^2 - (2y - 12) - 31 = 0 \\
&\Rightarrow \quad 4y^2 - 48y + 144 + y^2 - 2y + 12 - 31 = 0 \\
&\Rightarrow \quad 5y^2 - 50y + 125 = 0 \\
&\Rightarrow \quad y^2 - 10y + 25 = 0 \\
&\Rightarrow \quad (y - 5)(y - 5) = 0 \\
&\Rightarrow \quad y = 5 \\
&\Rightarrow \quad x = 2(5) - 12 = -2
\end{aligned}
$$

$\Rightarrow$ Point of contact $= (-2, 5)$

6. (i) $x - 2y - 1 = 0 \quad \cap \quad x^2 + y^2 + 2x - 8y - 8 = 0$

$$
\begin{aligned}
\Rightarrow \quad x = 2y + 1 \quad &\Rightarrow \quad (2y + 1)^2 + y^2 + 2(2y + 1) - 8y - 8 = 0 \\
&\Rightarrow \quad 4y^2 + 4y + 1 + y^2 + 4y + 2 - 8y - 8 = 0 \\
&\Rightarrow \quad 5y^2 - 5 = 0 \\
&\Rightarrow \quad y^2 - 1 = 0 \\
&\Rightarrow \quad (y - 1)(y + 1) = 0 \\
&\Rightarrow \quad y = 1 \quad \text{or} \quad y = -1 \\
&\Rightarrow \quad x = 2(1) + 1 = 3 \quad \text{or} \quad x = 2(-1) + 1 = -1 \\
&\Rightarrow \quad \text{L} = (3, 1) \quad \text{and} \quad \text{M} = (-1, -1)
\end{aligned}
$$

(ii) $\text{L} = (3, 1) \quad \text{M} = (-1, -1)$

$$\text{Midpoint} = \left(\frac{3 - 1}{2}, \frac{1 - 1}{2}\right) = (1, 0)$$

(iii) Centre $= (1, 0) \quad \text{L} = (3, 1)$

$\Rightarrow$ Radius $= \sqrt{(3 - 1)^2 = (1 - 0)^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$

$\Rightarrow$ Equation of circle: $(x - 1)^2 + (y - 0)^2 = (\sqrt{5})^2$

$\Rightarrow \quad (x - 1)^2 + y^2 = 5$

7. Circle: $x^2 + y^2 - 4x - 6y - 12 = 0$

$$
\begin{aligned}
x\text{-axis} \quad \Rightarrow \quad y = 0 \quad &\Rightarrow \quad x^2 + (0)^2 - 4x - 6(0) - 12 = 0 \\
&\Rightarrow \quad x^2 - 4x - 12 = 0 \\
&\Rightarrow \quad (x - 6)(x + 2) = 0 \\
&\Rightarrow \quad x = 6 \quad \text{or} \quad x = -2
\end{aligned}
$$

Points $= (6, 0)$ and $(-2, 0)$

Hence, length of intercept $= 6 - (-2) = 8$ units

8. Circle: $x^2 + y^2 - 4x + 6y - 7 = 0$

$$
\begin{aligned}
y\text{-axis} \quad \Rightarrow \quad x = 0 \quad &\Rightarrow \quad (0)^2 + y^2 - 4(0) + 6y - 7 = 0 \\
&\Rightarrow \quad y^2 + 6y - 7 = 0 \\
&\Rightarrow \quad (y + 7)(y - 1) = 0 \\
&\Rightarrow \quad y = -7 \quad \text{or} \quad y = 1
\end{aligned}
$$

Points $= (0, -7)$ and $(0, 1)$

Hence, length of chord $= 1 - (-7) = 8$ units

9. Circle: $x^2 + y^2 - 4x + 11y - 12 = 0$

x-axis $\Rightarrow$ $y = 0$ $\Rightarrow$ $x^2 + (0)^2 - 4x + 11(0) - 12 = 0$

$\Rightarrow$ $x^2 - 4x - 12 = 0$

$\Rightarrow$ $(x - 6)(x + 2) = 0$

$\Rightarrow$ $x = 6$ or $x = -2$

$\Rightarrow$ Positive x-axis $= (6, 0)$ $\Rightarrow$ $a = 6$

y-axis $\Rightarrow$ $x = 0$ $\Rightarrow$ $(0)^2 + y^2 - 4(0) + 11y - 12 = 0$

$\Rightarrow$ $y^2 + 11y - 12 = 0$

$\Rightarrow$ $(y - 1)(y + 12) = 0$

$\Rightarrow$ $y = 1$ or $y = -12$

$\Rightarrow$ Positive y-axis $= (0, 1)$ $\Rightarrow$ $b = 1$

10. Circle: $x^2 + y^2 - 4x - 8y - 5 = 0$

x-axis $\Rightarrow$ $y = 0$ $\Rightarrow$ $x^2 + (0)^2 - 4x - 8(0) - 5 = 0$

$\Rightarrow$ $x^2 - 4x - 5 = 0$

$\Rightarrow$ $(x + 1)(x - 5) = 0$

$\Rightarrow$ $x = -1$ or $x = 5$

Points $= (-1, 0)$ and $(5, 0)$

Length of intercept $= 5 - (-1) = 6$ units

11. Circle s_1: $x^2 + y^2 - 3x + 5y - 4 = 0$

Circle s_2: $x^2 + y^2 - x + 4y - 7 = 0$

Common chord is $s_1 - s_2 = 0$

$\Rightarrow$ $x^2 + y^2 - 3x + 5y - 4 - (x^2 + y^2 - x + 4y - 7) = 0$

$\Rightarrow$ $x^2 + y^2 - 3x + 5y - 4 - x^2 - y^2 + x - 4y + 7 = 0$

$\Rightarrow$ $-2x + y + 3 = 0$

$\Rightarrow$ $2x - y - 3 = 0$

Chord: $y = 2x - 3$ $\cap$ Circle: $x^2 + y^2 - 3x + 5y - 4 = 0$

$\Rightarrow$ $x^2 + (2x - 3)^2 - 3x + 5(2x - 3) - 4 = 0$

$\Rightarrow$ $x^2 + 4x^2 - 12x + 9 - 3x + 10x - 15 - 4 = 0$

$\Rightarrow$ $5x^2 - 5x - 10 = 0$

$\Rightarrow$ $x^2 - x - 2 = 0$

$\Rightarrow$ $(x - 2)(x + 1) = 0$

$\Rightarrow$ $x = 2$ or $x = -1$

$\Rightarrow$ $y = 2(2) - 3 = 1$ or $y = 2(-1) - 3 = -5$

Points $= (2, 1)$ and $(-1, -5)$

12. Common tangent: $s_1 - s_2 = 0$

$\Rightarrow$ $x^2 + y^2 + 14x - 10y - 26 - (x^2 + y^2 - 4x + 14y + 28) = 0$

$\Rightarrow$ $x^2 + y^2 + 14x - 10y - 26 - x^2 - y^2 + 4x - 14y - 28 = 0$

$\Rightarrow$ $18x - 24y - 54 = 0$

$\Rightarrow$ $3x - 4y - 9 = 0$

$\Rightarrow$ $3x = 4y + 9$ $\cap$ $x^2 + y^2 + 14x - 10y - 26 = 0$

$\Rightarrow$ $x = \dfrac{4y + 9}{3}$ $\Rightarrow$ $\left(\dfrac{4y + 9}{3}\right)^2 + y^2 + 14\left(\dfrac{4y + 9}{3}\right) - 10y - 26 = 0$

$\Rightarrow$ $\dfrac{16y^2 + 72y + 81}{9} + y^2 + \dfrac{56y + 126}{3} - 10y - 26 = 0$

$\Rightarrow$ $16y^2 + 72y + 81 + 9y^2 + 168y + 378 - 90y - 234 = 0$

$\Rightarrow$ $25y^2 + 150y + 225 = 0$

$\Rightarrow$ $y^2 + 6y + 9 = 0$

$\Rightarrow$ $(y + 3)(y + 3) = 0$

$\Rightarrow$ $y = -3$

$\Rightarrow$ $x = \dfrac{4(-3) + 9}{3} = \dfrac{-3}{3} = -1$

$\Rightarrow$ Point of intersection $= (-1, -3)$

13. Common chord: $s_1 - s_2 = 0$

$$\Rightarrow x^2 + y^2 + 4x - 2y - 5 - (x^2 + y^2 + 14x - 12y + 65) = 0$$
$$\Rightarrow x^2 + y^2 + 4x - 2y - 5 - x^2 - y^2 - 14x + 12y - 65 = 0$$
$$\Rightarrow -10x + 10y - 70 = 0$$
$$\Rightarrow x - y + 7 = 0$$
$$\Rightarrow y = x + 7 \quad \cap \quad x^2 + y^2 + 4x - 2y - 5 = 0$$
$$\Rightarrow x^2 + (x + 7)^2 + 4x - 2(x + 7) - 5 = 0$$
$$\Rightarrow x^2 + x^2 + 14x + 49 + 4x - 2x - 14 - 5 = 0$$
$$\Rightarrow 2x^2 + 16x + 30 = 0$$
$$\Rightarrow x^2 + 8x + 15 = 0$$
$$\Rightarrow (x + 3)(x + 5) = 0$$
$$\Rightarrow x = -3 \quad \text{or} \quad x = -5$$
$$\Rightarrow y = -3 + 7 = 4 \quad \text{or} \quad y = -5 + 7 = 2$$

$\Rightarrow$ Points of intersection $= (-3, 4)$ and $(-5, 2)$

14. Circle: $x^2 + y^2 + 2x - 8y + 4 = 0$

(i) Centre $= (-1, 4) =$ co-ordinates of the Polar star

(ii) $y = 1 \quad \cap \quad x^2 + y^2 + 2x - 8y + 4 = 0$

$$\Rightarrow x^2 + (1)^2 + 2x - 8(1) + 4 = 0$$
$$\Rightarrow x^2 + 2x - 3 = 0$$
$$\Rightarrow (x + 3)(x - 1) = 0$$
$$\Rightarrow x = -3 \quad \text{or} \quad x = 1$$

$\Rightarrow$ Rising: $(-3, 1)$; setting $(1, 1)$

Exercise 11.6

1. $s_1: x^2 + y^2 - 2x - 15 = 0 \quad \Rightarrow \quad$ centre $= (1, 0)$

and radius $= \sqrt{(-1)^2 + (0)^2 + 15} = \sqrt{16} = 4$

$s_2: x^2 + y^2 - 14x - 16y + 77 = 0 \quad \Rightarrow \quad$ centre $= (7, 8)$

and radius $= \sqrt{(-7)^2 + (-8)^2 - 77}$

$= \sqrt{49 + 64 - 77} = \sqrt{36} = 6$

Distance between centres $= \sqrt{(7 - 1)^2 + (8 - 0)^2}$

$= \sqrt{(6)^2 + (8)^2} = \sqrt{100} = 10$

Sum of 2 radii $= 4 + 6 = 10$

Hence, the circles touch externally.

2. $s_1: x^2 + y^2 + 4x - 6y + 12 = 0 \quad \Rightarrow \quad$ centre $= (-2, 3)$

and radius $= \sqrt{(2)^2 + (-3)^2 - 12} = \sqrt{1} = 1$

$s_2: x^2 + y^2 - 12x + 6y - 76 = 0 \quad \Rightarrow \quad$ centre $= (6, -3)$

and radius $= \sqrt{(-6)^2 + (3)^2 + 76}$

$= \sqrt{36 + 9 + 76} = \sqrt{121} = 11$

Distance between centres $= \sqrt{(6 + 2)^2 + (-3 - 3)^2}$

$= \sqrt{(8)^2 + (-6)^2} = \sqrt{100} = 10$

Difference between the 2 radii $= 11 - 1 = 10$

Hence, the circles touch internally.

3. $s_1: x^2 + y^2 - 4x - 2y - 20 = 0 \quad \Rightarrow \quad$ centre $= (2, 1)$

and radius $= \sqrt{(-2)^2 + (-1)^2 + 20}$

$= \sqrt{4 + 1 + 20} = \sqrt{25} = 5$

$s_2: x^2 + y^2 - 16x - 18y + 120 = 0 \quad \Rightarrow \quad$ centre $= (8, 9)$

and radius $= \sqrt{(-8)^2 + (-9)^2 - 120}$

$= \sqrt{64 + 81 - 120} = \sqrt{25} = 5$

Distance between centres $= \sqrt{(8 - 2)^2 + (9 - 1)^2}$

$= \sqrt{(6)^2 + (8)^2} = \sqrt{100} = 10$

Sum of the 2 radii $= 5 + 5 = 10$

Hence, the circles touch externally.

4. $s_1: x^2 + y^2 - 16y + 32 = 0 \quad \Rightarrow \quad$ centre $= (0, 8)$

and radius $= \sqrt{(0)^2 + (-8)^2 - 32}$

$= \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$

$s_2: x^2 + y^2 - 18x + 2y + 32 = 0 \quad \Rightarrow \quad$ centre $= (9, -1)$

and radius $= \sqrt{(-9)^2 + (1)^2 - 32}$

$= \sqrt{81 + 1 - 32} = \sqrt{50} = 5\sqrt{2}$

Distance between centres $= \sqrt{(9 - 0)^2 + (-1 - 8)^2}$

$= \sqrt{(9)^2 + (-9)^2} = \sqrt{162} = 9\sqrt{2}$

Sum of the 2 radii $= 4\sqrt{2} + 5\sqrt{2} = 9\sqrt{2}$

Hence, the circles touch externally.

5. (i) $s_1: x^2 + y^2 - 4x - 6y + 5 = 0 \quad \Rightarrow \quad$ centre $= (2, 3)$

and radius $= \sqrt{(-2)^2 + (-3)^2 - 5}$

$= \sqrt{4 + 9 - 5} = \sqrt{8} = 2\sqrt{2}$

$s_2: x^2 + y^2 - 6x - 8y + 23 = 0 \quad \Rightarrow \quad$ centre $= (3, 4)$

and radius $= \sqrt{(-3)^2 + (-4)^2 - 23}$

$= \sqrt{9 + 16 - 23} = \sqrt{2}$

Distance between centres $= \sqrt{(3 - 2)^2 + (4 - 3)^2}$

$= \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

Difference between the radii $= 2\sqrt{2} - \sqrt{2} = \sqrt{2}$

Hence, the circles touch internally.

(ii) Common tangent: $s_1 - s_2 = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y + 5 - (x^2 + y^2 - 6x - 8y + 23) = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y + 5 - x^2 - y^2 + 6x + 8y - 23 = 0$

$\Rightarrow \quad 2x + 2y - 18 = 0$

$\Rightarrow \quad x + y - 9 = 0$

(iii) $y = -x + 9 \ \cap \ x^2 + y^2 - 4x - 6y + 5 = 0$

$\Rightarrow \quad x^2 + (-x + 9)^2 - 4x - 6(-x + 9) + 5 = 0$

$\Rightarrow \quad x^2 + x^2 - 18x + 81 - 4x + 6x - 54 + 5 = 0$

$\Rightarrow \quad 2x^2 - 16x + 32 = 0$

$\Rightarrow \quad x^2 - 8x + 16 = 0$

$\Rightarrow \quad (x - 4)(x - 4) = 0$

$\Rightarrow \quad x = 4$

$\Rightarrow \quad y = -4 + 9 = 5$

Point of contact $= (4, 5)$

6. (i) $r^2 = (3)^2 + (4)^2 = 9 + 16 = 25$

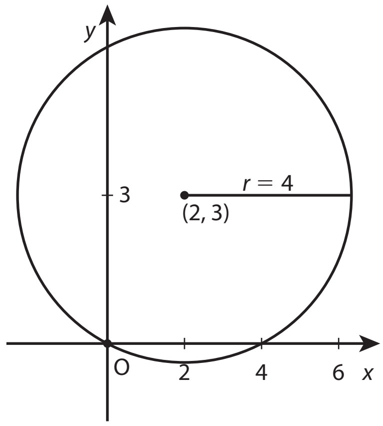
$\Rightarrow \quad r = \sqrt{25} = 5$

(ii) Centre $= (3, 0)$, radius $= 5$

Equation of circle: $(x - 3)^2 + (y - 0)^2 = (5)^2 = 25$

$\Rightarrow \quad (x - 3)^2 + y^2 = 25$

7. (i) Sketch circle: centre $= (2, 3)$, radius $= 4$



(ii) Equation of circle: $(x - 2)^2 + (y - 3)^2 = (4)^2 = 16$

$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 - 16 = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y - 3 = 0$

y-axis $\Rightarrow \quad x = 0$

$\Rightarrow \quad (0)^2 + y^2 - 4(0) - 6y - 3 = 0$

$\Rightarrow \quad y^2 - 6y - 3 = 0$

$$\Rightarrow \quad y = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 + 12}}{2} = \frac{6 \pm \sqrt{48}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$$

Points on y-axis $= (0, 3 - 2\sqrt{3})$ and $(0, 3 + 2\sqrt{3})$

Distance between points $= \sqrt{(0 - 0)^2 + [3 + 2\sqrt{3} - (3 - 2\sqrt{3})]^2}$

$= \sqrt{0 + (3 + 2\sqrt{3} - 3 + 2\sqrt{3})^2}$

$= \sqrt{(4\sqrt{3})^2} = 4\sqrt{3}$

(iii) x-axis $\Rightarrow \quad y = 0$

$\Rightarrow \quad x^2 + (0)^2 - 4x - 6(0) - 3 = 0$

$\Rightarrow \quad x^2 - 4x - 3 = 0$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 + 12}}{2}$$

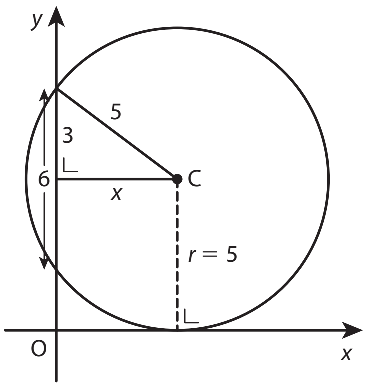
$$= \frac{4 \pm \sqrt{28}}{2} = \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}$$

Points on x-axis $= (2 - \sqrt{7}, 0)$ and $(2 + \sqrt{7}, 0)$

$\Rightarrow$ Length of intercept $= (2 + \sqrt{7}) - (2 - \sqrt{7})$

$= 2 + \sqrt{7} - 2 + \sqrt{7} = 2\sqrt{7}$

8. (i) $x^2 + (3)^2 = (5)^2$



$\Rightarrow \quad x^2 + 9 = 25$

$\Rightarrow \quad x^2 = 25 - 9 = 16$

$\Rightarrow \quad x = \sqrt{16} = 4$

$\Rightarrow \quad$ centre $= (4, 5)$

(ii) Equation of circle: $(x - 4)^2 + (y - 5)^2 = 5^2$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 - 10y + 25 = 25$

$\Rightarrow \quad x^2 + y^2 - 8x - 10y + 16 = 0$

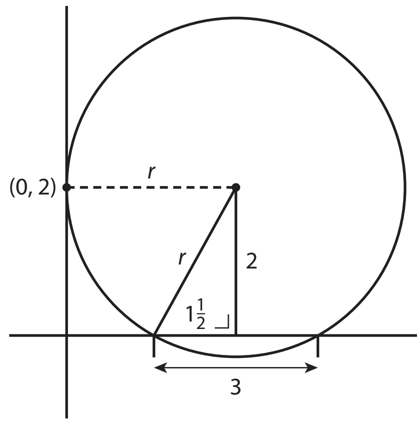
9. $r^2 = \left(1\frac{1}{2}\right)^2 + (2)^2$

$= 6\frac{1}{4}$

$\Rightarrow \quad r = \sqrt{6\frac{1}{4}} = \frac{5}{2}$

$\Rightarrow \quad$ Centre $= \left(\frac{5}{2}, 2\right)$

Equation of circle: $\left(x - \frac{5}{2}\right)^2 + (y - 2)^2 = \left(\frac{5}{2}\right)^2$



10. (i) A$(-1, -4)$ B$(2, 0)$

$\Rightarrow \quad |AB| = \sqrt{(2+1)^2 + (0+4)^2}$

$= \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

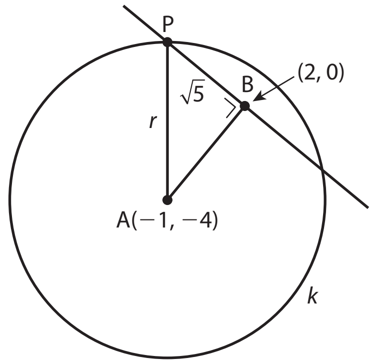
Radius $= |AP|$

$\Rightarrow \quad |AP|^2 = (\sqrt{5})^2 + (5)^2$

$= 5 + 25 = 30$

$\Rightarrow$ Radius $= |AP| = \sqrt{30}$

(ii) Equation of circle: $(x+1)^2 + (y+4)^2 = (\sqrt{30})^2 = 30$



11. (i) $x^2 + (4)^2 = (5)^2$

$\Rightarrow \quad x^2 + 16 = 25$

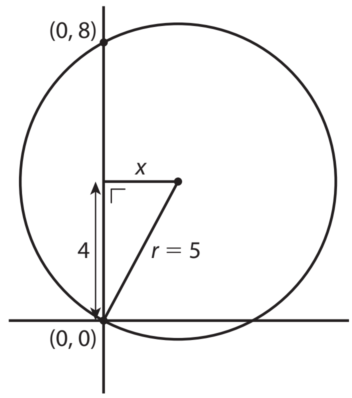
$\Rightarrow \quad x^2 = 25 - 16 = 9$

$\Rightarrow \quad x = \sqrt{9} = 3$

$\Rightarrow \quad$ Centre $= (3, 4)$

(ii) Equation of circle: $(x-3)^2 + (y-4)^2 = (5)^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = 25$



12. $s_1: x^2 + y^2 - 6x + 4y - 12 = 0$

$\Rightarrow \quad$ Centre $= (3, -2)$ and radius $= \sqrt{(-3)^2 + (2)^2 + 12}$

$= \sqrt{9 + 4 + 12} = \sqrt{25} = 5$

$s_2: x^2 + y^2 + 12x - 20y + k = 0$

$\Rightarrow \quad$ Centre $= (-6, 10)$ and radius $= \sqrt{(6)^2 + (-10)^2 - k}$

$= \sqrt{136 - k}$

Distance between centres $= \sqrt{(-6-3)^2 + (10+2)^2}$

$= \sqrt{81 + 144} = \sqrt{225} = 15$

Circles touch externally $\Rightarrow \quad 5 + \sqrt{136 - k} = 15$

$\Rightarrow \quad \sqrt{136 - k} = 15 - 5 = 10$

$\Rightarrow \quad (\sqrt{136 - k})^2 = (10)^2$

$\Rightarrow \quad 136 - k = 100$

$\Rightarrow \quad k = 36$

Exercise 11.7

1. Centre $= (-g, -f) = (3, -4)$

Circle touching the x-axis $\Rightarrow$ $r = |-f| = |-4| = 4$

Equation of circle: $(x - 3)^2 + (y + 4)^2 = (4)^2 = 16$

2 Centre $= (-g, -f) = (-3, 2)$

Circle touching the y-axis $\Rightarrow$ $r = |-g| = |-3| = 3$

Equation of circle k: $(x + 3)^2 + (y - 2)^2 = (3)^2 = 9$

3. Centre $= (5, y) = (-g, -f)$

(i) Circle touching the x-axis and y-axis $\Rightarrow$ $r = |-g| = |-f|$

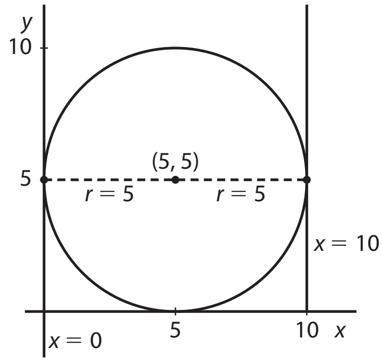
$\Rightarrow |5| = |y|$ $\Rightarrow$ $y = 5$

(ii) Centre $= (5, 5)$, radius $= 5$

Equation of circle: $(x - 5)^2 + (y - 5)^2 = (5)^2 = 25$

(iii) y-axis is a tangent $\Rightarrow$ $x = 0$

2nd tangent parallel to the y-axis is: $x = 10$



4. (i) Sketch of the circle

(ii) Diameter $= 8$ $\Rightarrow$ radius $= 4$

(iii) Circle touching the x-axis

$\Rightarrow r = |-f| = 4$

$\Rightarrow$ $-f = 4$

Centre $(-g, -f)$ lies on the line: $2x - 3y = 0$

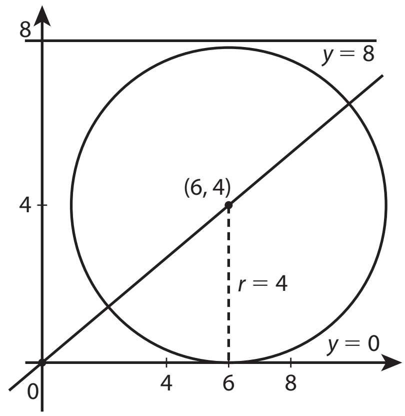
$\Rightarrow$ $-2g - 3(4) = 0$

$\Rightarrow$ $-2g = 12$

$\Rightarrow$ $-g = 6$

$\Rightarrow$ Centre $= (-g, -f) = (6, 4)$

(iv) Equation of circle: $(x - 6)^2 + (y - 4)^2 = (4)^2 = 16$



5. (i) Lines $x = 0$ and $x = 8$ are tangents to the circle

$\Rightarrow$ diameter $= 8$ $\Rightarrow$ radius $= 4$

(ii) Circle touching the y-axis

$\Rightarrow$ $r = |-g| = 4$ $\Rightarrow$ $x = 4$

Centre $(4, y)$ lies on the line: $2x - y - 3 = 0$

$\Rightarrow$ $2(4) - y - 3 = 0$

$\Rightarrow$ $8 - y - 3 = 0$

$\Rightarrow$ $y = 5$

$\Rightarrow$ Centre $= (4, 5)$

(iii) Equation of circle: $(x - 4)^2 + (y - 5)^2 = (4)^2 = 16$

6 x-axis is a tangent $\Rightarrow$ radius $= |-f|$

$\Rightarrow$ $\sqrt{g^2 + f^2 - c} = |-f|$

$\Rightarrow$ $g^2 + f^2 - c = f^2$

$\Rightarrow$ $g^2 - c = 0$ $\Rightarrow$ $g^2 = c$

x-axis is a tangent at $(4, 0)$ $\Rightarrow$ $-g = 4$ $\Rightarrow$ $g = -4$

$\Rightarrow$ $c = (4)^2 = 16$

Point (1, 3) $\Rightarrow$ $(1)^2 + (3)^2 + 2g(1) + 2f(3) + c = 0$

$\Rightarrow$ $2g + 6f + c = -10$

$\Rightarrow$ $2(-4) + 6f + 16 = -10$

$\Rightarrow$ $-8 + 6f + 16 = -10$

$\Rightarrow$ $6f = -18$

$\Rightarrow$ $f = -3$

$\Rightarrow$ Equation of circle: $x^2 + y^2 + 2(-4)x + 2(-3)y + 16 = 0$

$\Rightarrow$ $x^2 + y^2 - 8x - 6y + 16 = 0$

7. y-axis is a tangent $\Rightarrow$ radius $= |-g|$

$\Rightarrow$ $\sqrt{g^2 + f^2 - c} = |-g|$

$\Rightarrow$ $g^2 + f^2 - c = g^2$

$\Rightarrow$ $f^2 - c = 0$ $\Rightarrow$ $f^2 = c$

y-axis is a tangent at $(0, -3)$ $\Rightarrow$ $-f = -3$ $\Rightarrow$ $f = 3$

$\Rightarrow$ $c = (3)^2 = 9$

Point (4, 1) $\Rightarrow$ $(4)^2 + (1)^2 + 2g(4) + 2f(1) + c = 0$

$\Rightarrow$ $16 + 1 + 8g + 2f + c = 0$

$\Rightarrow$ $8g + 2f + c = -17$

$\Rightarrow$ $8g + 2(3) + 9 = -17$

$\Rightarrow$ $8g = -32$

$\Rightarrow$ $g = -4$

$\Rightarrow$ Equation of circle: $x^2 + y^2 + 2(-4)x + 2(3)y + 9 = 0$

$\Rightarrow$ $x^2 + y^2 - 8x + 6y + 9 = 0$

8. x-axis is a tangent

$\Rightarrow$ radius $= |-f|$

$\Rightarrow$ $\sqrt{g^2 + f^2 - c} = |-f|$

$\Rightarrow$ $g^2 + f^2 - c = f^2$

$\Rightarrow$ $g^2 = c$

$y = 0$ and $y = 10$ are tangents to the circle

$\Rightarrow$ diameter $= 10$ $\Rightarrow$ radius $= 5$

$\Rightarrow$ centre $(-g, -f)$ lies on line $y = 5$

$\Rightarrow$ $-f = 5$ $\Rightarrow$ $f = -5$

Point (1, 5) $\Rightarrow$ $(1)^2 + (5)^2 + 2g(1) + 2f(5) + c = 0$

$\Rightarrow$ $2g + 10f + c = -26$

$f = -5$ $\Rightarrow$ $2g + 10(-5) + c = -26$

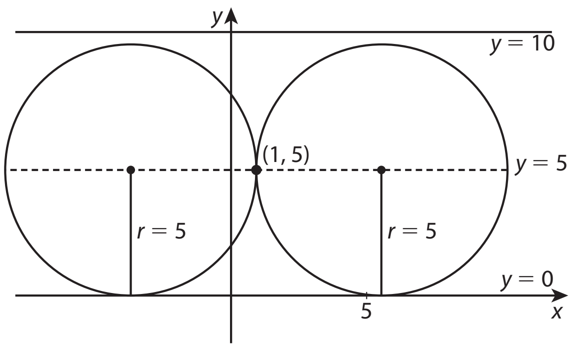
$\Rightarrow$ $2g + c = 24$

$g^2 = c$ $\Rightarrow$ $2g + g^2 - 24 = 0$

$\Rightarrow$ $g^2 + 2g - 24 = 0$

$\Rightarrow$ $(g + 6)(g - 4) = 0$

$\Rightarrow$ $g = -6$ or $g = 4$



Centre $= (6, 5)$, radius $= 5$

Equation of circle: $(x - 6)^2 + (y - 5)^2 = (5)^2 = 25$

OR centre $= (-4, 5)$, radius $= 5$

Equation of circle: $(x + 4)^2 + (y - 5)^2 = (5)^2 = 25$

Revision Exercise 11 (Core)

1. (i) Point (1, 2), centre $(-1, 5)$ $\Rightarrow$ length of radius $= \sqrt{(-1 - 1)^2 + (5 - 2)^2}$

$= \sqrt{4 + 9} = \sqrt{13}$

(ii) Equation of circle: $(x + 1)^2 + (y - 5)^2 = (\sqrt{13})^2 = 13$

2. Circle: $x^2 + y^2 - 2x - 4y - 9 = 0$

$\Rightarrow$ Centre $= (1, 2)$ and radius $= \sqrt{(-1)^2 + (-2)^2 + 9}$

$= \sqrt{1 + 4 + 9} = \sqrt{14}$

New circle: centre $= (0, 0)$, radius $= \sqrt{14}$

Equation of circle: $x^2 + y^2 = (\sqrt{14})^2 = 14$

3. Centre $= (2, 3) \Rightarrow$ circle touches x-axis at $(2, 0)$

$\Rightarrow$ radius $= 3$

Equation of circle: $(x - 2)^2 + (y - 3)^2 = (3)^2 = 9$

4. $x^2 + y^2 = 25 \;\Rightarrow\;$ centre $= (0, 0)$ and radius $= \sqrt{25} = 5$

Line: $3x - 4y + 25 = 0$

$\Rightarrow$ Perpendicular distance $= \dfrac{|3(0) - 4(0) + 25|}{\sqrt{(3)^2 + (-4)^2}}$

$= \dfrac{25}{\sqrt{25}} = \dfrac{25}{5} = 5 =$ radius

Hence, the line is a tangent to the circle.

5. $A(-1, -3), B(3, 1) \Rightarrow$ centre $=$ midpoint $= \left(\dfrac{-1+3}{2}, \dfrac{-3+1}{2}\right)$

$= (1, -1)$

Radius $= \sqrt{(-1 - 1)^2 + (-3 + 1)^2}$

$= \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

Equation of circle: $(x - 1)^2 + (y + 1)^2 = (2\sqrt{2})^2 = 8$

6. Circle: $(x - 5)^2 + y^2 = 36$

x-axis $\Rightarrow y = 0 \Rightarrow \quad (x - 5)^2 + (0)^2 = 36$

$\Rightarrow \quad x^2 - 10x + 25 - 36 = 0$

$\Rightarrow \quad x^2 - 10x - 11 = 0$

$\Rightarrow \quad (x - 11)(x + 1) = 0$

$\Rightarrow \quad x = 11 \quad$ or $\quad x = -1$

$\Rightarrow P = (11, 0);\; Q = (-1, 0)$

7. Circle: $x^2 + y^2 - 2x + 4y + 4 = 0$

$\Rightarrow$ Centre $= (1, -2)$

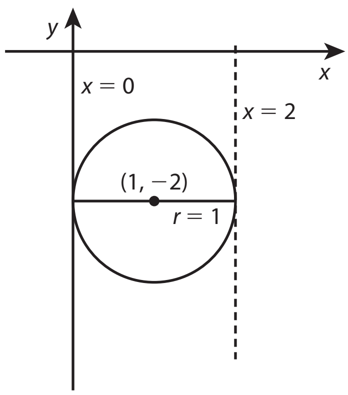
and radius $= \sqrt{(-1^2) + (2)^2 - 4}$

$= \sqrt{1 + 4 - 4}$

$= \sqrt{1} = 1$

Line $x = k$ is parallel to the y-axis

$\Rightarrow k = 0, 2$



8. Centres $(8, 5)$ and $(2, -3)$.

Distance between centres $= \sqrt{(2 - 8)^2 + (-3 - 5)^2}$

$= \sqrt{(-6)^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

Circles touch externally $\;\Rightarrow\;$ sum of the two radii $= 10$

$\Rightarrow$ radius of k_1 + radius of $k_2 = 10$

$\Rightarrow \quad 6$ + radius of $k_2 = 10$

$\Rightarrow$ radius of $k_2 = 10 - 6 = 4$

9. Circle: $(x - 5)^2 + (y + 2)^2 = 30$

Point $(0, 0) \quad \Rightarrow (0 - 5)^2 + (0 + 2)^2$

$= 25 + 4 = 29 < 30$

$\Rightarrow$ point is inside the circle.

10. A = (1, 6) B = (11, 0)

$$\text{Centre} = \text{midpoint} = \left(\frac{1+11}{2}, \frac{6+10}{2}\right) = (6, 8)$$

$$\text{Radius} = \sqrt{(6-1)^2 + (8-6)^2} = \sqrt{(5)^2 + (2)^2} = \sqrt{25+4} = \sqrt{29}$$

Equation of circle: $(x-6)^2 + (y-8)^2 = (\sqrt{29})^2 = 29$

11. From the equation of the path of the first satellite,

$$x^2 + y^2 = 2250000,$$

whose centre is (0, 0) and whose radius is $r_1 = \sqrt{2250000} = 1500$, the distance travelled by the first satellite is

$$2\pi r_1 = 2\pi(1500) = 9424.8 \text{ km.}$$

The radius of the path of the second satellite is then

$$r_2 = 1500 + 200 = 1700,$$

and so the distance travelled by the second satellite is

$$2\pi r_2 = 2\pi(1700) = 10681.4 \text{ km.}$$

Thus the extra distance travelled by the second satellite is

$$10681.4 - 9424.8 = 1256.6$$

or 1257 km, correct to the nearest km.

Revision Exercise 11 (Advanced)

1. Circle: $x^2 + y^2 - 6x - 2y - 3 = 0$

$$\text{Centre} = (3, 1) \text{ and radius} = \sqrt{(-3)^2 + (-1)^2 + 3}$$
$$= \sqrt{9+1+3} = \sqrt{13}$$

$$\text{Centre } (3, 1), \text{ Point } (5, 4) \Rightarrow \text{slope radius} = \frac{4-1}{5-3} = \frac{3}{2}$$

$$\Rightarrow \text{perpendicular slope} = \frac{-2}{3}$$

$$\Rightarrow \text{equation of tangent: } y - 4 = \frac{-2}{3}(x-5)$$

$$\Rightarrow \quad 3y - 12 = -2x + 10$$

$$\Rightarrow \quad 2x + 3y - 22 = 0$$

2. (i) Centre = (5, −1), line l: $3x - 4y + 11 = 0$

$$\text{Perpendicular distance} = \frac{|3(5) - 4(-1) + 11|}{\sqrt{(3)^2 + (-4)^2}}$$
$$= \frac{|15 + 4 + 11|}{\sqrt{25}} = \frac{30}{5} = 6 = \text{radius}$$

(ii) Centre = (5, −1), line: $x + py + 1 = 0$, radius = 6

$$\text{Perpendicular distance} = \frac{|15 + p(-1) + 1|}{\sqrt{(1)^2 + (p)^2}} = 6$$

$$\Rightarrow \frac{|6-p|}{\sqrt{1+p^2}} = 6$$
$$\Rightarrow |6-p| = 6\sqrt{1+p^2}$$
$$\Rightarrow (6-p)^2 = 36(1+p^2)$$
$$\Rightarrow 36 - 12p + p^2 = 36 + 36p^2$$
$$\Rightarrow 35p^2 + 12p = 0$$
$$\Rightarrow p(35p + 12) = 0$$
$$\Rightarrow p = 0 \quad \text{or} \quad 35p = -12$$
$$p = \frac{-12}{35}$$

3. Circle: $x^2 + y^2 + 4x - 3y - 12 = 0$

$\Rightarrow$ Centre $= \left(-2, \frac{3}{2}\right)$ and radius $= \sqrt{(2)^2 + \left(\frac{-3}{2}\right)^2 + 12}$

$= \sqrt{4 + \frac{9}{4} + 12} = \sqrt{18\frac{1}{4}}$

Tangent: $8x + 3y + k = 0$

$\Rightarrow$ Perpendicular distance $= \dfrac{|8(-2) + 3\left(\frac{3}{2}\right) + k|}{\sqrt{(8)^2 + (3)^2}} = \sqrt{18\frac{1}{4}}$

$\Rightarrow \dfrac{\left|-16 + \frac{9}{2} + k\right|}{\sqrt{64 + 9}} = \sqrt{\dfrac{73}{4}}$

$\Rightarrow \left|-11\frac{1}{2} + k\right| = \sqrt{\dfrac{73}{4}} \cdot \sqrt{73}$

$\Rightarrow \left|-11\frac{1}{2} + k\right| = \dfrac{73}{2}$

$\Rightarrow \quad -11\frac{1}{2} + k = -\dfrac{73}{2}$ or $-11\frac{1}{2} + k = \dfrac{73}{2}$

$\Rightarrow k = -\dfrac{73}{2} + \dfrac{23}{2} = -25$ or $k = \dfrac{73}{2} + \dfrac{23}{2} = 48$

4. (i) Circle k: $x^2 + y^2 + px - 2y + 5 = 0$

Point A(5, 2) $\Rightarrow (5)^2 + (2)^2 + p(5) - 2(2) + 5 = 0$

$\Rightarrow 25 + 4 + 5p - 4 + 5 = 0$

$\Rightarrow 5p + 30 = 0$

$\Rightarrow 5p = -30 \Rightarrow p = -6$

(ii) Line: $x - y - 1 = 0$ meets circle: $x^2 + y^2 - 6x - 2y + 5 = 0$

$\Rightarrow x = y + 1 \quad \Rightarrow (y + 1)^2 + y^2 - 6(y + 1) - 2y + 5 = 0$

$\Rightarrow y^2 + 2y + 1 + y^2 - 6y - 6 - 2y + 5 = 0$

$\Rightarrow 2y^2 - 6y = 0$

$\Rightarrow y^2 - 3y = 0$

$\Rightarrow y(y - 3) = 0$

$\Rightarrow y = 0$ or $y = 3$

$\Rightarrow x = 0 + 1 = 1$ or $x = 3 + 1 = 4$

$\Rightarrow$ Points of intersection $= (1, 0)$ and $(4, 3)$

5. (i) c_1: $x^2 + y^2 + 2x - 2y - 23 = 0$

Centre $= (-1, 1)$

Radius $= \sqrt{(1)^2 + (-1)^2 + 23} = \sqrt{25} = 5$

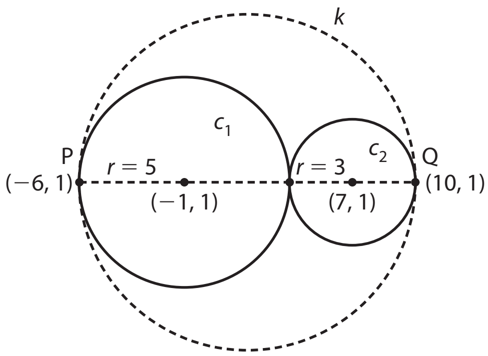
c_2: $x^2 + y^2 - 14x - 2y + 41 = 0$

Centre $= (7, 1)$

Radius $= \sqrt{(-7)^2 + (-1)^2 - 41} = \sqrt{9} = 3$

Distance between centres $= \sqrt{(7 + 1)^2 + (1 - 1)^2}$

$= \sqrt{8^2 + 0^2} = \sqrt{64} = 8$



Sum of the two radii $= 5 + 3 = 8$

Hence, the circle touch externally.

(ii) P $= (-6, 1)$, Q $= (10, 1)$

Centre of circle k = midpoint [PQ] $= \left(\dfrac{-6 + 10}{2}, \dfrac{1 + 1}{2}\right) = (2, 1)$

Diameter $c_1 = 10$, diameter $c_2 = 6$

Diameter of circle $k = 10 + 6 = 16$

$\Rightarrow$ Radius of circle $k = \dfrac{16}{2} = 8$

Equation of circle k: $(x - 2)^2 + (y - 1)^2 = (8)^2 = 64$

6. Circle: $x^2 + y^2 - 10y + 20 = 0$

(i) Centre $C = (0, 5)$

(ii) Radius $= \sqrt{(0)^2 + (-5)^2 - 20} = \sqrt{25 - 20} = \sqrt{5}$

(iii) $y = 2x \cap x^2 + y^2 - 10y + 20 = 0$

$$\Rightarrow \quad x^2 + (2x)^2 - 10(2x) + 20 = 0$$
$$\Rightarrow \quad x^2 + 4x^2 - 20x + 20 = 0$$
$$\Rightarrow \quad 5x^2 - 20x + 20 = 0$$
$$\Rightarrow \quad x^2 - 4x + 4 = 0$$
$$\Rightarrow \quad (x - 2)(x - 2) = 0$$
$$\Rightarrow \quad x = 2 \quad \Rightarrow \quad y = 2(2) = 4$$

One point of contact $= (2, 4)$

$\Rightarrow$ Line is a tangent to the circle.

7. $x^2 + y^2 = 4 \quad \Rightarrow \quad$ centre $= (0, 0)$ and radius $= \sqrt{4} = 2$

$x^2 + y^2 - 8x - 6y + 16 = 0$

$\Rightarrow$ Centre $= (4, 3)$ and radius $= \sqrt{(-4)^2 + (-3)^2 - 16}$

$= \sqrt{16 + 9 - 16} = \sqrt{9} = 3$

Distance between centres $= \sqrt{(4 - 0)^2 + (3 - 0)^2}$

$= \sqrt{16 + 9} = \sqrt{25} = 5$

Sum of the two radii $= 2 + 3 = 5$

Hence, the circles touch externally.

Common tangent: $s_1 - s_2 = 0$

$$\Rightarrow \quad x^2 + y^2 - 4 - (x^2 + y^2 - 8x - 6y + 16) = 0$$
$$\Rightarrow \quad x^2 + y^2 - 4 - x^2 - y^2 - 8x + 6y - 16 = 0$$
$$\Rightarrow \quad 8x + 6y - 20 = 0$$
$$\Rightarrow \quad 4x + 3y - 10 = 0$$

8. Circle: $x^2 + y^2 + 2gx + 2fy + 7 = 0$

(i) Point $(2, 5)$

$$\Rightarrow \quad (2)^2 + (5)^2 + 2g(2) + 2f(5) + 7 = 0$$
$$\Rightarrow \quad 4 + 25 + 4g + 10f + 7 = 0$$
$$\Rightarrow \quad 4g + 10f = -36$$
$$\Rightarrow \quad 2g + 5f = -18$$

Point $(-2, 1)$

$$\Rightarrow \quad (-2)^2 + (1)^2 + 2g(-2) + 2f(1) + 7 = 0$$
$$\Rightarrow \quad 4 + 1 - 4g + 2f + 7 = 0$$
$$\Rightarrow \quad -4g + 2f = -12$$
$$\Rightarrow \quad 2g - f = 6$$

(ii) $2g + 5f = -18$

$\underline{2g - f = 6}$

$6f = -24$

$f = -4 \quad \Rightarrow \quad 2g - (-4) = 6$

$2g = 2 \quad \Rightarrow \quad g = 1$

(iii) Circle: $x^2 + y^2 + 2(1)x + 2(-4)y + 7 = 0$

$\Rightarrow \quad x^2 + y^2 + 2x - 8y + 7 = 0$

Centre $= (-1, 4)$ and radius $= \sqrt{(1)^2 + (-4)^2 - 7} = \sqrt{10}$

9. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

$(0, 0)$: $\quad c = 0$

$(4, 2)$: $\quad 16 + 4 + 8g + 4f + 0 = 0$

$8g + 4f = -20$

$2g + f = -5 \qquad$ **... 1**

$(-g, -f)$ is on the line $x + y = 1$:

$$-g - f = 1 \qquad \textbf{... 2}$$

Adding **1** and **2**,

$$g = -4$$

1: $-8 + f = -5$

$$f = 3$$

Thus the equation of the circle is $x^2 + y^2 - 8x + 6y = 0$, or $(x - 4)^2 + (y + 3)^2 = 25$.

10. Circle: $(x + 3)^2 + (y - 2)^2 = 25$

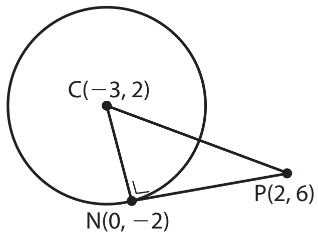
(i) Centre C $= (-3, 2)$

(ii) Radius $= \sqrt{25} = 5$

(iii) Point N $(0, -2)$ $\Rightarrow (0 + 3)^2 + (-2 - 2)^2 = 25$

$\Rightarrow (3)^2 + (-4)^2 = 9 + 16 = 25 = 25$

$\Rightarrow$ Point N $(0, -2)$ lies on the circle.



$|CP| = \sqrt{(2 + 3)^2 + (6 - 2)^2} = \sqrt{25 + 16} = \sqrt{41}$

Radius $= |CN| = 5$

$\Rightarrow |NP|^2 + |CN|^2 = |CP|^2$

$\Rightarrow |NP|^2 + (5)^2 = (\sqrt{41})^2$

$\Rightarrow |NP|^2 + 25 = 41$

$\Rightarrow |NP|^2 = 41 - 25 = 16$

$\Rightarrow |NP|^2 = \sqrt{16} = 4$

11. Centre $(-g, -f)$ on line $x - 2y - 1 = 0$

$\Rightarrow -g - 2(-f) - 1 = 0$

$\Rightarrow -g + 2f - 1 = 0$

Circle touching the x-axis

$\Rightarrow g^2 = c$

$y = 0$ and $y = 6$ are parallel tangents $\Rightarrow$ equation of diameter is $y = 3$

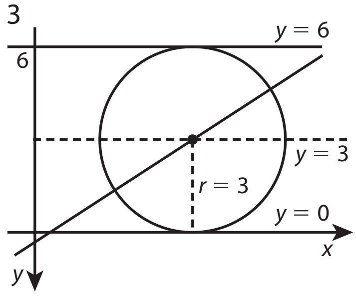
$\Rightarrow -f = 3 \Rightarrow f = -3$

$\Rightarrow -g + 2(-3) - 1 = 0$

$\Rightarrow -g - 6 - 1 = 0$

$\Rightarrow -g = 7 \Rightarrow g = -7$

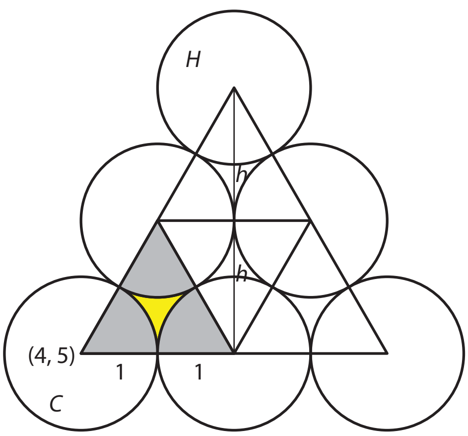
$\Rightarrow c = g^2 = (-7)^2 = 49$

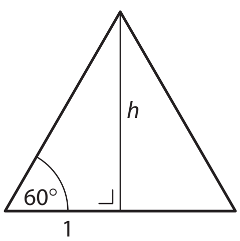


Equation of circle: $x^2 + y^2 + 2(-7)x + 2(-3)y + 49 = 0$

$\Rightarrow x^2 + y^2 - 14x - 6y + 49 = 0$

12. (i) The centre of C is (4, 5) and its radius is 1.





From the triangle,

$$\tan 60° = \frac{h}{1}$$

$$h = \sqrt{3}$$

The co-ordinates of the centre of H are then:

x co-ordinate: $4 + 2 = 6$

y co-ordinate: $5 + 2h = 5 + 2\sqrt{3}$

The radius of H is also 1.

The equation of H is then

$$(x - 6)^2 + (y - (5 + 2\sqrt{3}))^2 = 1$$

or $(x - 6)^2 + (y - 5 - 2\sqrt{3})^2 = 1$

(ii) Area of one triangle $= \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$

Area of one sector $= \frac{1}{2}(1)^2\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$

Area of three sectors $= 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$

Area of yellow shaded region $= \sqrt{3} - \frac{\pi}{2} = 0.16125$

Area of trapped region is 4 times the area of the yellow shaded region, i.e.

$$4(0.16125) = 0.645$$

Revision Exercises 11 (Extended Response Questions)

1. (i) Circle: $x^2 + y^2 - 8x - 7y + 12 = 0$

y-axis $\Rightarrow$ $x = 0$ $\Rightarrow (0)^2 + y^2 - 8(0) - 7y + 12 = 0$

$\Rightarrow$ $y^2 - 7y + 12 = 0$

$\Rightarrow$ $(y - 3)(y - 4) = 0$

$\Rightarrow$ $y = 3$ or $y = 4$

Points $= (0, 3)$ and $(0, 4)$

$\Rightarrow$ Length of the intercept $= 4 - 3 = 1$

(ii) Circle has centre $= \left(4, 3\frac{1}{2}\right)$ and radius $= \sqrt{(-4)^2 + \left(-3\frac{1}{2}\right)^2 - 12}$

$$= \sqrt{16 + 12\frac{1}{4} - 12} = \sqrt{16\frac{1}{4}}$$

Point $(9, 2)$, centre $= \left(4, 3\frac{1}{2}\right) \Rightarrow$ hypotenuse $= \sqrt{(4 - 9)^2 + \left(3\frac{1}{2} - 2\right)^2}$

$$= \sqrt{25 + 2\frac{1}{4}} = \sqrt{27\frac{1}{4}}$$

$\Rightarrow$ Length of tangent $= \sqrt{\left(\sqrt{27\frac{1}{4}}\right)^2 - \left(16\frac{1}{4}\right)^2}$

$$= \sqrt{27\frac{1}{4} - 16\frac{1}{4}} = \sqrt{11}$$

(iii) The circle has centre $\left(4, \frac{7}{2}\right)$ and radius $\sqrt{\frac{65}{4}}$.

Under the translation with components 4 and $-\frac{7}{2}$, the co-ordinates of the centre of the image circle are:

x co-ordinate: $4 + 4 = 8$

y co-ordinate: $\frac{7}{2} - \frac{7}{2} = 0$

The image circle has centre (8, 0) and radius $\sqrt{\frac{65}{4}}$. Thus its equation is

$$(x-8)^2+(y-0)^2=\frac{65}{4}$$

or $\quad x^2+y^2-16x+64=\frac{65}{4}$

or $\quad 4x^2+4y^2-64x+256=65$

or $\quad 4x^2+4y^2-64x+191=0$

2. (i) Line: $3x-4y+1=0$

Equation of parallel line: $3x-4y+c=0$

(ii) Circle: $x^2+y^2-8x+2y-8=0$

Centre $=(4,-1)$ and radius $=\sqrt{(-4)^2+(1)^2+8}$

$=\sqrt{16+1+8}=\sqrt{25}=5$

(iii) Perpendicular distance $=\dfrac{|3(4)-4(-1)+c|}{\sqrt{(3)^2+(-4)^2}}=5$

$\Rightarrow \quad \dfrac{|12+4+c|}{\sqrt{25}}=5$

$\Rightarrow \quad \dfrac{|16+c|}{5}=5$

$\Rightarrow \quad |16+c|=25$

$\Rightarrow \quad 16+c=25 \quad$ or $\quad 16+c=-25$

$\Rightarrow \quad c=25-16=9 \quad$ or $\quad c=-25-16=-41$

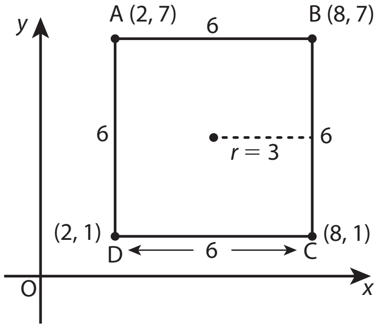
Tangents: $3x-4y+9=0$ or $3x-4y-41=0$

3. (i) C = (8, 1); D = (2, 1)

(ii) A = (2, 7), C = (8, 1)

Centre = midpoint $=\left(\dfrac{2+8}{2},\dfrac{7+1}{2}\right)$

$=(5,4)$



(iii) Diameter = |AB| = 6

Radius $=\dfrac{6}{2}=3$

Equation of circle: $(x-5)^2+(y-4)^2=(3)^2=9$

4. (i) x-axis a tangent:

$r=|-f|$

$r^2=f^2$

$g^2+f^2-c=f^2$

$g^2=c$

y-axis a tangent:

$r=|-g|$

$r^2=g^2$

$g^2+f^2-c=g^2$

$f^2=c$

Thus if both axes are tangents,

$g^2=f^2=c$

(ii) Let the equation of the circle be

$x^2+y^2+2gx+2fy+c=0$

[1] $(-3, 6)$ is on the circle.

$9+36-6g+12f+c=0$

$-6g+12f+c=-45 \quad$ **... 1**

[2] $(-6, 3)$ is on the circle.

$$36 + 9 - 12g + 6f + c = 0$$

$$-12g + 6f + c = -45 \quad \textbf{... 2}$$

[3] y-axis is a tangent.

$$f^2 = c \quad \textbf{... 3}$$

Solving,

1: $-6g + 12f + c = -45$

2 $\times -1$: $\underline{12g - 6f - c = 45}$

$6g + 6f \qquad = 0$

$g = -f \quad \textbf{... 4}$

1: $6f + 12f + f^2 = -45$

$f^2 + 18f + 45 = 0$

$(f + 3)(f + 15) = 0$

$f = -3 \quad \text{or} \quad f = -15$

$f = -3$: $\quad g = 3, c = 9$

one circle is $x^2 + y^2 + 6x - 6y + 9 = 0$

$f = -15$: $\quad g = 15, c = 225$

the other circle is $x^2 + y^2 + 30x - 30y + 225 = 0$

(iii) As was stated in the beginning of the question, the x-axis is a tangent to both these circles.

(iv) The radius of the first circle is

$$r_1 = \sqrt{(3)^2 + (-3)^2 - 9} = 3$$

The radius of the second circle is

$$r_2 = \sqrt{(15)^2 + (-15)^2 - 225} = 15$$

Thus the scale factor of the enlargement is

$$k = \frac{r_2}{r_1} = \frac{15}{3} = 5$$

(v) The circumference of the first circle is $2\pi r_1 = 2\pi(3) = 6\pi$.

The circumference of the second circle is

$2\pi r_2 = 2\pi(15) = 30\pi$.

As $30\pi = 5 \times (6\pi) = k \times (6\pi)$,

the circumference of the larger circle is $k \times$ circumference of the smaller circle.

5. Circle: $(x - 20)^2 + (y - 18)^2 = 16$

$\Rightarrow$ centre A $= (20, 18)$, radius $= \sqrt{16} = 4$

$|AD|^2 + (12)^2 = (20)^2$

$\Rightarrow |AD|^2 + 144 = 400$

$\Rightarrow |AD|^2 = 400 - 144 = 256$

$\Rightarrow |AD| = \sqrt{256} = 16$

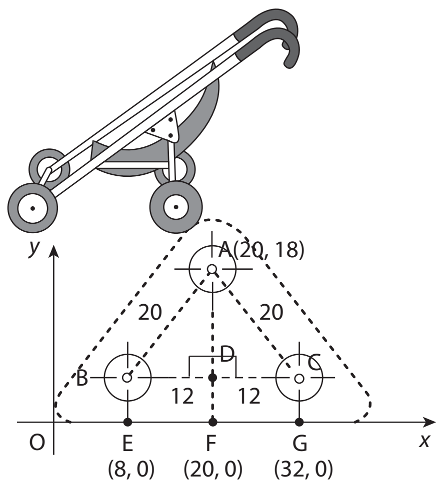
$|AF| = 18 \quad \Rightarrow \quad |DF| = 18 - 16 = 2$

centre B $= (8, 2)$, radius $= 4$

equation of circle: $(x - 8)^2 + (y - 2)^2 = (4)^2 = 16$

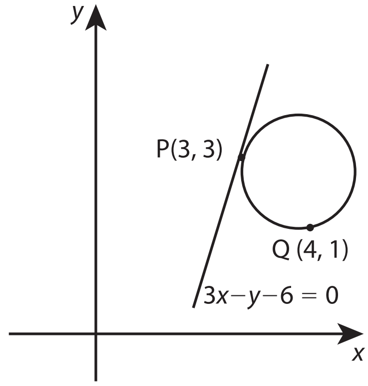
centre C $= (32, 2)$, radius $= 4$

equation of circle: $(x - 32)^2 + (y - 2)^2 = (4)^2 = 16$



6. (i) The line and the points are shown below.

(ii) The circle is shown opposite.



(iii) Point (3, 3) $\Rightarrow \quad (3)^2 + (3)^2 + 2g(3) + 2f(3) + c = 0$

$\Rightarrow \quad 6g + 6f + c = -18$

Point (4, 1) $\Rightarrow \quad (4)^2 + (1)^2 + 2g(4) + 2f(1) + c = 0$

$$\begin{aligned} \Rightarrow \quad & 8g + 2f + c = -17 \\ & \underline{6g + 6f + c = -18} \\ \Rightarrow \quad & 2g - 4f = 1 \end{aligned}$$

Tangent: $3x - y - 6 = 0 \Rightarrow \text{slope} = -\frac{a}{b} = -\frac{3}{-1} = 3$

Perpendicular slope $= -\frac{1}{3}$, point (3, 3)

$\Rightarrow$ equation of diameter: $y - 3 = -\frac{1}{3}(x - 3)$

$$\begin{aligned} &\Rightarrow 3y - 9 = -x + 3 \\ &\Rightarrow x + 3y = 12 \end{aligned}$$

Centre $(-g, -f) \Rightarrow \quad -g - 3f = 12$

$$\begin{aligned} \Rightarrow \quad & -2g - 6f = 24 \\ & -2g - 6f = 24 \\ & \underline{2g - 4f = 1} \text{ (adding)} \\ \Rightarrow \quad & -10f = 25 \\ \Rightarrow \quad & f = \frac{-25}{10} = \frac{-5}{2} \\ \Rightarrow \quad & 2g - 4\left(-\frac{5}{2}\right) = 1 \\ \Rightarrow \quad & 2g + 10 = 1 \\ \Rightarrow \quad & 2g = -9 \\ \Rightarrow \quad & g = \frac{-9}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & 6\left(\frac{-9}{2}\right) + 6\left(\frac{-5}{2}\right) + c = -18 \\ \Rightarrow \quad & -27 - 15 + c = -18 \\ \Rightarrow \quad & c = -18 + 42 = 24 \end{aligned}$$

Equation of circle: $x^2 + y^2 + 2\left(\frac{-9}{2}\right)x + 2\left(\frac{-5}{2}\right)y + 24 = 0$

$\Rightarrow \quad x^2 + y^2 - 9x - 5y + 24 = 0$

(iv) For the circle, $g = -\frac{9}{2}$, $f = -\frac{5}{2}$ and $c = 24$.

Thus the centre of the circle is

$$(-g, -f) = \left(\frac{9}{2}, \frac{5}{2}\right)$$

and its radius is

$$\sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{9}{2}\right)^2 + \left(-\frac{5}{2}\right)^2 - 24} = \frac{\sqrt{10}}{2}.$$

The centre of the image circle is then

$$\left(\frac{9}{2}, \frac{5}{2}\right) \xrightarrow{-\frac{3}{2}, \frac{1}{2}} (3, 3) \xrightarrow{-\frac{3}{2}, \frac{1}{2}} \left(\frac{3}{2}, \frac{7}{2}\right) = (h, k)$$

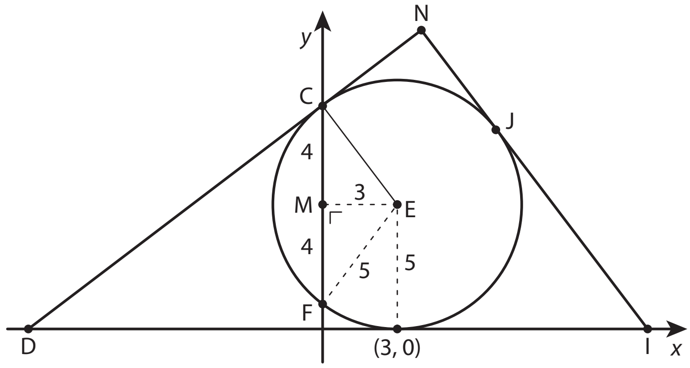
The radius of the image circle is also $\frac{\sqrt{10}}{2}$.

The equation of the image circle is then

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{10}{4}$$

or $\quad (x - 1.5)^2 + (y - 3.5)^2 = 2.5.$

7. (i)



From the diagram:

$|FM| = |MC| = 4, |EM| = 3$

and so the radius of the circle is $r = \sqrt{3^2 + 4^2} = 5$. The centre of the circle is E $= (3, 5) = (h, k)$. Thus the equation of the circle is

$$(x - 3)^2 + (y - 5)^2 = 25.$$

(ii) From the diagram, C $= (0, 9)$. Then

$$\text{slope of CE} = \frac{5 - 9}{3 - 0} = -\frac{4}{3}$$

$$\text{slope of DC} = \frac{3}{4}$$

Equation of DC is

$$(y - 9) = \frac{3}{4}(x - 0)$$

$$4y - 36 = 3x$$

or $\quad 3x - 4y + 36 = 0.$ **... 1**

D is where this line cuts the x-axis. Let $y = 0$. Then

$3x + 36 = 0$

$x = -12$

Thus D $= (-12, 0)$.

(iii) The equation of a line perpendicular to DC is of the form $4x + 3y + k = 0$.

The perpendicular distance from E $= (3, 5)$ to this line is $r = 5$. Thus

$$\frac{|4(3) + 3(5) + k|}{\sqrt{4^2 + 3^2}} = 5$$

$|k + 27| = 25$

$k + 27 = 25$ or $k + 27 = -25$

$k = -2 \quad$ or $\quad k = -52$

As $k = -2$ gives a tangent near the origin, $k = -52$. Thus the equation of the tangent at J is

$4x + 3y - 52 = 0.$ **... 2**

I is where this line intersects the x-axis. Let $y = 0$. Then

$4x - 52 = 0$

$x = 13$

Thus I $= (13, 0)$.

(iv) To find the co-ordinates of N:

1 $\times$ 3: $\quad 9x - 12y = -108$

2 $\times$ 4: $\quad \underline{16x + 12y = 208}$

$25x = 100$

$x = 4$

2: $\quad 16 + 3y = 52$

$3y = 36$

$y = 12$

Thus N $= (4, 12)$.

Then $|DI| = 13 + 12 = 25$

$|DN| = \sqrt{(4 + 12)^2 + (12 - 0)^2} = 20$

$|NI| = \sqrt{25^2 - 20^2} = 15$

Hence the perimeter of the triangle is $25 + 20 + 15 = 60$.

8. Circle: $x^2 + y^2 - 2ax - 2by + b^2 = 0$

(i) $2g = -2a \Rightarrow g = -a$

$2f = -2b \Rightarrow f = -b$

$\Rightarrow$ centre $(-g, -f) = (a, b)$

and radius $= \sqrt{(-a)^2 + (-b)^2 - b^2}$

$= \sqrt{a^2 + b^2 - b^2} = \sqrt{a^2} = a$

(ii) Circle touches the y-axis $\Rightarrow f^2 = c$

$\Rightarrow (-b)^2 = b^2$

$\Rightarrow b^2 = b^2$ true

(iii) Point $(1, 2)$ $\Rightarrow (1)^2 + (2)^2 - 2a(1) - 2b(2) + b^2 = 0$

$\Rightarrow 1 + 4 - 2a - 4b + b^2 = 0$

$\Rightarrow -2a - 4b + b^2 = -5$

Point $(2, 3)$ $\Rightarrow (2)^2 + (3)^2 - 2a(2) - 2b(3) + b^2 = 0$

$\Rightarrow 4 + 9 - 4a - 6b + b^2 = 0$

$\Rightarrow -4a - 6b + b^2 = -13$

$\Rightarrow \underline{-2a - 4b + b^2 = -5}$ (subtracting)

$\Rightarrow -2a - 2b = -8$

$\Rightarrow a + b = 4$

$\Rightarrow a = 4 - b$

$\Rightarrow -2(4 - b) - 4b + b^2 = -5$

$\Rightarrow -8 + 2b - 4b + b^2 = -5$

$\Rightarrow b^2 - 2b - 3 = 0$

$(b + 1)(b - 3) = 0$

$\Rightarrow b = -1$ or $b = 3$

$\Rightarrow a = 4 - (-1) = 5$ or $a = 4 - 3 = 1$

Centre $= (5, -1)$, radius $= 5$

Circle: $\Rightarrow (x - 5)^2 + (y + 1)^2 = (5)^2 = 25$

and centre $= (1, 3)$, radius $= 1$

Circle: $(x - 1)^2 + (y - 3)^2 = (1)^2 = 1$

(iv) Distance between centres $= \sqrt{(1 - 5)^2 + (3 + 1)^2}$

$= \sqrt{16 + 16} + \sqrt{32} = 4\sqrt{2}$

Chapter 12

Exercise 12.1

1. (i) $3x - 5 > x + 3$

$\Rightarrow \quad 2x > 8$

$\Rightarrow \quad x > 4$

(ii) $6x - 5 \leqslant 2x - 1$

$\Rightarrow \quad 4x \leqslant 4$

$\Rightarrow \quad x \leqslant 1$

(iii) $1 - 3x > 10$

$\Rightarrow \quad -3x > 9$

$\Rightarrow \quad 3x < -9$

$\Rightarrow \quad x < -3$

2. (i) $\frac{x}{2} + 2 < 7, x \in N$

$\Rightarrow \quad \frac{x}{2} < 5$

$\Rightarrow \quad x < 10$

(ii) $\frac{1}{6}(x - 1) \geqslant \frac{1}{3}(x - 4), x \in Z$

$\Rightarrow \quad 1(x - 1) \geqslant 2(x - 4)$

$\Rightarrow \quad x - 1 \geq 2x - 8$

$\Rightarrow \quad -x \geqslant -7$

$\Rightarrow \quad x \leqslant 7$

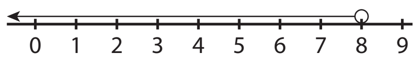
(iii) $\frac{4 - x}{2} > \frac{2 - x}{3}, x \in R$

$\Rightarrow \quad 3(4 - x) > 2(2 - x)$

$\Rightarrow \quad 12 - 3x > 4 - 2x$

$\Rightarrow \quad -x > -8$

$\Rightarrow \quad x < 8$

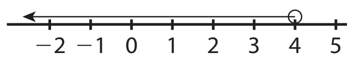


3. (i) $12x - 3(x - 3) < 45, x \in R$

$\Rightarrow \quad 12x - 3x + 9 < 45$

$\Rightarrow \quad 9x < 36$

$\Rightarrow \quad x < 4$



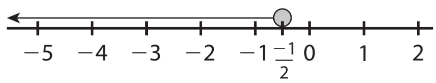
(ii) $x(x - 4) \geqslant x^2 + 2, x \in R$

$\Rightarrow x^2 - 4x \geqslant x^2 + 2$

$\Rightarrow -4x \geqslant 2$

$\Rightarrow \quad 4x \leqslant -2$

$\Rightarrow \quad x \leqslant -\frac{1}{2}$



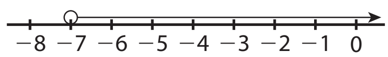
(iii) $x - 2(5 + 2x) < 11$

$\Rightarrow \quad x - 10 - 4x < 11$

$\Rightarrow \quad -3x < 21$

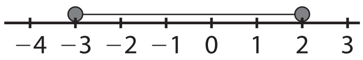
$\Rightarrow \quad 3x > -21$

$\Rightarrow \quad x > -7$



4. (i) $-2 \leqslant x + 1 \leqslant 3, x \in R$

$\Rightarrow \quad -3 \leqslant x \leqslant 2$

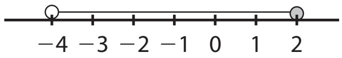


(ii) $13 > 1 - 3x \geqslant 7, x \in R$

$\Rightarrow \quad 12 > -3x \geqslant 6$

$\Rightarrow \quad -12 < 3x \leqslant 6$

$\Rightarrow \quad -4 < x \leqslant 2$

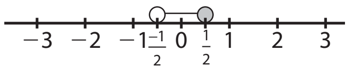


(iii) $3 \geq 4x + 1 > -1$

$\Rightarrow \quad 2 \geq 4x > -2$

$\Rightarrow \quad \frac{1}{2} \geq x > -\frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2} < x \leqslant \frac{1}{2}$



5. (i) $3 > \frac{3}{5}(x - 2) > 0$

$\Rightarrow \quad 15 > 3x - 6 > 0$

$\Rightarrow \quad 21 > 3x > 6$

$\Rightarrow \quad 7 > x > 2$

$\Rightarrow \quad 2 < x < 7$

(ii) $-4 \leqslant \frac{2}{5}(1 - 3x) \leqslant 1$

$\Rightarrow -20 \leqslant 2 - 6x \leqslant 5$

$\Rightarrow -22 \leqslant -6x \leqslant 3$

$\Rightarrow \quad 22 \geqslant 6x \geqslant -3$

$\Rightarrow \quad 3\frac{2}{3} \geqslant x \geqslant -\frac{1}{2}$

$\Rightarrow -\frac{1}{2} \leqslant x \leqslant 3\frac{2}{3}$

(iii) $3 \leqslant 2 - \frac{x}{7} < 4$

$\Rightarrow 21 \leqslant 14 - x \leqslant 28$

$\Rightarrow 7 \leqslant -x \leqslant 14$

$\Rightarrow -7 \geqslant x \geqslant -14$

$\Rightarrow -14 \leqslant x \leqslant -7$

6. $3(x - 2) > x - 4$ and $4x + 12 > 2x + 17, x \in R$

$\Rightarrow \quad 3x - 6 > x - 4 \qquad \Rightarrow \quad 2x > 5$

$\Rightarrow \quad 2x > 2 \qquad \Rightarrow \quad x > 2\frac{1}{2}$

$\Rightarrow \quad x > 1$

$\Rightarrow \quad \text{ANS}: x > 2\frac{1}{2}$

7. (i) $2x - 5 < x - 1, x \in R$

$\Rightarrow \quad x < 4$

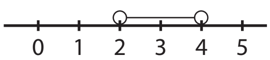
(ii) $7(x + 1) > 23 - x, x \in R$

$\Rightarrow \quad 7x + 7 > 23 - x$

$\Rightarrow \quad 8x > 16$

$\Rightarrow \quad x > 2$

(iii) $2 < x < 4, x \in R$



8. (i) $2x - 3 > 2, x \in R$

$\Rightarrow \quad 2x > 5$

$\Rightarrow \quad x > 2\frac{1}{2}$

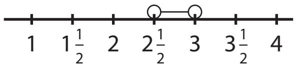
(ii) $3(x + 2) < 12 + x, x \in R$

$\Rightarrow \quad 3x + 6 < 12 + x$

$\Rightarrow \quad 2x < 6$

$\Rightarrow \quad x < 3$

(iii) $2\frac{1}{2} < x < 3, x \in R$



9. (i) $15 - x < 2(11 - x), x \in Z$

$\Rightarrow \quad 15 - x < 22 - 2x$

$\Rightarrow \quad x < 7$

(ii) $5(3x - 1) > 12x + 19, x \in Z$

$\Rightarrow \quad 15x - 5 > 12x + 19$

$\Rightarrow \quad 3x > 24$

$\Rightarrow \quad x > 8$

(iii) $\therefore$ Null set

10. (i) $3x + 8 \leqslant 20, x \in N$

$\Rightarrow 3x \leqslant 12$

$\Rightarrow x \leqslant 4$

(ii) $2(3x - 7) \geq x + 6, x \in N$

$\Rightarrow \quad 6x - 14 \geqslant x + 6$

$\Rightarrow \quad 5x \geqslant 20$

$\Rightarrow \quad x \geqslant 4$

(iii) $x = 4$

11. Width $= x$, Length $= x + 1$

Perimeter $= 2(x) + 2(x + 1) \leqslant 38$

$\Rightarrow 2x + 2x + 2 \leqslant 38$

$\Rightarrow \quad 4x \leqslant 36$

$\Rightarrow \quad x \leqslant 9$

Width $= 9$ m, length $= 10$ m

12. $100 < 2^n < 200$

$\Rightarrow \quad 2^{6.645} < 2^n < 2^{7.645}$

$\Rightarrow \quad 6.645 < n < 7.645$

Hence $a = 6.645$, $b = 7.645$, $n = 7$

13. Example 1: $4 > 3 > 0$ Thus $4^2 > 3^2$

Example 2: $4 > 3 > 0$ Thus $4^{-2} < 3^{-2}$, as $\frac{1}{16} < \frac{1}{9}$.

(i) if $a < b < 0$ and $n(odd) > 0$, then $a^n < b^n$.

(ii) if $a < b < 0$ and $n(even) > 0$, then $a^n > b^n$.

(iii) if $a < b < 0$ and $n(odd) < 0$, then $a^n > b^n$.

(iv) if $a < b < 0$ and $n(even) < 0$, then $a^n < b^n$.

14. $5 - 3x < -10 \quad \cap \quad 4x + 6 < 32, \quad x \in Z$

$\Rightarrow \quad -3x < -15 \quad \cap \quad 4x < 26$

$\Rightarrow \quad 3x > 15 \quad \cap \quad x < 6\frac{1}{2}$

$\Rightarrow \quad x > 5 \quad \cap \quad x < 6\frac{1}{2}, \quad x \in Z$

$\Rightarrow \quad x = 6$

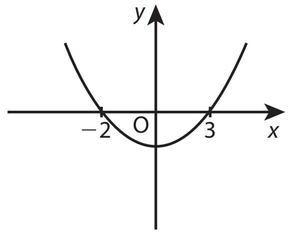
Exercise 12.2

1. (i) Let $x^2 - x - 6 = 0$

$\Rightarrow \quad (x + 2)(x - 3) = 0$

$\Rightarrow \quad$ Roots: $x = -2, 3$

Hence $\quad x^2 - x - 6 \geqslant 0 \quad \Rightarrow \quad x \leqslant -2 \quad \text{or} \quad x \geqslant 3$

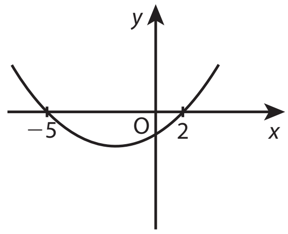


(ii) Let $x^2 + 3x - 10 = 0$

$\Rightarrow \quad (x + 5)(x - 2) = 0$

$\Rightarrow \quad$ Roots: $x = -5, 2$

Hence $\quad x^2 + 3x - 10 \leqslant 0 \quad \Rightarrow \quad -5 \leqslant x \leqslant 2$

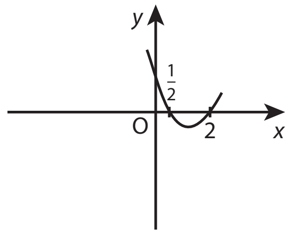


(iii) Let $2x^2 - 5x + 2 = 0$

$\Rightarrow \quad (2x - 1)(x - 2)=0$

$\Rightarrow \quad$ Roots: $x = \frac{1}{2}, 2$

Hence $\quad 2x^2 - 5x + 2 < 0 \quad \Rightarrow \quad \frac{1}{2} < x < 2$

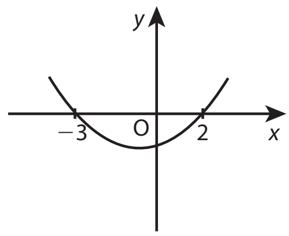


2. (i) Let $6 - x - x^2 = 0$

$\Rightarrow \quad (3 + x)(2 - x) = 0$

$\Rightarrow \quad$ Roots: $x = -3, 2$

Hence $\quad 6 - x - x^2 \geq 0 \quad \Rightarrow \quad -3 \leqslant x \leqslant 2$

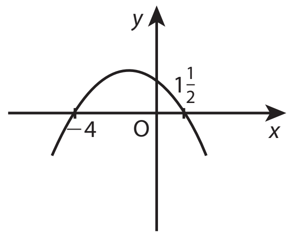


(ii) Let $12 - 5x - 2x^2 = 0$

$\Rightarrow \quad (4 + x)(3 - 2x) = 0$

$\Rightarrow \quad$ Roots: $x = -4, 1\frac{1}{2}$

Hence $\quad 12 - 5x - 2x^2 > 0 \quad \Rightarrow \quad -4 < x < 1\frac{1}{2}$



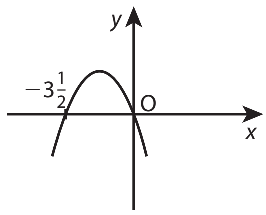
(iii) Let $-2x^2 - 7x = 0$

$\Rightarrow \quad 2x^2 + 7x = 0$

$\Rightarrow \quad x(2x + 7) = 0$

$\Rightarrow \quad$ Roots: $x = 0, -3\frac{1}{2}$

Hence $\quad -2x^2 - 7x \geqslant 0 \quad \Rightarrow \quad -3\frac{1}{2} \leqslant x \leqslant 0$

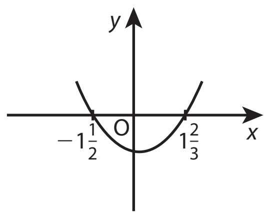


3. (i) Let $6x^2 - x - 15 = 0$

$\Rightarrow \quad (2x + 3)(3x - 5) = 0$

$\Rightarrow \quad$ Roots: $x = -1\frac{1}{2}, 1\frac{2}{3}$

Hence $\quad 6x^2 - x - 15 > 0 \quad \Rightarrow \quad x < -1\frac{1}{2} \quad \text{or} \quad x > 1\frac{2}{3}$

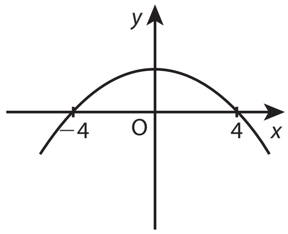


(ii) Let $16 - x^2 = 0$

$\Rightarrow \quad (4 + x)(4 - x) = 0$

$\Rightarrow \quad$ Roots: $x = -4, 4$

Hence $\quad 16 - x^2 \leqslant 0 \quad \Rightarrow \quad x \leqslant -4 \quad$ or $\quad x \geq 4$



(iii) Let $2(x^2 - 6) = 5x$

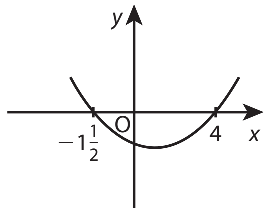
$\Rightarrow \quad 2x^2 - 12 = 5x$

$\Rightarrow \quad 2x^2 - 5x - 12 = 0$

$\Rightarrow \quad (2x + 3)(x - 4) = 0$

$\Rightarrow \quad$ Roots: $x = -1\frac{1}{2}, 4$

Hence $\quad 2(x^2 - 6) \geq 5x \quad \Rightarrow \quad x \leqslant -1\frac{1}{2} \quad$ or $\quad x \geq 4$



4. Let $(4 - x)(1 - x) = x + 11$

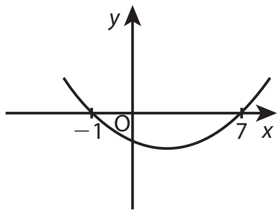
$\Rightarrow \quad 4 - 5x + x^2 - x - 11 = 0$

$\Rightarrow \quad x^2 - 6x - 7 = 0$

$\Rightarrow \quad (x + 1)(x - 7) = 0$

$\Rightarrow \quad$ Roots: $x = -1, 7$

Hence $\quad (4 - x)(1 - x) < x + 11 \quad \Rightarrow \quad -1 < x < 7$



5. Let $x^2 - 6x + 2 = 0$

$$\Rightarrow \quad \text{Roots:} \quad x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$$

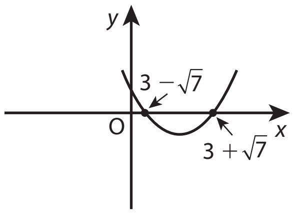
$$= \frac{6 \pm \sqrt{36 - 8}}{2}$$

$$= \frac{6 \pm \sqrt{28}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{2}$$

$$= 3 - \sqrt{7}, 3 + \sqrt{7}$$

Hence $\quad x^2 - 6x + 2 \leqslant 0 \quad \Rightarrow \quad 3 - \sqrt{7} \leqslant x \leqslant 3 + \sqrt{7}$



6. $x^2 + (k + 1)x + 1 = 0$

Real Roots $\quad \Rightarrow \quad b^2 - 4ac \geq 0$

$\Rightarrow \quad (k + 1)^2 - 4(1)(1) \geqslant 0$

$\Rightarrow \quad k^2 + 2k + 1 - 4 \geqslant 0$

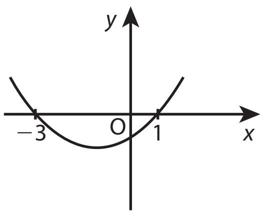
$\Rightarrow \quad k^2 + 2k + 1 - 3 \geqslant 0$

Solve $k^2 + 2k - 3 = 0$

$\Rightarrow \quad (k + 3)(k - 1) = 0$

$\Rightarrow \quad$ Roots $= -3, 1$

Hence $\quad k^2 + 2k - 3 \geqslant 0 \quad \Rightarrow \quad k \leqslant -3$ or $k \geqslant 1$



7. $kx^2 + 4x + 3 + k = 0$

Real Roots $\quad \Rightarrow \quad b^2 - 4ac \geqslant 0$

$\Rightarrow \quad (4)^2 - 4(k)(3 + k) \geqslant 0$

$\Rightarrow \quad 16 - 12k - 4k^2 \geqslant 0$

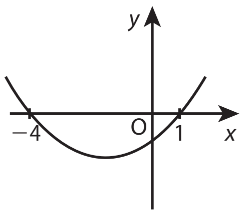
$\Rightarrow \quad k^2 + 3k - 4 \leqslant 0$

solve $k^2 + 3k - 4 = 0$

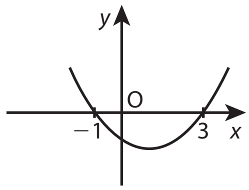
$\Rightarrow \quad (k + 4)(k - 1) = 0$

$\Rightarrow \quad$ Roots: $\quad k = -4, 1$

Hence $\quad k^2 + 3k - 4 \leqslant 0 \quad \Rightarrow \quad -4 \leqslant k \leqslant 1$



8. $px^2 + (p+3)x + p = 0$



Real Roots $\Rightarrow \quad b^2 - 4ac \geqslant 0$

$\Rightarrow \quad (p+3)^2 - 4(p)(p) \geqslant 0$

$\Rightarrow \quad p^2 + 6p + 9 - 4p^2 \geqslant 0$

$\Rightarrow \quad -3p^2 + 6p + 9 \geqslant 0$

$\Rightarrow \quad p^2 - 2p - 3 \leqslant 0$

Solve $p^2 - 2p - 3 = 0$

$\Rightarrow \quad (p+1)(p-3) = 0$

$\Rightarrow \quad$ Roots: $\quad p = -1, 3$

Hence $\quad p^2 - 2p - 3 \leqslant 0 \quad \Rightarrow \quad -1 \leqslant p \leqslant 3$

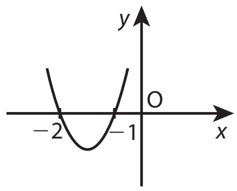
$x = -2 \quad \Rightarrow \quad p(-2)^2 + (p+3)(-2) + p = 0$

$\Rightarrow \quad 4p - 2p - 6 + p = 0$

$\Rightarrow \quad 3p = 6$

$\Rightarrow \quad p = 2$

9. (i) $\dfrac{x+3}{x+2} < 2, x \neq -2$



$\Rightarrow \quad \dfrac{x+3}{x+2}(x+2)^2 < 2(x+2)^2$

$\Rightarrow \quad (x+3)(x+2) < 2(x^2 + 4x + 4)$

$\Rightarrow \quad x^2 + 5x + 6 < 2x^2 + 8x + 8$

$\Rightarrow \quad x^2 + 3x + 2 > 0$

Solve $x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x+2)(x+1) = 0$

$\Rightarrow \quad$ Roots $x = -2, -1$

Hence $\quad x^2 + 3x + 2 > 0 \quad \Rightarrow \quad x < -2 \quad$ or $\quad x > -1$

(ii) $\dfrac{x+5}{x-3} > 1, x \neq 3$

$\Rightarrow \quad \dfrac{x+5}{x-3}(x-3)^2 > 1(x-3)^2$

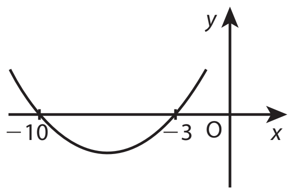
$\Rightarrow \quad (x+5)(x-3) > 1(x^2 - 6x + 9)$

$\Rightarrow \quad x^2 + 2x - 15 > x^2 - 6x + 9$

$\Rightarrow \quad 8x > 24$

$\Rightarrow \quad x > 3$

(iii) $\dfrac{2x-1}{x+3} > 3, x \neq 3$



$\Rightarrow \quad \dfrac{2x-1}{x+3}(x+3)^2 > 3(x+3)^2$

$\Rightarrow \quad (2x-1)(x+3) > 3(x^2 + 6x + 9)$

$\Rightarrow \quad 2x^2 + 5x - 3 > 3x^2 + 18x + 27$

$\Rightarrow \quad x^2 + 13x + 30 < 0$

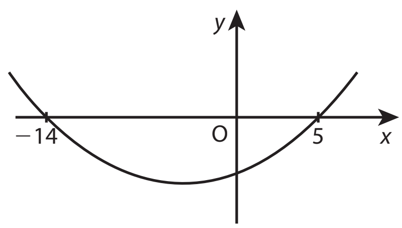
Solve $x^2 + 13x + 30 = 0$

$\Rightarrow \quad (x+10)(x+3) = 0$

$\Rightarrow \quad x = -10, -3$

Hence $\quad x^2 + 13x + 30 < 0 \quad \Rightarrow \quad -10 < x < -3$

10. (i) $\dfrac{3x+4}{x-5} > 2, x \neq 5$



$\Rightarrow \quad \dfrac{3x+4}{x-5}(x-5)^2 > 2(x-5)^2$

$\Rightarrow \quad (3x+4)(x-5) > 2(x^2 - 10x + 25)$

$\Rightarrow \quad 3x^2 - 11x - 20 > 2x^2 - 20x + 50$

$\Rightarrow \quad x^2 + 9x - 70 > 0$

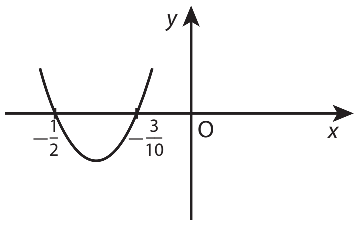
Solve $x^2 + 9x - 70 = 0$

$\Rightarrow$ $(x + 14)(x - 5) = 0$

$\Rightarrow$ $x = -14, 5$

Hence $x^2 + 9x - 70 > 0 \quad \Rightarrow \quad x < -14$ or $x > 5$

(ii) $\dfrac{1 - 2x}{4x + 2} > 2, x \neq -\dfrac{1}{2}$



$\Rightarrow$ $\dfrac{1 - 2x}{4x + 2}(4x + 2)^2 > 2(4x + 2)^2$

$\Rightarrow$ $(1 - 2x)(4x + 2) > 2(16x^2 + 16x + 4)$

$\Rightarrow$ $4x + 2 - 8x^2 - 4x > 32x^2 + 32x + 8$

$\Rightarrow$ $40x^2 + 32x + 6 < 0$

$\Rightarrow$ $20x^2 + 16x + 3 < 0$

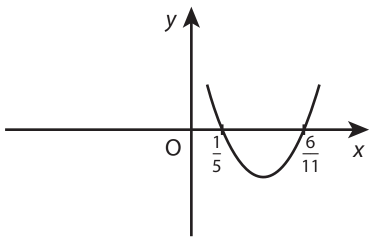
Solve $20x^2 + 16x + 3 = 0$

$\Rightarrow$ $(2x + 1)(10x + 3) = 0$

$\Rightarrow$ Roots: $x = -\dfrac{1}{2}, -\dfrac{3}{10}$

Hence $20x^2 + 16x + 3 < 0 \quad \Rightarrow \quad -\dfrac{1}{2} < x < -\dfrac{3}{10}$

(iii) $\dfrac{3 + 4x}{5x - 1} > 3, x \neq \dfrac{1}{5}$



$\Rightarrow$ $\dfrac{3 + 4x}{5x - 1}(5x - 1)^2 > 3(5x - 1)^2$

$\Rightarrow$ $(3 + 4x)(5x - 1) > 3(25x^2 - 10x + 1)$

$\Rightarrow$ $15x - 3 + 20x^2 - 4x > 75x^2 - 30x + 3$

$\Rightarrow$ $55x^2 - 41x + 6 > 0$

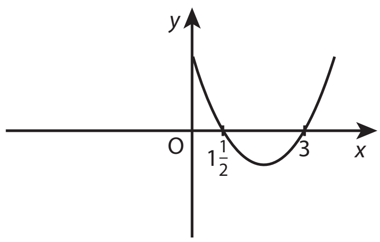
Solve $55x^2 - 41x + 6 = 0$

$\Rightarrow$ $(5x - 1)(11x - 6) = 0$

$\Rightarrow$ Roots: $x = \dfrac{1}{5}, \dfrac{6}{11}$

Hence $55x^2 - 41x + 6 < 0 \quad \Rightarrow \quad \dfrac{1}{5} < x < \dfrac{6}{11}$

11. (i) $\dfrac{x}{2x - 3} \leqslant 1, \quad x \neq \dfrac{3}{2}$



$\Rightarrow$ $\dfrac{x}{2x - 3}(2x - 3)^2 \leqslant 1(2x - 3)^2$

$\Rightarrow$ $x(2x - 3) \leqslant 1(4x^2 - 12x + 9)$

$\Rightarrow$ $2x^2 - 3x \leqslant 4x^2 - 12x + 9$

$\Rightarrow$ $2x^2 - 9x + 9 \geqslant 0$

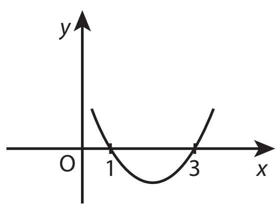
Solve $2x^2 - 9x + 9 = 0$

$\Rightarrow$ $(2x - 3)(x - 3) = 0$

$\Rightarrow$ Roots: $x = 1\dfrac{1}{2}, 3$

Hence $2x^2 - 9x + 9 \geqslant 0 \quad \Rightarrow \quad x \leqslant 1\dfrac{1}{2}$ or $x \geqslant 3$

(ii) $\dfrac{2x - 4}{x - 1} < 1, \quad x \neq 1$



$\Rightarrow$ $\dfrac{2x - 4}{x - 1}(x - 1)^2 < 1(x - 1)^2$

$\Rightarrow$ $(2x - 4)(x - 1) < 1(x^2 - 2x + 1)$

$\Rightarrow$ $2x^2 - 6x + 4 < x^2 - 2x + 1$

$\Rightarrow$ $x^2 - 4x + 3 < 0$

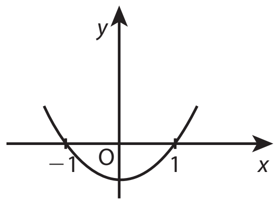
Solve $x^2 - 4x + 3 = 0$

$\Rightarrow$ $(x - 1)(x - 3) = 0$

$\Rightarrow$ $x = 1, 3$

Hence $x^2 - 4x + 3 < 0 \quad \Rightarrow \quad 1 < x < 3$

(iii) $\dfrac{x-5}{x-1} \leqslant 3, \quad x \neq 1$



$\Rightarrow \quad \dfrac{x-5}{x-1}(x-1)^2 \leqslant 3(x-1)^2$

$\Rightarrow \quad (x-5)(x-1) \leqslant 3(x^2 - 2x + 1)$

$\Rightarrow \quad x^2 - 6x + 5 \leqslant 3x^2 - 6x + 3$

$\Rightarrow \quad 2x^2 - 2 \geq 0$

$\Rightarrow \quad x^2 - 1 \geq 0$

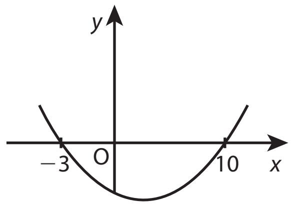
Solve $\quad x^2 - 1 = 0$

$\Rightarrow \quad (x+1)(x-1) = 0$

$\Rightarrow \quad$ Roots: $\quad x = -1, 1$

Hence $\quad x^2 - 1 \geqslant 0 \Rightarrow x \leqslant -1 \quad$ or $\quad x \geqslant 1$

12. (i) $\dfrac{2x-7}{x+3} < 1, \quad x \neq -3$



$\Rightarrow \quad \dfrac{2x-7}{x+3}(x+3)^2 < 1(x+3)^2$

$\Rightarrow \quad (2x-7)(x+3) < 1(x^2 + 6x + 9)$

$\Rightarrow \quad 2x^2 - x - 21 < x^2 + 6x + 9$

$\Rightarrow \quad x^2 - 7x - 30 < 0$

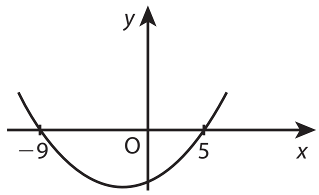
Solve $\quad x^2 - 7x - 30 = 0$

$\Rightarrow \quad (x+3)(x-10) = 0$

$\Rightarrow \quad$ Roots: $x = -3, 10$

Hence $\quad x^2 - 7x - 30 < 0 \quad \Rightarrow \quad -3 < x < 10$

(ii) $\dfrac{2x-3}{x-5} < \dfrac{3}{2}, \quad x \neq 5$



$\Rightarrow \quad \dfrac{2x-3}{x-5}(x-5)^2 < \dfrac{3}{2}(x-5)^2$

$\Rightarrow \quad (2x-3)(x-5) < \dfrac{3}{2}(x^2 - 10x + 25)$

$\Rightarrow \quad 2x^2 - 13x + 15 < \dfrac{3x^2 - 30x + 75}{2}$

$\Rightarrow \quad 4x^2 - 26x + 30 < 3x^2 - 30x + 75$

$\Rightarrow \quad x^2 + 4x - 45 < 0$

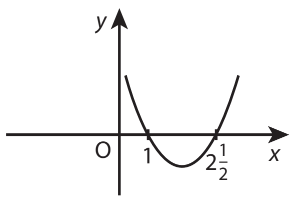
Solve $\quad x^2 + 4x - 45 = 0$

$\Rightarrow \quad (x+9)(x-5) = 0$

$\Rightarrow \quad x = -9, 5$

Hence $\quad x^2 + 4x - 45 < 0 \quad \Rightarrow \quad -9 < x < 5$

(iii) $\dfrac{x+2}{x-1} \leqslant 3, \quad x \neq 1$



$\Rightarrow \quad \dfrac{x+2}{x-1}(x-1)^2 \leqslant 3(x-1)^2$

$\Rightarrow \quad (x+2)(x-1) \leqslant 3(x^2 - 2x + 1)$

$\Rightarrow \quad x^2 + x - 2 \leqslant 3x^2 - 6x + 3$

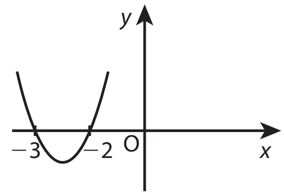
$\Rightarrow \quad 2x^2 - 7x + 5 \geqslant 0$

Solve $\quad 2x^2 - 7x + 5 = 0$

$\Rightarrow \quad (x-1)(2x-5) = 0$

$\Rightarrow \quad$ Roots: $x = 1, 2\frac{1}{2}$

Hence $\quad 2x^2 - 7x + 5 \geqslant 0 \quad \Rightarrow \quad x \leqslant 1 \quad$ or $\quad x \geqslant 2\frac{1}{2}$



13. From graph: $-3 > x > -2$

Using inequality $\Rightarrow 2x^2 + 4x > x^2 - x - 6, \quad x \in R$

$\Rightarrow x^2 + 5x + 6 > 0$

Solve $\quad x^2 + 5x + 6 = 0$

$\Rightarrow (x + 3)(x + 2) = 0$

$\Rightarrow$ Roots: $x = -3, -2$

Hence $\quad x^2 + 5x + 6 > 0 \Rightarrow x < -3 \quad$ or $\quad x > -2$

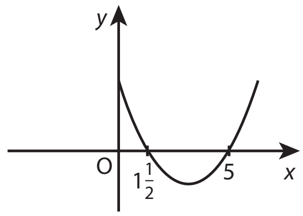
14. No real roots if $b^2 - 4ac < 0$

$x^2 + x + 1 = 0 \quad \Rightarrow \quad b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$

Hence $\quad x^2 + x + 1 > 0$

$\Rightarrow \quad x^2 + x + \frac{1}{4} + \frac{3}{4} > 0$

$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0 \qquad$ True



15. (i) $f(t) \leqslant 4 \Rightarrow -11 + 13t - 2t^2 \leqslant 4$

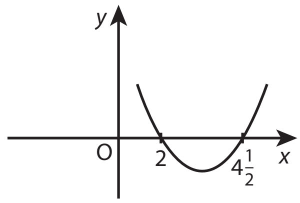
$\Rightarrow \quad 2t^2 - 13t + 15 \geq 0$

Solve $2t^2 - 13t + 15 = 0$

$\Rightarrow \quad (2t - 3)(t - 5) = 0$

$\Rightarrow t = 1\frac{1}{2}, 5$

Hence $\quad 2t^2 - 13t + 15 \geq 0 \quad \Rightarrow \quad t \leqslant 1\frac{1}{2} \quad$ or $\quad t \geqslant 5$



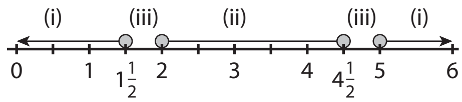
(ii) $f(t) \geq 7 \quad \Rightarrow \quad -11 + 13t - 2t^2 \geqslant 7$

$\Rightarrow \quad 2t^2 - 13t + 18 \leqslant 0$

$\Rightarrow \quad (t - 2)(2t - 9) = 0$

$\Rightarrow \quad$ Roots: $t = 2, 4\frac{1}{2}$

Hence $\quad 2t^2 - 13t + 18 \leqslant 0 \quad \Rightarrow \quad 2 \leqslant t \leqslant 4\frac{1}{2}$



(iii) $4 < f(t) < 7 \Rightarrow$

$1\frac{1}{2} < t < 2 \quad$ and $\quad 4\frac{1}{2} < t < 5$

16. (i) $x < -3 \quad$ or $\quad x > -\frac{1}{2}$

(ii) $x \leqslant 1 \quad$ or $\quad x \geq 3$

(iii) $-1\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$

(iv) $-1 < x < 5$

17. Length $= x \quad$ and $\quad$ Width $= x - 3$

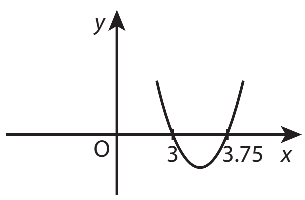
Ratio $< 5 \quad \Rightarrow \quad \frac{x}{x-3} < 5$

$\Rightarrow \quad \frac{x}{x-3}(x - 3)^2 < 5(x - 3)^2$

$\Rightarrow \quad x(x - 3) < 5(x^2 - 6x + 9)$

$\Rightarrow \quad x^2 - 3x < 5x^2 - 30x + 45$

$\Rightarrow \quad 4x^2 - 27x + 45 > 0$



Solve $4x^2 - 27x + 45 = 0$

$\Rightarrow \quad (x - 3)(4x - 15) = 0$

$\Rightarrow \quad$ Roots: $x = 3, 3.375$

Hence $\quad 4x^2 - 27x + 45 > 0 \quad \Rightarrow \quad x < 3 \quad$ or $\quad x > 3.75$

Since $x < 3$ is not valid, hence $\quad$ (i) Length > 3.75

(ii) Width > 0.75

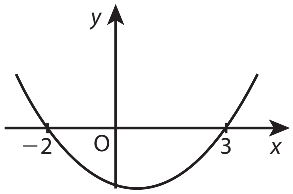
18. Positive graphs $\Rightarrow$ No real roots

$\Rightarrow \; b^2 - 4ac < 0$

$x^2 - 2px + p + 6 = 0 \;\Rightarrow\; (-2p)^2 - 4(1)(p + 6) < 0$

$\Rightarrow \; 4p^2 - 4p - 24 < 0$

$\Rightarrow \; p^2 - p - 6 < 0$



Solve $\quad p^2 - p - 6 = 0$

$\Rightarrow \; (p + 2)(p - 3) = 0$

$\Rightarrow$ Roots: $\quad p = -2, 3$

Hence $\quad p^2 - p - 6 < 0 \;\Rightarrow\; -2 < p < 3$

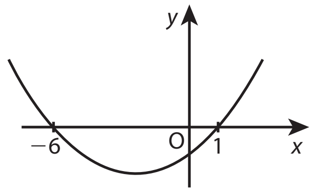
19. (i) Perimeter $< 50 \;\Rightarrow\; 2(x + 3) + 2(x + 2) < 50$

$\Rightarrow \; 2x + 6 + 2x + 4 < 50$

$\Rightarrow \; 4x < 50 - 10$

$\Rightarrow \; 4x < 40$

$\Rightarrow \; x < 10$



(ii) Area $> 12 \;\Rightarrow\; (x + 3)(x + 2) > 12$

$\Rightarrow \; x^2 + 5x + 6 > 12$

$\Rightarrow \; x^2 + 5x - 6 > 0$

Solve $\quad x^2 + 5x - 6 = 0$

$\Rightarrow \; (x + 6)(x - 1) = 0$

$\Rightarrow \; x = -6$ (Not valid), $\quad x = 1$ (valid)

Hence $\quad x^2 + 5x - 6 > 0 \;\Rightarrow\; x > 1$

(iii) $1 < x < 10$, by combining (i) and (ii) above

20. Use Pythagoras $\Rightarrow \; h^2 = x^2 + 3^2 = x^2 + 9$

$\Rightarrow \; h = \sqrt{x^2 + 9}$

Perimeter $> 8 \;\Rightarrow\; \sqrt{x^2 + 9} + x + 3 > 8$

$\Rightarrow \; \sqrt{x^2 + 9} > -x + 5$

$\Rightarrow \; x^2 + 9 > (-x + 5)^2$

$\Rightarrow \; x^2 + 9 > x^2 - 10x + 25$

$\Rightarrow \; 10x > 16$

$\Rightarrow \; x > 1.6$

Perimeter $< 12 \;\Rightarrow\; \sqrt{x^2 + 9} + x + 3 < 12$

$\Rightarrow \; \sqrt{x^2 + 9} < -x + 9$

$\Rightarrow \; x^2 + 9 < (-x + 9)^2$

$\Rightarrow \; x^2 + 9 < x^2 - 18x + 81$

$\Rightarrow \; 18x < 72$

$\Rightarrow \; x < 4$

Hence $\quad 1.6 < x < 4$

Since $x \in Z \;\Rightarrow\; x = 2$ m $\quad$ or $\quad 3$ m

Exercise 12.3

1. (i) $x + 3 = -1$ OR $x + 3 = 1$

$\Rightarrow \; x = -4$ OR $x = -2$

(ii) $x - 2 = -4$ OR $x - 2 = 4$

$\Rightarrow \; x = -2$ OR $x = 6$

(iii) $2x - 1 = -5$ OR $2x - 1 = 5$

$\Rightarrow \; 2x = -4$ OR $2x = 6$

$\Rightarrow \; x = -2$ OR $x = 3$

(iv) $3x - 2 = -x$ OR $3x - 2 = x$

$\Rightarrow$ $4x = 2$ OR $2x = 2$

$\Rightarrow$ $x = \frac{1}{2}$ OR $x = 1$

(v) $2(x - 3) = -2$ OR $2(x - 3) = 2$

$\Rightarrow$ $x - 3 = -1$ OR $x - 3 = 1$

$\Rightarrow$ $x = 2$ OR $x = 4$

(vi) $x - 5 = -(x + 1)$ OR $x - 5 = +(x + 1)$

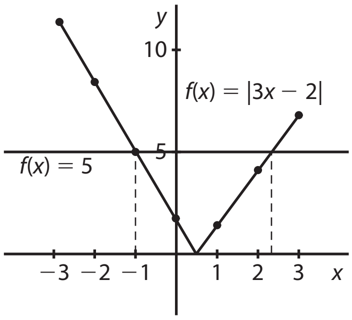
$\Rightarrow$ $x - 5 = -x - 1$ OR $x - 5 = x + 1$

$\Rightarrow$ $2x = 4$ OR $-5 = +1$ (Not valid)

$\Rightarrow$ $x = 2$

2. Copy and complete the following table and hence sketch a graph of $f(x) = |3x - 2|$

x	-3	-2	-1	0	1	2	3
$f(x) = \|3x - 2\|$	11	8	5	2	1	4	7



Solve $|3x - 2| = 5$.

$\Rightarrow x = -1, 2\frac{1}{3}$

3. $f(x) = |x|$, $g(x) = |x - 4|$, $h(x) = |x + 3|$

$f(-2) = |-2| = 2 \Rightarrow$ point $(-2, 2) \in f(x)$

$g(2) = |2 - 4| = |-2| = 2 \Rightarrow$ point $(2, 2) \in g(x)$

$h(-5) = |-5 + 3| = |-2| = 2 \Rightarrow$ point $(-5, 2) \in h(x)$

4. 2 points on $f(x)$ are $(-1, 0)$ and $(0, 1)$

$f(-1) = |a(-1) + b| = 0$ and $f(0) = |a(0) + b| = 1$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 1$

Hence, $-a + 1 = 0$

$\Rightarrow$ $a = 1$

Hence, $f(x) = |x + 1|$.

2 points on $g(x)$ are $(-1, 0)$ and $(0, 2)$

$g(-1) = |a(-1) + b| = 0$ and $g(0) = |a(0) + b| = 2$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 2$

Hence $-a + 2 = 0$

$\Rightarrow$ $a = 2$

Hence, $g(x) = |2x + 2|$

2 points on $h(x)$ are $(-1, 0)$ and $(0, 3)$

$h(-1) = |a(-1) + b| = 0$ and $h(0) = |a(0) + b| = 3$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 3$

Hence $-a + 3 = 0$

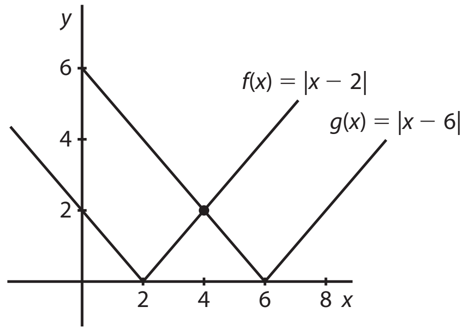
$\Rightarrow$ $a = 3$

Hence, $h(x) = |3x + 3|$

$x = -2 \Rightarrow f(-2) = |-2 + 1| = 1$ $\Rightarrow (-2, 1) \in f(x)$

$x = -2 \Rightarrow g(-2) = |2(-2) + 2| = 2$ $\Rightarrow (-2, 2) \in g(x)$

$x = -2 \Rightarrow h(-2) = |3(-2) + 3| = 3$ $\Rightarrow (-2, 3) \in h(x)$



5. $f(x) = |x - 2|$

$\Rightarrow f(0) = |0 - 2| = 2 \quad (0, 2) \in f(x)$

$\Rightarrow f(2) = |2 - 2| = 0 \quad (2, 0) \in f(x)$

$\Rightarrow f(4) = |4 - 2| = 2 \quad (4, 2) \in f(x)$

$g(x) = |x - 6|$

$\Rightarrow g(0) = |0 - 6| = 6 \quad (0, 6) \in g(x)$

$\Rightarrow g(6) = |6 - 6| = 0 \quad (6, 0) \in g(x)$

$\Rightarrow g(8) = |8 - 6| = 2 \quad (8, 2) \in g(x)$

Hence, $|x - 2| = |x - 6|$ at $x = 4$.

Solve $|x - 2| = |x - 6|$

$\Rightarrow x - 2 = -(x - 6)$ OR $x - 2 = +(x - 6)$

$\Rightarrow x - 2 = -x + 6$ OR $x - 2 = x - 6$

$\Rightarrow 2x = 8$ OR $-2 = -6$ (Not valid)

$\Rightarrow x = 4$

6. (i) $|x - 6| < 2$

$\Rightarrow -2 < x - 6 < 2$

$\Rightarrow 4 < x < 8$

(ii) $|x + 2| \leqslant 4$

$\Rightarrow -4 \leqslant x + 2 \leqslant 4$

$\Rightarrow -6 \leqslant x \leqslant 2$

(iii) $|2x - 1| \geqslant 5$

$\Rightarrow -5 \geqslant 2x - 1 \geqslant 5$

$\Rightarrow -4 \geqslant 2x \geqslant 6 \qquad \Rightarrow -4 \geqslant 2x$ or $2x \geqslant 6$

$\Rightarrow 2x \leqslant -4$ or $x \geqslant 3$

$\Rightarrow x \leqslant -2$ or $x \geqslant 3$

(iv) $|2x - 1| \geqslant 11$

$\Rightarrow -11 \geqslant 2x - 1 \geqslant 11$

$\Rightarrow -10 \geqslant 2x \geqslant 12 \qquad \Rightarrow -10 \geqslant 2x$ or $2x \geqslant 12$

$\Rightarrow 2x \leqslant -10$ or $x \geqslant 6$

$\Rightarrow x \leqslant -5$ or $x \geqslant 6$

(v) $|3x + 5| < 4$

$\Rightarrow -4 < 3x + 5 < 4$

$\Rightarrow -9 < 3x < -1$

$\Rightarrow -3 < x < -\frac{1}{3}$

(vi) $|x - 4| < 3$

$\Rightarrow -3 < x - 4 < 3$

$\Rightarrow 1 < x < 7$

7. (i) $|2x - 1| \geqslant 7$

$\Rightarrow -7 \geqslant 2x - 1 \geqslant 7$

$\Rightarrow -6 \geqslant 2x \geqslant 8$

$\Rightarrow -3 \geqslant x \geqslant 4$

(ii) $|3x + 4| \leqslant |x + 2|$

$\Rightarrow (3x + 4)^2 \leqslant (x + 2)^2$

$\Rightarrow 9x^2 + 24x + 16 \leqslant x^2 + 4x + 4$

$\Rightarrow 8x^2 + 20x + 12 \leqslant 0$

$\Rightarrow 2x^2 + 5x + 3 \leqslant 0$

Solve $2x^2 + 5x + 3 = 0$

$\Rightarrow (2x + 3)(x + 1) = 0$

$\Rightarrow$ Roots: $x = -1\frac{1}{2}, -1$

Hence, $2x^2 + 5x + 3 \leqslant 0 \Rightarrow -1\frac{1}{2} \leqslant x \leqslant -1$

(iii) $2|x - 1| \leqslant |x + 3|$

$\Rightarrow [2(x - 1)]^2 \leqslant (x + 3)^2$

$\Rightarrow 4(x^2 - 2x + 1) \leqslant x^2 + 6x + 9$

$\Rightarrow 4x^2 - 8x + 4 \leqslant x^2 + 6x + 9$

$\Rightarrow 3x^2 - 14x - 5 \leqslant 0$

Solve $3x^2 - 14x - 5 = 0$

$\Rightarrow (3x + 1)(x - 5) = 0$

$\Rightarrow$ Roots: $x = -\frac{1}{3}, 5$

Hence, $3x^2 - 14x - 5 \leqslant 0 \Rightarrow -\frac{1}{3} \leqslant x \leqslant 5$

8. $f(x) = |x| - 4$

$f(-4) = |-4| - 4 = 4 - 4 = 0$ $\quad (-4, 0) \in f(x)$

$f(0) = |0| - 4 = -4$ $\quad (0, -4) \in f(x)$

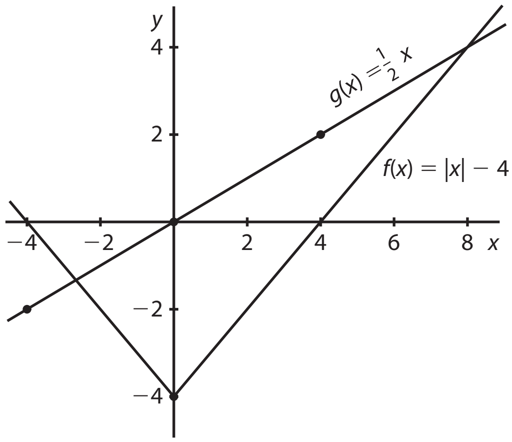
$f(4) = |4| - 4 = 4 - 4 = 0$ $\quad (4, 0) \in f(x)$

$g(x) = \frac{1}{2}x$

$g(-4) = \frac{1}{2}(-4) = -2$ $\quad (-4, -2) \in g(x)$

$g(0) = \frac{1}{2}(0) = 0$ $\quad (0, 0) \in g(x)$

$g(4) = \frac{1}{2}(4) = 2$ $\quad (4, 2) \in g(x)$



$f(x) \leqslant g(x) \Rightarrow |x| - 4 \leqslant \frac{1}{2}x$

Solve $|x| = \frac{1}{2}x + 4$

$\Rightarrow x^2 = \frac{1}{4}x^2 + 4x + 16$

$\Rightarrow 4x^2 = x^2 + 16x + 64$

$\Rightarrow 3x^2 - 16x - 64 = 0$

$\Rightarrow (3x + 8)(x - 8) = 0$

$\Rightarrow$ Roots $x = -2\frac{2}{3}, 8$

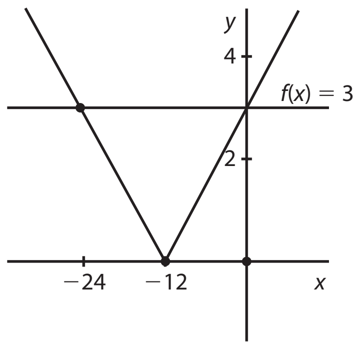
Hence, $3x^2 - 16x - 64 \leqslant 0 \Rightarrow -2\frac{2}{3} \leqslant x \leqslant 8$

9. $f(x) = \left|\frac{1}{4}x + 3\right|$

$f(-24) = \left|\frac{1}{4}(-24) + 3\right| = |-6 + 3| = |-3| = 3 \Rightarrow (-24, 3) \in f(x)$

$f(-12) = \left|\frac{1}{4}(-12) + 3\right| = |-3 + 3| = 0 \Rightarrow (-12, 0) \in f(x)$

$f(0) = \left|\frac{1}{4}(0) + 3\right| = |0 + 3| = 3 \Rightarrow (0, 3) \in f(x)$



Hence, $\left|\frac{1}{4}x + 3\right| \geqslant 3 \Rightarrow x \leqslant -24$ or $x \geqslant 0$

10. $|1 + 2x| < |x + 2|$

$\Rightarrow (1 + 2x)^2 < (x + 2)^2$

$\Rightarrow 1 + 4x + 4x^2 < x^2 + 4x + 4$

$\Rightarrow 3x^2 - 3 < 0$

$\Rightarrow x^2 - 1 < 0$

Solve $x^2 - 1 = 0$

$\Rightarrow (x + 1)(x - 1) = 0$

$\Rightarrow$ Roots: $x = -1, 1$

Hence, $x^2 - 1 < 0 \Rightarrow -1 < x < 1$

11. $\left|\dfrac{1}{1 + 2x}\right| = 1 \Rightarrow \left(\dfrac{1}{1 + 2x}\right)^2 = (1)^2$

$\Rightarrow \dfrac{1}{1 + 4x + 4x^2} = \dfrac{1}{1}$

$\Rightarrow 1 + 4x + 4x^2 = 1$

$\Rightarrow 4x^2 + 4x = 0$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow$ Roots: $x = 0, -1$

Solve $\left|\dfrac{1}{1 + 2x}\right| < 1$

Hence, $\dfrac{1}{1 + 4x + 4x^2} < 1$

$\Rightarrow 1 + 4x + 4x^2 > 1$

$\Rightarrow 4x^2 + 4x > 0$

$\Rightarrow x^2 + x > 0$

$\Rightarrow -1 > x > 0$

12. (i) $-4 < x < 2$

(ii) $x < -4$ or $x > 2$

(iii) $1\frac{1}{4} < x < 3\frac{1}{2}$

(iv) $2 < x < 3$

(v) $1\frac{1}{4} < x < 2$

(vi) $2 < x < 3$

(vii) $3 < x < 3\frac{1}{2}$

13. (i) $\dfrac{x}{2x - 1} < -2$

$\Rightarrow + \left(\dfrac{x}{2x - 1}\right) < -2$

$\Rightarrow \dfrac{x}{2x - 1}(2x - 1)^2 < -2(2x - 1)^2$

$\Rightarrow x(2x - 1) < -2(4x^2 - 4x + 1)$

$\Rightarrow 2x^2 - x < -8x^2 + 8x - 2$

$\Rightarrow 10x^2 - 9x + 2 < 0$

Solve $10x^2 - 9x + 2 = 0$

$\Rightarrow (5x - 2)(2x - 1) = 0$

$\Rightarrow$ Roots: $x = \dfrac{2}{5}, \dfrac{1}{2}$

Hence, $10x^2 - 9x + 2 < 0 \Rightarrow \dfrac{2}{5} < x\dfrac{1}{2}$

(ii) $|x-3| = 2|x-1|$

$\Rightarrow (x-3)^2 = [2(x-1)]^2$

$\Rightarrow x^2 - 6x + 9 = 4(x^2 - 2x + 1)$

$\Rightarrow x^2 - 6x + 9 = 4x^2 - 8x + 4$

$\Rightarrow 3x^2 - 2x - 5 = 0$

$\Rightarrow (x+1)(3x-5) = 0$

$\Rightarrow$ Roots: $x = -1, 1\frac{2}{3}$

(iii) $|x-1| - |2x+1| > 0$

$\Rightarrow |x-1| > |2x+1|$

$\Rightarrow (x-1)^2 > (2x+1)^2$

$\Rightarrow x^2 - 2x + 1 > 4x^2 + 4x + 1$

$\Rightarrow 3x^2 + 6x < 0$

$\Rightarrow x^2 + 2x < 0$

Solve $x^2 + 2x = 0$

$\Rightarrow x(x+2) = 0$

$\Rightarrow$ Roots: $x = -2, 0$

Hence, $x^2 + 2x < 0 \Rightarrow -2 < x < 0$

Exercise 12.4

1. Proof: Assume x and y are both positive integers and $x^2 - y^2 = (x+y)(x-y)$.

If $x > y$, then $(x+y)$ and $(x-y)$ are positive integers; hence, $(x+y)(x-y)$ is a positive integer $\neq 1$.

If $x < y$, then $(x+y)$ is a positive integer and $(x-y)$ is a negative integer; hence, $(x+y)(x-y)$ is a negative integer $\neq 1$.

Hence, there is a contradiction in both cases.

$\therefore$ There are no positive integer solutions to $x^2 - y^2 = 1$.

2. Proof: Assume $(a+b)$ is a rational number.

Hence, $(a+b)$ can be written as $\frac{p}{q}$ where $p, q \in Z, q \neq 0$.

Since "a" is a rational number $= \frac{m}{n}$ where $m, n \in Z, n \neq 0$

Then, $a + b = \frac{p}{q}$

$\Rightarrow \frac{m}{n} + b = \frac{p}{q}$

$\Rightarrow b = \frac{p}{q} - \frac{m}{n}$

$= \frac{pn - mq}{qn}$ which is rational.

This is a contradiction as b is a irrational number; hence, $a + b$ is an irrational number.

3. Proof: Assume x and y are both positive integers
and $x^2 - y^2 = (x + y)(x - y)$.
If $x > y$, then $(x + y)$ and $(x - y)$ they are positive integers;
hence, $(x + y)(x - y)$ is a positive integer $\neq 10$.
Note: $x + y = 5$ and $x + y = 10$

$$\begin{array}{rcl} x + y = 5 & \quad & x + y = 10 \\ \underline{x - y = 2} & & \underline{x - y = 1} \\ 2x = 7 & & 2x = 11 \\ \Rightarrow x = 3\frac{1}{2} \notin Z & & \Rightarrow x = 5\frac{1}{2} \in Z \end{array}$$

If $x < y$, then $(x + y)$ is a positive integer
and $(x - y)$ is a negative integer;
hence, $(x + y)(x - y)$ is a negative integer $\neq 10$.
Hence, there is a contradiction in both cases.
$\therefore$ There are no positive integer solutions to $x^2 - y^2 = 10$.

4. Proof: a divides b $\Rightarrow$ $b = k(a), k \in N$
b divides c $\Rightarrow$ $c = m(b), m \in N$
$\Rightarrow$ $c = m[k(a)]$
$\Rightarrow$ $c = mk(a), m, k \in N$
Then a divides c.

5. Proof: a divides b $\Rightarrow$ $b = k(a), k \in N$
a divides c $\Rightarrow$ $c = m(a), m \in N$
hence, $(b + c) = k(a) + m(a)$
$= (k + m)(a), k, m \in N$
Then a divides $(b + c)$.

6. Proof: $a^2 + b^2 \geqslant 2ab$
$\Rightarrow$ $a^2 - 2ab + b^2 \geqslant 0$
$\Rightarrow$ $(a - b)(a - b) \geqslant 0$
$\Rightarrow$ $(a - b)^2 \geqslant 0$, true for $a, b \in R$
Hence, $a^2 + b^2 \geqslant 2ab$.

7. Proof: a is a rational number
Hence, a can be written as $\frac{p}{q}$ where $p, q \in Z, q \neq 0$.
b is a rational number.
Hence, b can be written as $= \frac{m}{n}$ where $m, n \in Z, n \neq 0$.
Hence, $a + b = \frac{p}{q} + \frac{m}{n} = \frac{pn + mq}{qn}$
$=$ a rational number, as $p, q, m, n \in Z$
$qn \neq 0$.

8. Proof: x is an odd number $= 2a + 1$, where $a \in N$.
y is an odd number $= 2b + 1$, where $b \in N$.
Hence, $x + y = 2a + 1 + 2b + 1$
$= 2a + 2b + 2$
$= 2(a + b + 1)$
Hence, $x + y$ has a factor of 2.
$\Rightarrow x + y$ must be even.
Hence, the sum of 2 odd numbers is always even.

Exercise 12.5

1. (i) $a^2 + 2ab + b^2 \geq 0$

$\Rightarrow \quad (a + b)(a + b) \geq 0$

$\Rightarrow \quad (a + b)^2 \geq 0$, true for $a, b \in R$

(ii) $a^2 + 2ab + b^2 \geq 0$

$\Rightarrow \quad a^2 + 2ab + b^2 + b^2 \geq 0$

$\Rightarrow \quad (a + b)^2 + (b)^2 \geq 0$, true for $a, b \in R$

2. $(a + b)^2 \geq 4ab$

$\Rightarrow \quad a^2 + 2ab + b^2 - 4ab \geq 0$

$\Rightarrow \quad a^2 - 2ab + b^2 \geq 0$

$\Rightarrow \quad (a - b)^2 \geq 0$, true for $a, b \in R$

3. $-(a^2 + 2ab + b^2) \leq 0$

$\Rightarrow \quad a^2 + 2ab + b^2 \geq 0$

$\Rightarrow \quad (a + b)^2 \geq 0$, true for $a, b \in R$

4. (i) $a + \dfrac{1}{a} \geq 2$

$\Rightarrow \quad a^2 + 1 \geq 2a$

$\Rightarrow \quad a^2 - 2a + 1 \geq 0$

$\Rightarrow \quad (a - 1)^2 \geq 0$, true for $a > 0$ and $a \in R$

(ii) $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{2}{a + b}$

$$(a)(b)(a + b) \cdot \frac{1}{a} + (a)(b)(a + b) \cdot \frac{1}{b} \geq (a)(b)(a + b) \cdot \frac{2}{a + b}$$

$\Rightarrow \quad b(a + b) + a(a + b) \geq 2ab$

$\Rightarrow \quad ab + b^2 + a^2 + ab \geq 2ab$

$\Rightarrow \quad a^2 + b^2 \geq 0$, true for $a > 0, b > 0$ and $a, b \in R$

5. $a^2 - 6a + 9 + b^2 \geq 0$

$\Rightarrow \quad (a - 3)(a - 3) + b^2 \geq 0$

$\Rightarrow \quad (a - 3)^2 + b^2 \geq 0$, true for $a, b \in R$

6. (i) $x^2 + 6x + 9 \geq 0$

$\Rightarrow \quad (x + 3)^2 \geq 0$, true for $x \in R$

(ii) $x^2 - 10x + 25 \geq 0$

$\Rightarrow \quad (x - 5)^2 \geq 0$, true for $x \in R$

(iii) $x^2 + 4x + 6 \geq 0$

$\Rightarrow \quad x^2 + 4x + 4 + 2 \geq 0$

$\Rightarrow \quad (x + 2)^2 + 2 \geq 0$, true for $x \in R$

(iv) $x^2 - 6x + 10 \geq 0$

$\Rightarrow \quad x^2 - 6x + 9 + 1 \geq 0$

$\Rightarrow \quad (x - 3)^2 + 1 \geq 0$, true for $x \in R$

(v) $4x^2 + 12x + 11 \geq 0$

$\Rightarrow \quad x^2 + 3x + \dfrac{11}{4} \geq 0$

$\Rightarrow \quad x^2 + 3x + \dfrac{9}{4} + \dfrac{2}{4} \geq 0$

$\Rightarrow \quad \left(x + 1\dfrac{1}{2}\right)^2 + \dfrac{1}{2} \geq 0$, true for $x \in R$

(vi) $4x^2 - 4x + 2 \geq 0$

$\Rightarrow \quad x^2 - x + \frac{1}{2} \geq 0$

$\Rightarrow \quad x^2 - x + \frac{1}{4} + \frac{1}{4} \geq 0$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \geq 0,$ true for $x \in R$

7. (i) $-x^2 + 10x - 25 \geq 0$

$\Rightarrow \quad x^2 - 10x + 25 \geq 0$

$\Rightarrow \quad (x - 5)^2 \geq 0,$ true for $x \in R$

(ii) $-x^2 - 4x - 7 \leq 0$

$\Rightarrow \quad x^2 + 4x + 7 \geq 0$

$\Rightarrow \quad x^2 + 4x + 4 + 3 \geq 0$

$\Rightarrow \quad (x + 2)^2 + 3 \geq 0,$ true for $x \in R$

8. (i) $p^2 + 4q^2 \geq 4pq$

$\Rightarrow \quad p^2 - 4pq + 4q^2 \geq 0$

$\Rightarrow \quad (p - 2q)(p - 2q) \geq 0$

$\Rightarrow \quad (p - 2q)^2 \geq 0,$ true for $p, q \in R$

(ii) $(p + q)^2 \leq 2(p^2 + q^2)$

$\Rightarrow \quad p^2 + 2pq + q^2 \leq 2p^2 + 2q^2$

$\Rightarrow \quad -p^2 + 2pq - q^2 \leq 0$

$\Rightarrow \quad p^2 - 2pq + q^2 \geq 0$

$\Rightarrow \quad (p - q)^2 \geq 0,$ true for $p, q \in R$

9. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Proof: $a^3 + b^3 > a^2b + ab^2$

$\Rightarrow \quad a^3 + b^3 - a^2b - ab^2 > 0$

$\Rightarrow \quad (a + b)(a^2 - ab + b^2) - ab(a + b) > 0$

$\Rightarrow \quad (a + b)(a^2 - ab + b^2 - ab) > 0$

$\Rightarrow \quad (a + b)(a - b)^2 > 0,$ true for $a > 0, b > 0$

10. Given $a^2 + b^2 \geq 2ab$.

(i) $a^2 + c^2 \geq 2ac$

(ii) $b^2 + c^2 \geq 2bc$

$a^2 + b^2 \geq 2ab$

$a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

Add: $\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ac$

$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ac$

11. $\frac{p + q}{2} > \sqrt{pq}$

$\Rightarrow \quad p + q > 2\sqrt{pq}$

$\Rightarrow \quad (p + q)^2 > (2\sqrt{pq})^2$

$\Rightarrow \quad p^2 + 2pq + q^2 > 4pq$

$\Rightarrow \quad p^2 - 2pq + q^2 > 0$

$\Rightarrow \quad (p - q)^2 > 0 \ldots$ true

12. $(ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2)$

$\Rightarrow \quad a^2x^2 + 2axby + b^2y^2 \leq a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$

$\Rightarrow \quad -a^2y^2 + 2axby - b^2x^2 \leq 0$

$\Rightarrow \quad a^2y^2 - 2axby + b^2x^2 \geq 0$

$\Rightarrow \quad (ay - bx)^2 \geq 0,$ true for $a, b, x, y \in R$

13. $a^4 + b^4 \geq 2a^2b^2$

$\Rightarrow \quad a^4 - 2a^2b^2 + b^4 \geq 0$

$\Rightarrow \quad (a^2 - b^2)(a^2 - b^2) \geq 0$

$\Rightarrow \quad (a^2 - b^2)^2 \geq 0,$ true for $a, b \in R$

14. $(a + 2b)\left(\dfrac{1}{a} + \dfrac{1}{2b}\right) \geq 4$

$\Rightarrow \quad a \cdot \dfrac{1}{a} + \dfrac{a}{2a} + \dfrac{2b}{a} + 2b \cdot \dfrac{1}{2b} \geq 4$

$\Rightarrow \quad 1 + \dfrac{a}{2b} + \dfrac{2b}{a} + 1 \geq 4$

$\Rightarrow \quad \dfrac{a}{2b} + \dfrac{2b}{a} \geq 2$

$\Rightarrow \quad 2ab \cdot \dfrac{a}{2b} + 2ab \cdot \dfrac{2b}{a} \geq 2ab\,.\,2 \ldots$ since $a > 0, b > 0$

$\Rightarrow \quad a^2 + 4b^2 \geq 4ab$

$\Rightarrow \quad a^2 - 4ab + 4b^2 \geq 0$

$\Rightarrow \quad (a - 2b)^2 \geq 0,$ true for a, b positive

15. $\dfrac{a}{(a + 1)^2} \leq \dfrac{1}{4}$

$\Rightarrow \quad \dfrac{a}{(a + 1)^2} \cdot (a + 1)^2 \leq \dfrac{1}{4}(a + 1)^2$

$\Rightarrow \quad a \leq \dfrac{a^2 + 2a + 1}{4}$

$\Rightarrow \quad 4a \leq a^2 + 2a + 1$

$\Rightarrow \quad -a^2 + 2a - 1 \leq 0$

$\Rightarrow \quad a^2 - 2a + 1 \geq 0$

$\Rightarrow \quad (a - 1)^2 \geq 0,$ true for $a \in R$

16. (i) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$

$= (a^2 + b^2)(a + b)(a - b)$

(ii) $a^5 - a^4b - ab^4 + b^5$

$= a^4(a - b) - b^4(a - b)$

$= (a - b)(a^4 - b^4)$

$= (a - b)(a^2 + b^2)(a + b)(a - b)$

$= (a^2 + b^2)(a + b)(a - b)^2$

(iii) $a^5 + b^5 > a^4b + ab^4$

$\Rightarrow \quad a^5 - a^4b - ab^4 + b^5 > 0$

$\Rightarrow \quad (a^2 + b^2)(a + b)(a - b)^2 > 0,$ true if a and b are positive unequal real numbers.

17. Given: $a^2 + b^2 = 1$ and $c^2 + d^2 = 1.$

$a^2 + b^2 \geq 2ab \Rightarrow a^2 - 2ab + b^2 \geq 0$

$\Rightarrow \quad (a - b)^2 \geq 0$

Hence, $(a - b)^2 + (c - d)^2 \geq 0$

$\Rightarrow \quad a^2 - 2ab + b^2 + c^2 - 2cd + d^2 \geq 0$

$\Rightarrow \quad 1 - 2ab + 1 - 2cd \geq 0$

$\Rightarrow \quad -2ab - 2cd \geq -2$

$\Rightarrow \quad 2ab + 2cd \leq 2$

$\Rightarrow \quad ab + cd \leq 1$

18. $\sqrt{ab} > \dfrac{2ab}{a+b}$

$\Rightarrow \quad (a+b).\sqrt{ab} > (a+b)\dfrac{2ab}{a+b}$ if a and b are positive and unequal

$\Rightarrow \quad (a+b)\sqrt{ab} > 2ab$

$\Rightarrow \quad (a+b) > \dfrac{2ab}{\sqrt{ab}}$

$\Rightarrow \quad a+b > 2\sqrt{ab}$

$\Rightarrow \quad (a+b)^2 > (2\sqrt{ab})^2$

$\Rightarrow \quad a^2 + 2ab + b^2 > 4ab$

$\Rightarrow \quad a^2 - 2ab + b^2 > 0$

$\Rightarrow \quad (a-b)^2 > 0,$ true if a and b are positive and unequal.

19. $a + \dfrac{9}{a+2} \geq 4$

$\Rightarrow \quad a(a+2) + \dfrac{9}{a+2}(a+2) \geq 4(a+2)$, where $(a+2) > 0$

$\Rightarrow \quad a^2 + 2a + 9 \geq 4a + 8$

$\Rightarrow \quad a^2 - 2a + 1 \geq 0$

$\Rightarrow \quad (a-1)^2 \geq 0,$ true where $(a+2) > 0$

20. If $\dfrac{a}{b} > \dfrac{c}{d}$,

$\Rightarrow \quad \dfrac{a}{b}(bd) > \dfrac{c}{d}(bd)$ if a, b, c, d are positive numbers

$\Rightarrow \quad ad > cd$

Hence, $\dfrac{a+c}{b+d} > \dfrac{c}{d}$

$\Rightarrow \quad \dfrac{(a+c)}{(b+d)} \cdot (d)(b+d) > \dfrac{c}{d}(d)(b+d)$ if a, b, c, d are positive numbers

$\Rightarrow \quad ad + cd > cb + cd$

$\Rightarrow \quad ad > cb,$ true

21. $(a^3 - b^3)(a-b) = (a-b)(a^2 + ab + b^2)(a-b)$

$= (a-b)^2(a^2 + ab + b^2)$

If $a > b \Rightarrow (a > b) > 0$.

Hence, $(a-b)^2$ is positive,

and $a^2 + ab + b^2 > a^2 - 2ab + b^2 = (a-b)^2 > 0$

Hence, $a^2 + ab + b^2$ is a positive, true if a, b, c, d are positive numbers.

Hence, $a^4 + b^4 \geq a^3b + ab^3$

$\Rightarrow \quad a^4 - a^3b - ab^3 + b^4 \geq 0$

$\Rightarrow \quad a^3(a-b) - b^3(a-b) \geq 0$

$\Rightarrow \quad (a-b)(a^3 - b^3) \geq 0$

$\Rightarrow \quad (a-b)(a-b)(a^2 + ab + b^2) \geq 0$

$\Rightarrow \quad (a-b)^2(a^2 + ab + b^2) \geq 0,$ true for $a, b \in R$ and $a > b$

Exercise 12.6

1. (i) $a^2 \times a^3 = a^{2+3} = a^5$

(ii) $x \, . \, x \, . \, x^2 = x^{1+1+2} = x^4$

(iii) $2x^3 \times 3x^3 = 6x^{3+3} = 6x^6$

(iv) $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

(v) $\dfrac{x^4}{x^5} = x^{4-5} = x^{-1}$

(vi) $a^0 = 1$

(vii) $\sqrt[3]{27} = 3$

(viii) $(a^3)^2 = a^6$

(ix) $\dfrac{(x^3)^2}{x^3} = \dfrac{x^6}{x^3} = x^{6-3} = x^3$

(x) $(3ab)^2 = 9a^2b^2$

2. (i) $\sqrt[3]{64} = 4$

(ii) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(iii) $\dfrac{1}{2^{-3}} = 2^3 = 8$

(iv) $\dfrac{2^{-2}}{3^{-2}} = \dfrac{3^2}{2^2} = \dfrac{9}{4}$

(v) $\dfrac{1}{4^{-\frac{1}{2}}} = 4^{\frac{1}{2}} = 2$

3. (i) $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

(ii) $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

(iii) $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$

(iv) $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^3 = 27$

(v) $125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}} = 5^2 = 25$

4. (i) $\left(\dfrac{2}{3}\right)^{-2} = \dfrac{1}{\left(\frac{2}{3}\right)^2} = \dfrac{1}{\frac{4}{9}} = \dfrac{9}{4}$

(ii) $\left(\dfrac{4}{9}\right)^{-\frac{1}{2}} = \dfrac{1}{\left(\frac{4}{9}\right)^{\frac{1}{2}}} = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$

(iii) $\left(\dfrac{9}{25}\right)^{-\frac{3}{2}} = \dfrac{1}{\left(\frac{9}{25}\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\left(\frac{3}{5}\right)^2\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\frac{3}{5}\right)^3} = \dfrac{1}{\frac{27}{125}} = \dfrac{125}{27}$

(iv) $\left(\dfrac{27}{125}\right)^{-\frac{2}{3}} = \left(\left(\frac{3}{5}\right)^3\right)^{\frac{-2}{3}} = \left(\dfrac{3}{5}\right)^{-2} = \dfrac{1}{\left(\frac{3}{5}\right)^2} = \dfrac{1}{\frac{9}{25}} = \dfrac{25}{9}$

(v) $\left(3\dfrac{3}{8}\right)^{\frac{1}{3}} = \left(\dfrac{27}{8}\right)^{\frac{1}{3}} = \dfrac{(27)^{\frac{1}{3}}}{(8)^{\frac{1}{3}}} = \dfrac{3}{2}$

5. $\dfrac{4^2 \times 16^{\frac{1}{2}}}{64^{\frac{2}{3}} \times 4^3} = \dfrac{4^2 \times 4}{(4^3)^{\frac{2}{3}} \times 4^3} = \dfrac{4^3}{4^2 \times 4^3} = \dfrac{4^3}{4^5} = 4^{3-5} = 4^{-2}$

6. $\dfrac{3^{\frac{1}{4}} \times 3 \times 3^{\frac{1}{6}}}{\sqrt{3}} = \dfrac{3^{\frac{1}{4}+1+\frac{1}{6}}}{3^{\frac{1}{2}}} = \dfrac{3^{1\frac{5}{12}}}{3^{\frac{1}{2}}} = 3^{1\frac{5}{12}-\frac{1}{2}} = 3^{\frac{11}{12}}$

$3^P = 3^{\frac{11}{12}} \Rightarrow p = \dfrac{11}{12}$

7. (i) $\dfrac{(xy^2)^3 \times (x^2y)^{-2}}{xy} = \dfrac{x^3y^6 \,.\, x^{-4}y^{-2}}{xy} = \dfrac{x^{-1}y^4}{x^1y^1}$

$= \dfrac{y^{4-1}}{x^{1+1}} = \dfrac{y^3}{x^2}$

(ii) $\left(\dfrac{p^2q}{p^{-1}q^3}\right)^4 = \dfrac{p^8\,q^4}{p^{-4}\,q^{12}} = \dfrac{p^{8+4}}{q^{12-4}} = \dfrac{p^{12}}{q^8}$

(iii) $a^{\frac{1}{4}} \times a^{\frac{-5}{4}} = a^{\frac{1}{4}-\frac{5}{4}} = a^{-1} = \dfrac{1}{a}$

(iv) $\left(\dfrac{y^{-2}}{y^{-3}}\right)^{\frac{2}{3}} = (y^{-2+3})^{\frac{2}{3}} = (y^1)^{\frac{2}{3}} = y^{\frac{2}{3}}$

(v) $\dfrac{(a\sqrt{b})^{-3}}{\sqrt{a^3b}} = \dfrac{(a^1b^{\frac{1}{2}})^{-3}}{(a^3b^1)^{\frac{1}{2}}} = \dfrac{a^{-3}b^{-\frac{3}{2}}}{a^{\frac{3}{2}}b^{\frac{1}{2}}} = \dfrac{1}{a^{3+\frac{3}{2}}b^{\frac{1}{2}+\frac{3}{2}}} = \dfrac{1}{a^{\frac{9}{2}}b^2}$

(vi) $\dfrac{\sqrt[4]{x^7}}{\sqrt{x^3}} = \dfrac{x^{\frac{7}{4}}}{x^{\frac{3}{2}}} = x^{\frac{7}{4}-\frac{3}{2}} = x^{\frac{1}{4}}$

8. (i) $\dfrac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \dfrac{x^{-\frac{1}{2}}(x+1)}{x^{\frac{1}{2}}} = \dfrac{x+1}{x^{\frac{1}{2}+\frac{1}{2}}} = \dfrac{x+1}{x}$

(ii) $(x + x^{\frac{1}{2}})(x - x^{\frac{1}{2}}) = x^2 - x\,.\,x^{\frac{1}{2}} + x\,.\,x^{\frac{1}{2}} - x^{\frac{1}{2}+\frac{1}{2}}$

$= x^2 - x$

(iii) $\dfrac{\sqrt{x} + \sqrt{x^3}}{\sqrt{x}} = \dfrac{x^{\frac{1}{2}} + x^{\frac{3}{2}}}{x^{\frac{1}{2}}} = \dfrac{x^{\frac{1}{2}}(1+x)}{x^{\frac{1}{2}}} = 1 + x$

9. $\dfrac{(x-1)^{\frac{1}{2}} + (x-1)^{-\frac{1}{2}}}{(x-1)^{\frac{1}{2}}} = \dfrac{(x-1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}(x-1)^{-\frac{1}{2}}}{(x-1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$

$= \dfrac{(x-1)^1 + (x-1)^0}{(x-1)^1} = \dfrac{x-1+1}{x-1} = \dfrac{x}{x-1}$

10. $\sqrt{3^{2n+1}} \times \sqrt[3]{3^{-3n}} = (3^{2n+1})^{\frac{1}{2}} \times (3^{-3n})^{\frac{1}{3}}$

$= 3^{n+\frac{1}{2}} \times 3^{-n} = 3^{n+\frac{1}{2}-n} = 3^{\frac{1}{2}}$

$\Rightarrow 3^k = 3^{\frac{1}{2}} \Rightarrow k = \dfrac{1}{2}$

11. $220\,.\,2^{\frac{1}{2}}\,.\,2^{\frac{1}{12}}\,.\,2^{\frac{1}{12}} = 220\,.\left(2^{\frac{1}{4}}\right)$

$= 261\,.\,626$

$= 262$ Hz

12. $\dfrac{A_1}{A_2} = \dfrac{4\pi R_1^2}{4\pi R_2^2} = \dfrac{R_1^2}{R_2^2}$

$\left(\dfrac{V_1}{V_2}\right)^{\frac{2}{3}} = \left(\dfrac{\frac{4}{3}\pi R_1^3}{\frac{4}{3}\pi R_2^3}\right)^{\frac{2}{3}} = \dfrac{\left(\frac{4}{3}\right)^{\frac{2}{3}}(\pi)^{\frac{2}{3}}R_1^{3\left(\frac{2}{3}\right)}}{\left(\frac{4}{3}\right)^{\frac{2}{3}}(\pi)^{\frac{2}{3}}R_2^{3\left(\frac{2}{3}\right)}} = \dfrac{R_1^2}{R_2^2}$

Hence, $\dfrac{A_1}{A_2} = \left(\dfrac{V_1}{V_2}\right)^{\frac{2}{3}}$

Hence, $\dfrac{A_1}{A_2} = \left(\dfrac{162}{384}\right)^{\frac{2}{3}} = \left(\dfrac{27}{64}\right)^{\frac{2}{3}} = \dfrac{9}{16}$

13. $f(n) = 3^n \Rightarrow$ (i) $f(n+3) = 3^{n+3}$
and (ii) $f(n+1) = 3^{n+1}$

$$\Rightarrow f(n+3) - f(n+1) = 3^{n+3} - 3^{n+1}$$
$$= 3^n . 3^3 - 3^n . 3^1$$
$$= 3^n(27-3) = 24(3^n)$$
$$\Rightarrow k\, f(n) = k(3^n) = 24(3^n)$$
$$\Rightarrow \quad k = 24$$

14. $f(n) = 3^{n-1} \Rightarrow f(n+3) + f(n) = k\, f(n)$

$$\Rightarrow 3^{n+3-1} + 3^{n-1} = k(3^{n-1})$$
$$\Rightarrow 3^{n-1} . 3^3 + 3^{n-1} = k(3^{n-1})$$
$$\Rightarrow 3^{n-1}(27+1) = k(3^{n-1})$$
$$\Rightarrow \quad 28 = k$$

Exercise 12.7

1. (i) $2^x = 32$

$$\Rightarrow 2^x = 2^5 \Rightarrow x = 5$$

(ii) $16^x = 64$

$$\Rightarrow (4^2)^x = 4^3$$
$$\Rightarrow 4^{2x} = 4^3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$$

(iii) $25^x = 125$

$$\Rightarrow (5^2)^x = 5^3$$
$$\Rightarrow \quad 5^{2x} = 5^3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$$

(iv) $3^x = \frac{1}{27}$

$$\Rightarrow 3^x = \frac{1}{3^3}$$
$$\Rightarrow 3^x = 3^{-3} \Rightarrow x = -3$$

2. (i) $9^x = \frac{1}{27}$

$$\Rightarrow (3^2)^x = \frac{1}{3^3}$$
$$\Rightarrow \quad 3^{2x} = 3^{-3} \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}$$

(ii) $4^x = \frac{1}{32}$

$$\Rightarrow \quad (2^2)^x = \frac{1}{2^5}$$
$$\Rightarrow \quad 2^{2x} = 2^{-5} \Rightarrow 2x = -5 \Rightarrow x = -\frac{5}{2}$$

(iii) $4^{x-1} = 2^{x+1}$

$$\Rightarrow (2^2)^{x-1} = 2^{x+1}$$
$$\Rightarrow \quad 2^{2x-2} = 2^{x+1} \Rightarrow 2x - 2 = x + 1$$
$$\Rightarrow x = 3$$

(iv) $\frac{1}{9^x} = 27$

$$\Rightarrow \frac{1}{(3^2)^x} = 3^3$$
$$\Rightarrow \quad \frac{1}{3^{2x}} = 3^3$$
$$\Rightarrow 3^{-2x} = 3^3 \Rightarrow -2x = 3$$
$$\Rightarrow \quad x = -\frac{3}{2}$$

3. (i) $2^x = \frac{\sqrt{2}}{2}$

$$\Rightarrow 2^x = \frac{2^{\frac{1}{2}}}{2^1}$$
$$\Rightarrow 2^x = 2^{\frac{1}{2}-1} = 2^{-\frac{1}{2}}$$
$$\Rightarrow \quad x = -\frac{1}{2}$$

(ii) $25^x = \frac{125}{\sqrt{5}}$

$$\Rightarrow (5^2)^x = \frac{5^3}{5^{\frac{1}{2}}}$$
$$\Rightarrow \quad 5^{2x} = 5^{3-\frac{1}{2}} = 5^{\frac{5}{2}}$$
$$\Rightarrow \quad 2x = \frac{5}{2}$$
$$\Rightarrow \quad x = \frac{5}{4}$$

(iii) $\frac{1}{8^x} = \sqrt{2}$

$$\Rightarrow \frac{1}{(2^3)^x} = 2^{\frac{1}{2}}$$
$$\Rightarrow \quad \frac{1}{2^{3x}} = 2^{\frac{1}{2}}$$
$$\Rightarrow \quad 2^{-3x} = 2^{\frac{1}{2}}$$
$$\Rightarrow -3x = \frac{1}{2}$$
$$\Rightarrow \quad x = -\frac{1}{6}$$

(iv) $7^x = \frac{1}{\sqrt[3]{7}}$

$$\Rightarrow 7^x = \frac{1}{7^{\frac{1}{3}}}$$
$$\Rightarrow 7^x = 7^{-\frac{1}{3}}$$
$$\Rightarrow \quad x = -\frac{1}{3}$$

4. $\sqrt{32} = (2^5)^{\frac{1}{2}} = 2^{\frac{5}{2}}$

$\Rightarrow \quad 16^{x-1} = 2\sqrt{32}$

$\Rightarrow \quad (2^4)^{x-1} = 2^1 . 2^{\frac{5}{2}}$

$\Rightarrow \quad 2^{4x-4} = 2^{1+\frac{5}{2}} = 2^{\frac{7}{2}}$

$\Rightarrow \quad 4x - 4 = \frac{7}{2}$

$\Rightarrow \quad 4x = \frac{7}{2} + 4 = \frac{15}{2}$

$\Rightarrow \quad x = \frac{15}{8}$

5. $27^x = 9$ and $2^{x-y} = 64$

$\Rightarrow (3^3)^x = 3^2 \qquad \Rightarrow \quad 2^{x-y} = 2^6$

$\Rightarrow \quad 3^{3x} = 3^2 \qquad \Rightarrow \quad x - y = 6$

$\Rightarrow \quad 3x = 2$

$\Rightarrow \quad x = \frac{2}{3} \qquad \Rightarrow \quad \frac{2}{3} - y = 6$

$-y = 6 - \frac{2}{3} = \frac{16}{3}$

$\Rightarrow \quad y = -\frac{16}{3}$

6. (i) $2^{x+2} = 2^x . 2^2 = 4 . 2^x$

(ii) $2^x + 2^x = 2 . 2^x$

Hence, $2^x + 2^x = 2^{x+2}(c - 2)$

$\Rightarrow \quad 2 . 2^x = 4 . 2^x(c - 2)$

$\Rightarrow \quad 2 = 4c - 8$

$\Rightarrow \quad 4c = 2 + 8 = 10$

$\Rightarrow \quad c = \frac{10}{4} = \frac{5}{2}$

7. $3^x = y \quad \Rightarrow \quad 3^{2x} = (3^x)^2 = y^2$

Solve $3^{2x} - 12(3^x) + 27 = 0$

Let $y = 3^x \quad \Rightarrow \quad y^2 - 12y + 27 = 0$

$\Rightarrow \quad (y - 3)(y - 9) = 0$

$\Rightarrow \quad y = 3, \; y = 9$

Hence, $3^x = 3, \; 3^x = 9$

$\Rightarrow \quad 3^x = 3^1, \; 3^x = 3^2$

$\Rightarrow \quad x = 1, \; x = 2$

8. Solve $2^{2x} - 3(2^x) - 4 = 0$

Let $y = 2^x \Rightarrow y^2 - 3y - 4 = 0$

$\Rightarrow (y - 4)(y + 1) = 0$

$\Rightarrow \quad y = 4, \; y = -1$

Hence, $2^x = 4 = 2^2, \; 2^x = -1$ (Not valid)

$\Rightarrow x = 2$

9. (i) Solve $2^{2x} - 9(2^x) + 8 = 0$

Let $y = 2^x \Rightarrow \quad (y^2 - 9y + 8 = 0$

$\Rightarrow \quad (y - 1)(y - 8) = 0$

$\Rightarrow \quad y = 1, \; y = 8$

Hence, $2^x = 1 = 2^0, \; 2^x = 8 = 2^3$

$\Rightarrow x = 0, \qquad x = 3$

(ii) Solve $3^{2x} - 10(3^x) + 9 = 0$

Let $y = 3^x \Rightarrow y^2 - 10y + 9 = 0$

$\Rightarrow (y - 1)(y - 9) = 0$

$\Rightarrow y = 1, \; y = 9$

Hence, $3^x = 1 = 3^0, \; 3^x = 9 = 3^2$

$\Rightarrow x = 0, \qquad x = 2$

10. $y = 2^x \Rightarrow$ (i) $2^{2x} = (2^x)^2 = y^2$

(ii) $2^{2x+1} = (2^x)^2 \,.\, 2^1 = 2y^2$

(iii) $2^{2x+3} = 2^x \,.\, 2^3 = 8y$

Hence, solve $2^{2x+1} - 2^{x+3} - 2^x + 4 = 0$

Let $y = 2^x \Rightarrow 2y^2 - 8y - y + 4 = 0$

$\Rightarrow 2y^2 - 9y + 4 = 0$

$\Rightarrow (2y - 1)(y - 4) = 0$

$\Rightarrow 2y = 1, \; y = 4$

$\Rightarrow \quad y = \dfrac{1}{2}$

Hence $2^x = \dfrac{1}{2} = 2^{-1}, \; 2^x = 4 = 2^2$

$\Rightarrow x = -1, \; x = 2$

11. Solve $3 \,.\, 3^x + 3^{-x} = 4$

Let $y = 3^x \Rightarrow 3y + \dfrac{1}{y} = 4$

$\Rightarrow 3y^2 + 1 = 4y$

$\Rightarrow 3y^2 - 4y + 1 = 0$

$\Rightarrow (3y - 1)(y - 1) = 0$

$\Rightarrow y = \dfrac{1}{3}, \; y = 1$

Hence, $3^x = \dfrac{1}{3} = 3^{-1}, \; 3^x = 1 = 3^0$

$\Rightarrow x = -1, \; x = 0$

12. Solve $2(4^x) + 4^{-x} = 3$

Let $y = 4^x \Rightarrow 2y + \dfrac{1}{y} = 3$

$\Rightarrow 2y^2 + 1 = 3y$

$\Rightarrow 2y^2 - 3y + 1 = 0$

$\Rightarrow (2y - 1)(y - 1) = 0$

$\Rightarrow y = \dfrac{1}{2}, \; y = 1$

Hence, $4^x = \dfrac{1}{2}, \quad 4^x = 1$

$\Rightarrow (2^2)^x = 2^{-1}, \qquad (2^2)^x = 1 = 2^0$

$\Rightarrow 2^{2x} = 2^{-1} \qquad 2^{2x} = 2^0$

$\Rightarrow 2x = -1 \qquad \Rightarrow 2x = 0$

$\Rightarrow x = -\dfrac{1}{2} \qquad \Rightarrow x = 0$

13. Solve $3^x - 28 + 27(3^{-x}) = 0$

Let $y = 3^x \Rightarrow y - 28 + 27\left(\dfrac{1}{y}\right) = 0$

$\Rightarrow y^2 - 28y + 27 = 0$

$\Rightarrow (y - 1)(y - 27) = 0$

$\Rightarrow y = 1, \; y = 27$

Hence, $3^x = 1 = 3^0, \; 3^x = 27 = 3^3$

$\Rightarrow x = 0, \qquad x = 3$

14. Solve $2^{x+1} + 2(2^{-x}) - 5 = 0$

Let $2^x = y \Rightarrow 2y + 2\left(\frac{1}{y}\right) - 5 = 0$

$\Rightarrow 2y^2 + 2 - 5y = 0$

$\Rightarrow 2y^2 - 5y + 2 = 0$

$\Rightarrow (2y - 1)(y - 2) = 0$

$\Rightarrow y = \frac{1}{2}, \quad y = 2$

Hence, $2^x = \frac{1}{2} = 2^{-1}, \qquad 2^x = 2^1$

$\Rightarrow x = -1, \quad x = 1$

15. Solve $3^x + 81(3^{-x}) - 30 = 0$

Let $y = 3^x \Rightarrow y + 81\left(\frac{1}{y}\right) - 30 = 0$

$\Rightarrow y^2 + 81 - 30y = 0$

$\Rightarrow y^2 - 30y + 81 = 0$

$\Rightarrow (y - 3)(y - 27) = 0$

$\Rightarrow y = 3, \quad y = 27$

Hence, $3^x = 3^1, \quad 3^x = 27 = 3^3$

$\Rightarrow x = 1, \qquad x = 3$

Exercise 12.8

1. (i) B (ii) A (iii) D (iv) C

2. $A(n) = 1000 \times 2^{0.2n}$

(i) $A(0) = 1000 \times 2^{0.2(0)}$

$= 1000 \times 2^0$

$= 1000 \times 1 = 1000$ ha

(ii) (a) $A(10) = 1000 \times 2^{0.2(10)}$

$= 1000 \times 2^2$

$= 1000 \times 4 = 4000$ ha

(b) $A(12) = 1000 \times 2^{0.2(12)}$

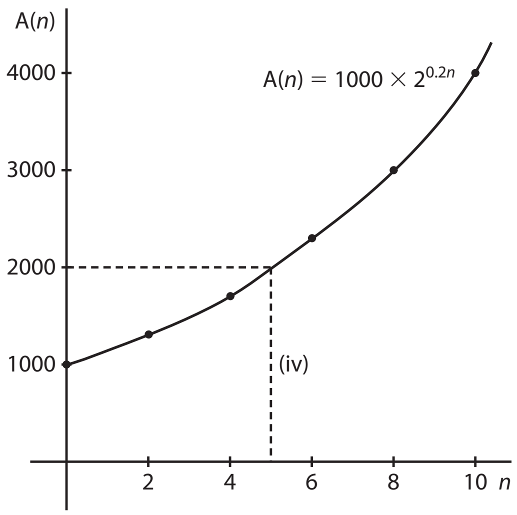
$= 1000 \times 2^{2.4}$

$= 1000(5.278) = 5278$ ha

(iii)

$n =$	0	2	4	6	8	10
$A(n) =$	1000	1320	1741	2297	3031	4000

(iv) 5 weeks

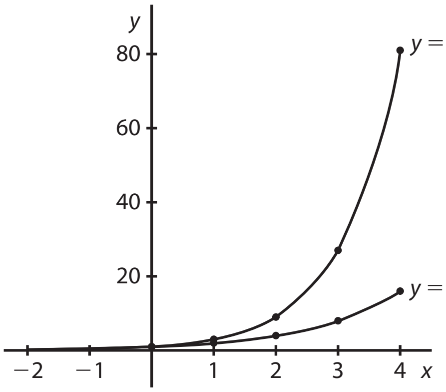


3. (i) Decreasing
(ii) Decreasing
(iii) Increasing
(iv) Decreasing

4. (i) $x = 0 \Rightarrow y = (0.6)2^0 = (0.6)(1) = 0.6$
(ii) $x = 0 \Rightarrow y = 3(2^{-0}) = 3(1) = 3$
(iii) $x = 0 \Rightarrow y = 8(2^0) = 8(1) = 8$
(iv) $x = 0 \Rightarrow y = 6(4^{-0}) = 6(1) = 6$

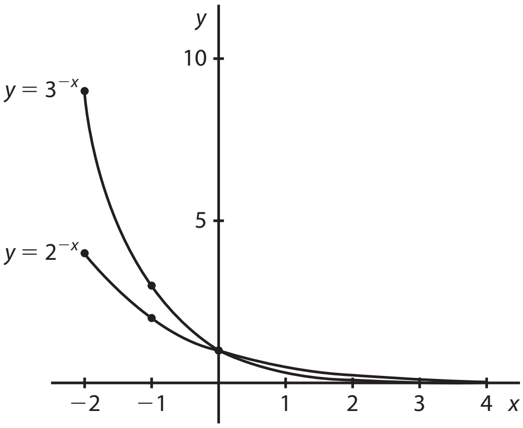
5. (i)

$x =$	-2	-1	0	1	2	3	4
$2^x =$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$3^x =$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81



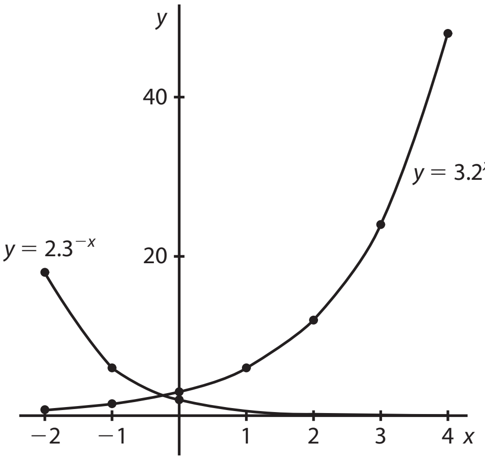
(ii)

$x =$	-2	-1	0	1	2	3	4
$2^{-x} =$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$3^{-x} =$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$



(iii)

$x =$	-2	-1	0	1	2	3	4
$3.2^x =$	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
$2.3^{-x} =$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$	$\frac{2}{81}$



(iv) $-2 \leqslant x < 0$

(v) $0 < x \leqslant 4$

(vi) $x = 0$

(vii) $0 < x \leqslant 4$

6. $D = 18(0.72)^T$

(i) Decay: Graph is decreasing

(ii) (a) $T = 5°C \Rightarrow D = 18(0.72)^5 = 3.48 = 3$ days

(b) $T = 2°C \Rightarrow D = 18(0.72)^2 = 9.33 = 9$ days

(c) $T = 0°C \Rightarrow D = 18(0.72)^0 = 18$ days

(iii) $18(0.72)^T \geqslant 5$

$\Rightarrow (0.72)^T \geqslant \frac{5}{18}$

$\Rightarrow \log(0.72)^T \geqslant \log\frac{5}{18}$

$\Rightarrow T(\log 0.72) \geqslant \log\frac{5}{18}$

$\Rightarrow T \geqslant \frac{\log\frac{5}{18}}{\log(0.72)}$

$\Rightarrow T \geqslant 3.899$

$\Rightarrow T = 3.9°C$

7. $P = 100(0.99988)^n$

(a) (i) $n = 200 \Rightarrow P = 100(0.99988)^{200} = 97.628 = 97.6\%$

(ii) $n = 500 \Rightarrow P = 100(0.99988)^{500} = 94.176 = 94.2\%$

(b) $n = 5000 \Rightarrow P = 100(0.99988)^{5000} = 54.879 = 54.9\%$

$n = 6000 \Rightarrow P = 100(0.99988)^{6000} = 48.673 = 48.7\%$

Hence, $100(0.99988)^n = 50$

$\Rightarrow (0.99988)^n = 0.5$

$\Rightarrow \log(0.99988)^n = \log(0.5)$

$\Rightarrow n\log(0.99988) = \log(0.5)$

$\Rightarrow n = \frac{\log(0.5)}{\log(0.99988)} = 5775.88 = 5780$ years

(c) $100(0.99988)^n = 79$

$\Rightarrow (0.99988)^n = 0.79$

$\Rightarrow \log(0.99988)^n = \log(0.79)$

$\Rightarrow n\log(0.99988) = \log(0.79)$

$$\Rightarrow n = \frac{\log(0.79)}{\log(0.99988)} = 1964.2 = 1964 \text{ years}$$

8. $A(n) = 1000 \times 2^{0.2n}$

(i) $A(0) = 1000 \times 2^{0.2(0)}$

$= 1000 \times 2^0$

$= 1000 \times 1 = 1000$ ha

(ii) (a) $A(5) = 1000 \times 2^{0.2(5)}$

$= 1000 \times 2^1 = 2000$ ha

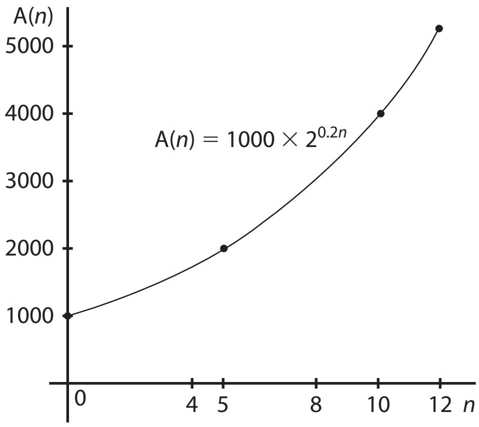
(b) $A(10) = 1000 \times 2^{0.2(10)}$

$= 1000 \times 2^2 = 4000$ ha

(c) $A(12) = 1000 \times 2^{0.2(12)}$

$= 1000 \times 2^{2.4}$

$= 1000(5.278) = 5278$ ha



9. $P(t) = 90 \times 3^{-0.25t} + 50$

(i)

$t =$	0	2	4	6	8	10
$90 \times 3^{-0.25t} + 50$	140	102	80	67	60	56

(ii) $P(0) = 90 \times 3^{-0.25(0)} + 50$

$= 90 \times 3^0 + 50$

$= 90 + 50 = 140$ beats/min

(iii) (a) $90 \times 3^{-0.25t} + 50 = 70$

$\Rightarrow 90 \times 3^{-0.25t} = 20$

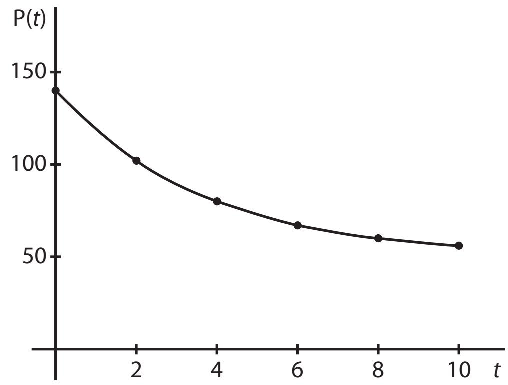
$\Rightarrow 3^{-0.25t} = \frac{20}{90} = \frac{2}{9}$

$\Rightarrow \log 3^{-0.25t} = \log\frac{2}{9}$

$\Rightarrow -0.25t(\log 3) = \log\frac{2}{9}$

$$\Rightarrow -0.25t = \frac{\log\frac{2}{9}}{\log 3} = -1.369$$

$$\Rightarrow t = \frac{-1.369}{-0.25} = 5.476 = 5.5 \text{ minutes}$$



(b) $90 \times 3^{-0.25t} + 50 = 55$

$\Rightarrow 90 \times 3^{-0.25t} = 5$

$\Rightarrow 3^{-0.25t} = \frac{5}{90} = \frac{1}{18}$

$\Rightarrow \log 3^{-0.25t} = \log\frac{1}{18}$

$\Rightarrow -0.25t(\log 3) = \log\frac{1}{18}$

$$\Rightarrow -0.25t = \frac{\log\frac{1}{18}}{\log 3} = -2.63$$

$$\Rightarrow t = \frac{-2.63}{-0.25} = 10.52 = 10.5 \text{ minutes}$$

(iv) 50 beats/min because $P(t) = 50$ as t gets larger using graph of $P(t)$.

10. $E \cong 10^{1.5M+4.8}$

$M = 7 \quad \Rightarrow \quad E = 10^{1.5(7)+4.8} = 10^{15.3}$

$M = 5 \quad \Rightarrow \quad E = 10^{1.5(5)+4.8} = 10^{12.3}$

$\Rightarrow$ No. of times $= \dfrac{10^{15.3}}{10^{12.3}} = 10^{15.3-12.3} = 10^3 = 1000$

11. $P(t) = 40b^t$

(i) $t = 0 \quad \Rightarrow \quad P(0) = 40b^0 = 40$

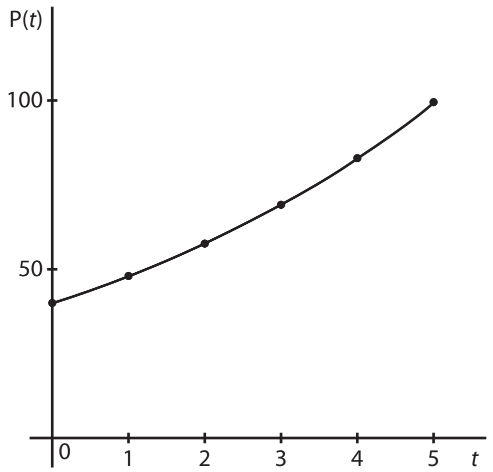
(ii) $t = 1 \quad \Rightarrow \quad P(1) = 40b^1 = 48$

$\Rightarrow \quad b = \dfrac{48}{40} = 1.2 > 1$

$\Rightarrow$ No. of flies is increasing

(iii)

$t =$	0	1	2	3	4	5
$P(t) = 40(1.2)^t$	40	48	57.6	69.1	82.9	99.5



Exercise 12.9

1. (i) $\log_2 4 = 2$

(ii) $\log_3 81 = 4$

(iii) $\log_{10} 1000 = 3$

(iv) $\log_2 64 = 6$

2. (i) $\log_8 16 = x \quad \Rightarrow \quad 8^x = 16$

$\Rightarrow \quad (2^3)^x = 2^4$

$\Rightarrow \quad 2^{3x} = 2^4$

$\Rightarrow \quad 3x = 4 \quad \Rightarrow \quad x = \dfrac{4}{3}$

(ii) $\log_9 27 = x \quad \Rightarrow \quad 9^x = 27$

$\Rightarrow \quad (3^2)^x = 3^3$

$\Rightarrow \quad 3^{2x} = 3^3$

$\Rightarrow \quad 2x = 3 \quad \Rightarrow \quad x = \dfrac{3}{2}$

(iii) $\log_{16} 32 = x \quad \Rightarrow \quad 16^x = 32$

$\Rightarrow \quad (2^4)^x = 2^5$

$\Rightarrow \quad 2^{4x} = 2^5$

$\Rightarrow \quad 4x = 5 \quad \Rightarrow \quad x = \dfrac{5}{4}$

(iv) $\log_{\frac{1}{2}} 8 = x \Rightarrow \left(\frac{1}{2}\right)^x = 8$

$\Rightarrow (2^{-1})^x = 2^3$

$\Rightarrow 2^{-x} = 2^3$

$\Rightarrow -x = 3 \Rightarrow x = -3$

(v) $\log_{\frac{1}{3}} 81 = x \Rightarrow \left(\frac{1}{3}\right)^x = 81$

$\Rightarrow (3^{-1})^x = 3^4$

$\Rightarrow 3^{-x} = 3^4$

$\Rightarrow -x = 4 \Rightarrow x = -4$

3. (i) $\log_{\frac{1}{3}} 27 = x \Rightarrow \left(\frac{1}{3}\right)^x = 27$

$\Rightarrow (3^{-1})^x = 3^3$

$\Rightarrow 3^{-x} = 3^3$

$\Rightarrow -x = 3 \Rightarrow x = -3$

(ii) $\log_{\sqrt{2}} 4 = x \Rightarrow (\sqrt{2})^x = 4$

$\Rightarrow (2^{\frac{1}{2}})^x = 2^2$

$\Rightarrow 2^{\frac{1}{2}x} = 2^2$

$\Rightarrow \frac{1}{2}x = 2$

$\Rightarrow x = 4$

(iii) $\log_8 x = 2$

$\Rightarrow 8^2 = x$

$\Rightarrow x = 64$

(iv) $\log_{64} x = \frac{1}{2}$

$\Rightarrow 64^{\frac{1}{2}} = x$

$\Rightarrow x = 8$

4. (i) $\log_2 x = -1$

$\Rightarrow 2^{-1} = x$

$\Rightarrow x = \frac{1}{2}$

(ii) $\log_3 \sqrt{27} = x$

$\Rightarrow 3^x = \sqrt{27} = (3^3)^{\frac{1}{2}} = 3^{\frac{3}{2}}$

$\Rightarrow x = \frac{3}{2}$

(iii) $\log_x 2 = 2 \Rightarrow x^2 = 2$

$\Rightarrow x = 2^{\frac{1}{2}} = \sqrt{2}$

(iv) $\log_2 (0.5) = x \Rightarrow 2^x = \frac{1}{2} = 2^{-1}$

$\Rightarrow x = -1$

5. (i) $\log_4 2 + \log_4 32 = \log_4 (2) . (32)$

$= \log_4 64 = x$

$\Rightarrow 4^x = 64 = 4^3$

$\Rightarrow x = 3$

(ii) $\log_6 9 + \log_6 8 - \log_6 2 = \log_6 \frac{(9)(8)}{(2)}$

$= \log_6 36 = x$

$\Rightarrow 6^x = 36 = 6^2$

$\Rightarrow x = 2$

(iii) $\log_6 4 + 2\log_6 3 = \log_6 4 + \log_6 3^2$
$= \log_6 4 + \log_6 9 = \log_6 36 = 2$

6. (i) $\log_3 2 + 2\log_3 3 - \log_3 18$
$= \log_3 2 + \log_3 3^2 - \log_3 18$
$= \log_3 2 + \log_3 9 - \log_3 18$
$= \log_3 \frac{(2)(9)}{18} = \log_3 1 = 0$

(ii) $\log_8 72 - \log_8 \left(\frac{9}{8}\right)$
$= \log_8 72 - (\log_8 9 - \log_8 8)$
$= \log_8 72 - \log_8 9 + \log_8 8$
$= \log_8 \frac{(72)(8)}{(9)} = \log_8 64 = 2$

7. $\log_3 5 = a$
(i) $\log_3 15 = \log_3 5\,.\,3 = \log_3 5 + \log_3 3 = a + 1$
(ii) $\log_3 \left(\frac{5}{3}\right) = \log_3 5 - \log_3 3 = a - 1$
(iii) $\log_3 \left(8\frac{1}{3}\right) = \log_3\left(\frac{25}{3}\right) = \log_3 25 - \log_3 3$
$= \log_3 5^2 - 1$
$= 2\log_3 5 - 1 = 2a - 1$
(iv) $\log_3 \left(\frac{25}{27}\right) = \log_3 25 - \log_3 27$
$= \log_3 (5)^2 - \log_3 (3)^3$
$= 2\log_3 5 - 3\log_3 3 = 2a - 3$
(v) $\log_3 75 = \log_3 (25)\,.\,(3) = \log_3 25 + \log_3 3$
$= \log_3 5^2 + 1$
$= 2\log_3 5 + 1$
$= 2a + 1$

8. (i) $2^x = 200$
$\Rightarrow \log 2^x = \log 200$
$\Rightarrow x\log 2 = \log 200$
$\Rightarrow x = \frac{\log 200}{\log 2} = 7.643 = 7.64$

(ii) $5^x = 500$
$\Rightarrow \log 5^x = \log 500$
$\Rightarrow x\log 5 = \log 500$
$\Rightarrow x = \frac{\log 500}{\log 5} = 3.861 = 3.86$

(iii) $3^{x+1} = 25$
$\Rightarrow \log 3^{(x+1)} = \log 25$
$\Rightarrow (x+1)\log 3 = \log 25$
$\Rightarrow x + 1 = \frac{\log 25}{\log 3} = 2.929$
$\Rightarrow x = 1.929 = 1.93$

(iv) $5^{2x+3} = 51$
$\Rightarrow \log 5^{(2x+3)} = \log 51$
$\Rightarrow (2x+3)\log 5 = \log 51$
$\Rightarrow 2x + 3 = \frac{\log 51}{\log 5} = 2.4429$
$\Rightarrow 2x = -0.5571$
$\Rightarrow x = -0.2785 = -0.279$

9. (i) $y = 2^{x+1} + 3$

$$\Rightarrow 2^{x-1} = y - 3$$
$$\Rightarrow \quad \log(2^{x-1}) = \log(y-3)$$
$$\Rightarrow \quad (x-1)\log 2 = \log(y-3)$$
$$\Rightarrow \quad x - 1 = \frac{\log(y-3)}{\log 2}$$
$$\Rightarrow \quad x = \frac{\log(y-3)}{\log 2} + 1$$

(ii) $y = 8 \quad \Rightarrow \quad x = \dfrac{\log(8-3)}{\log 2} + 1 = \dfrac{\log 5}{\log 2} + 1 = 2.321928 + 1$

$$= 3.321928$$
$$= 3.3219$$

10. $\log_{10} x = 1 + a \Rightarrow 10^{1+a} = x$

$\log_{10} y = 1 - a \Rightarrow 10^{1-a} = y$

$\Rightarrow xy = 10^{1+a} . 10^{1-a} = 10^{1+a+1-a}$

$= 10^2 = 100$

11. $p = \log_a \dfrac{21}{4} = \log_a 21 - \log_a 4$

$= \log_a 7.3 - \log_a 4$

$= \log_a 7 + \log_a 3 - \log_a 4$

$q = \log_a \dfrac{7}{3} = \log_a 7 - \log_a 3$

$\Rightarrow \quad p + q = \log_a 7 + \log_a 3 - \log_a 4 + \log_a 7 - \log_a 3$

$= 2\log_a 7 - \log_a 4$

$2r = 2\log_a \dfrac{7}{2} = 2[\log_a 7 - \log_a 2]$

$= 2\log_a 7 - 2\log_a 2$

$= 2\log_a 7 - \log_a 2^2 = 2\log_a 7 - \log_a 4$

Hence, $\quad p + q = 2r$

12. If $\log_a x = 4$ and $\log_a y = 5$

(i) $\log_a x^2 y = \log_a x^2 + \log_a y$

$= 2\log_a x + \log_a y = 2(4) + 5 = 13$

(ii) $\log_a axy = \log_a a + \log_a y$

$= 1 + 4 + 5 = 10$

(iii) $\log_a \dfrac{\sqrt{x}}{y} = \log_a \sqrt{x} - \log_a y$

$= \log_a x^{\frac{1}{2}} - \log_a y$

$= \dfrac{1}{2}\log_a x - \log_a y$

$= \dfrac{1}{2}(4) - 5 = -3$

13. Prove $\log_{25} x = \dfrac{1}{2}\log_5 x$

Proof $\log_{25} x = \dfrac{\log_5 x}{\log_5 25}$

$= \dfrac{\log_5 x}{2} = \dfrac{1}{2}\log_5 x$

14. (i) $\log_{10} 4 = 0.60205999 = 0.602$

(ii) $\log_{10} 27 = 1.43136 = 1.43$

(iii) $\log_{10} 356 = 2.5514 = 2.55$

(iv) $\log_{10} 5600 = 3.748 = 3.75$

(v) $\log_{10} 29\,000 = 4.462 = 4.46$
(vi) $\log_{10} 35\,0000 = 5.544 = 5.54$
(vii) $\log_{10} 38\,70\,000 = 6.5877 = 6.59$

15. Minimum $= 10^3 = 1000$
Maximum $= 10^4 = 10\,000$

16. $\log_3 15 = \dfrac{\log_{10} 15}{\log_{10} 3} = \dfrac{1.17609}{0.47712} = 2.464977$

$\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} = \dfrac{0.69897}{0.30103} = 2.3219$

Hence, $\log_3 15 - \log_2 5 = 2.464977 - 2.3219$
$= 0.143077 = 0.143$

17. (i) $\log_{27} 81 = \dfrac{\log_3 81}{\log_3 27} = \dfrac{4}{3}$

(ii) $\log_{32} 8 = \dfrac{\log_2 8}{\log_2 32} = \dfrac{3}{5}$

18. Show: $\log_b a = \dfrac{1}{\log_a b}$

Proof: $\log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b}$

19. $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \dfrac{1}{\log_5 x} = \log_x 2 + \log_x 3 + \log_x 5$
$= \log_x (2)(3)(5)$
$= \log_x 30$
$= \dfrac{1}{\log_{30} x}$

20. $\log_r p = \log_r 2 + 3 \log_r q$
$\Rightarrow \log_r p = \log_r 2 + \log_r q^3$
$\Rightarrow \log_r p = \log_r 2q^3$
$\Rightarrow p = 2q^3$

21. $\log_3 a + \log_9 a = \dfrac{3}{4}$
$\Rightarrow \log_3 a + \dfrac{\log_3 a}{\log_3 9} = \dfrac{3}{4}$
$\Rightarrow \log_3 a + \dfrac{\log_3 a}{2} = \dfrac{3}{4}$
$\Rightarrow 2 \log_3 a + \log_3 a = \dfrac{3}{2}$
$\Rightarrow 3\log_3 a = \dfrac{3}{2}$
$\Rightarrow \log_3 a = \dfrac{1}{2}$
$\Rightarrow 3^{\frac{1}{2}} = a \Rightarrow a = \sqrt{3}$

22. $3 \ln 41.5 - \ln 250 = 3(3.7256934) - 5.52146$
$= 11.17708 - 5.52146$
$= 5.6556$
$= 5.66$

23. Solve $\log_2 (x-2) + \log_2 x = 3$

$\Rightarrow \quad \log_2 (x-2)(x) = 3$

$\Rightarrow \quad x^2 - 2x = 2^3 - 8$

$\Rightarrow \quad x^2 - 2x - 8 = 0$

$\Rightarrow \quad (x-4)(x+2) = 0$

$\Rightarrow \quad x = 4, \qquad x = -2$ (Not valid)

24. $\log_{10} (x^2 + 6) - \log_{10} (x^2 - 1) = 1$

$\Rightarrow \quad \log_{10} \dfrac{x^2+6}{x^2-1} = 1$

$\Rightarrow \quad \dfrac{x^2+6}{x^2-1} = 10^1 = 10$

$\Rightarrow \quad 10x^2 - 10 = x^2 + 6$

$\Rightarrow \quad 9x^2 - 16 = 0$

$\Rightarrow \quad (3x+4)(3x-4) = 0$

$\Rightarrow \quad x = -\dfrac{4}{3}, x = \dfrac{4}{3} \quad \Rightarrow \quad x = \pm\dfrac{4}{3}$

25. $\log 2x - \log (x-7) = \log 3$

$\Rightarrow \quad \log \dfrac{2x}{x-7} = \log 3$

$\Rightarrow \quad \dfrac{2x}{x-7} = 3$

$\Rightarrow \quad 3x - 21 = 2x$

$\Rightarrow \quad x = 21$

26. $\log (2x+3) + \log(x-2) = 2 \log x$

$\Rightarrow \quad \log (2x+3)(x-2) = \log x^2$

$\Rightarrow \quad 2x^2 - x - 6 = x^2$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad (x-3)(x+2) = 0$

$\Rightarrow \quad x = 3, x = -2$ (Not valid)

27. $\log_{10}(17 - 3x) + \log_{10} x = 1$

$\Rightarrow \quad \log_{10}(17-3x)(x) = 1$

$\Rightarrow \quad 10^1 = 17x - 3x^2$

$\Rightarrow \quad 3x^2 - 17x + 10 = 0$

$\Rightarrow \quad (3x-2)(x-5) = 0 \quad \Rightarrow x = \dfrac{2}{3}, 5$

28. $\log_{10}(x^2 - 4x - 11) = 0$

$\Rightarrow \quad 10^0 = x^2 - 4x - 11$

$\Rightarrow \quad x^2 - 4x - 11 = 1$

$\Rightarrow \quad x^2 - 4x - 12 = 0$

$\Rightarrow \quad (x-6)(x+2) = 0$

$\Rightarrow \quad x = 6, -2$

29. $2 \log_2 x = y$ and $\log_2(2x) = y + 4$

$\Rightarrow \quad \log_2 x^2 = y \quad \Rightarrow \quad 2^{y+4} = 2x$

$\Rightarrow \quad 2^y = x^2 \quad \Rightarrow \quad 2^y . 2^4 = 2x$

$\Rightarrow \quad x^2 . 16 = 2x$

$\Rightarrow \quad 16x^2 - 2x = 0$

$\Rightarrow \quad 8x^2 - x = 0$

$\Rightarrow \quad x(8x-1) = 0$

$\Rightarrow \quad x = \dfrac{1}{8}, x = 0$ (Not valid)

30. $\log_6 x + \log_6 y = 1$

$\Rightarrow \quad \log_6 x \, . \, y = 1 \qquad \Rightarrow \quad 6 = xy$

$\Rightarrow \quad x = \dfrac{6}{y}$

Solve $\log_6 x + \log_6 y = 1 \quad \cap \quad 5x + y = 17$

$\Rightarrow x = \dfrac{6}{y} \qquad \Rightarrow \quad 5\left(\dfrac{6}{y}\right) + y = 17$

$\Rightarrow \quad \dfrac{30}{y} + y = 17$

$\Rightarrow \quad 30 + y^2 = 17y$

$\Rightarrow \quad y^2 - 17y + 30 = 0$

$(y - 2)(y - 15) = 0$

$\Rightarrow \quad y = 2, \quad y = 15$

$\Rightarrow \quad x = \dfrac{6}{2} = 3, x = \dfrac{6}{15} = \dfrac{2}{5}$

31. (i) Solve $4 \log_x 2 - \log_2 x - 3 = 0$

$\Rightarrow \quad 4 \log_x 2 - \dfrac{\log_x x}{\log_x 2} - 3 = 0$

$\Rightarrow \quad 4 \log_x 2 - \dfrac{1}{\log_x 2} - 3 = 0$

Let $y = \log_x 2 \quad \Rightarrow \quad 4y - \dfrac{1}{y} - 3 = 0$

$\Rightarrow \quad 4y^2 - 1 - 3y = 0$

$\Rightarrow \quad 4y^2 - 3y - 1 = 0$

$\Rightarrow \quad (4y + 1)(y - 1) = 0$

$\Rightarrow \quad y = -\dfrac{1}{4}, y = 1$

Hence, $\quad \log_x 2 = -\dfrac{1}{4}, \log_x 2 = 1$

$\Rightarrow \quad x^{-\frac{1}{4}} = 2, x^1 = 2$

$\Rightarrow \quad x = 2^{-4} = \dfrac{1}{16}, x = 2$

(ii) Solve $2 \log_4 x + 1 = \log_x 4$

$\Rightarrow \quad 2 \log_4 x + 1 = \dfrac{\log_4 4}{\log_4 x}$

$\Rightarrow \quad 2 \log_4 x + 1 = \dfrac{1}{\log_4 x}$

Let $y = \log_4 x \quad \Rightarrow \quad 2y + 1 = \dfrac{1}{y}$

$\Rightarrow \quad 2y^2 + y = 1$

$\Rightarrow \quad 2y^2 + y - 1 = 0$

$\Rightarrow \quad (2y - 1)(y + 1) = 0$

$\Rightarrow \quad y = \dfrac{1}{2}, y = -1$

Hence, $\log_4 x = \dfrac{1}{2}, \qquad \log_4 x = -1$

$\Rightarrow \quad 4^{\frac{1}{2}} = x, \qquad 4^{-1} = x$

$\Rightarrow \quad x = \sqrt{4} = 2, \qquad x = \dfrac{1}{4}$

Exercise 12.10

1. Property 1 $\Rightarrow$ $\log_a 1 = 0$ $\Rightarrow a^0 = 1$, true

Property 2 $\Rightarrow$ $\log_2 2 = 1$ $\Rightarrow 2^1 = 2$, true

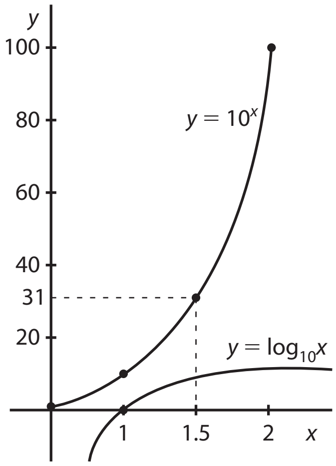
$\Rightarrow$ $\log_e e = 1$ $\Rightarrow e^1 = e$, true

$\Rightarrow$ $\log_{10} 10 = 1$ $\Rightarrow 10^1 = 10$, true

Property 5 $\Rightarrow$ $\log_a 0 = y$ $\Rightarrow a^y = 0 \Rightarrow$ No solution here

2.

$x =$	0	1	2
$y = 10^x =$	1	10	100
$y = \log_{10} x =$	undef.	0	0.3



3. (i)

$y =$

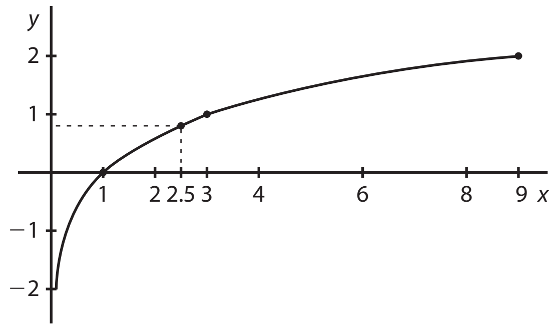
$x =$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x =$	-2	-1	0	1	2

(ii) Graph $y = \log_3 x$

(iii) $\log_3 2.5 = 0.8$, using graph

(iv) $\log_3 2.5 = \dfrac{\log_{10} 2.5}{\log_{10} 3}$

$= \dfrac{0.39794}{0.47712} = 0.834$



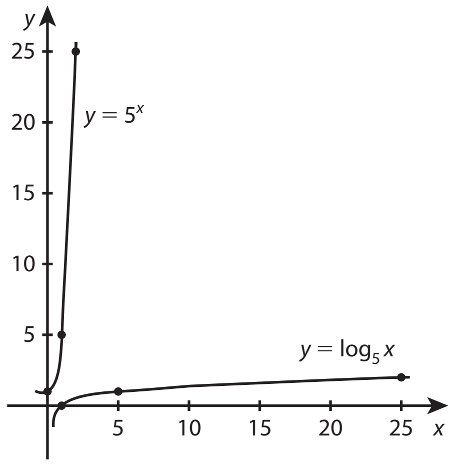
4. (i)

$x =$	0	1	2
$y = 5^x =$	1	5	25

(ii)

$x =$	0	1	5	10	25
$y = \log_5 x =$	undef.	0	1	1.4	2

(iii) One graph is the inverse of the other.

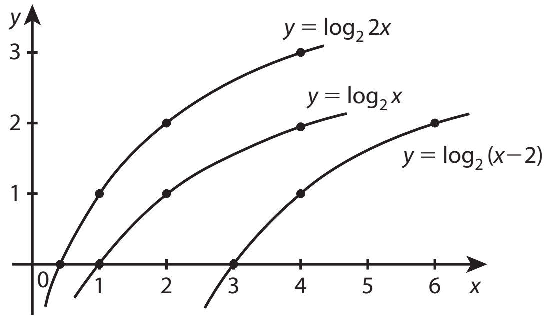


5.

$x =$	0	1	2	4
$y = \log_2 x =$	undef.	0	1	2

$x =$	0	1	2	4
$2x =$	0	2	4	8
$y = \log_2 2x =$	undef.	1	2	3

$x =$	2	3	4	6
$x - 2 =$	0	1	2	4
$y = \log_2 (x - 2) =$	undef.	0	1	2

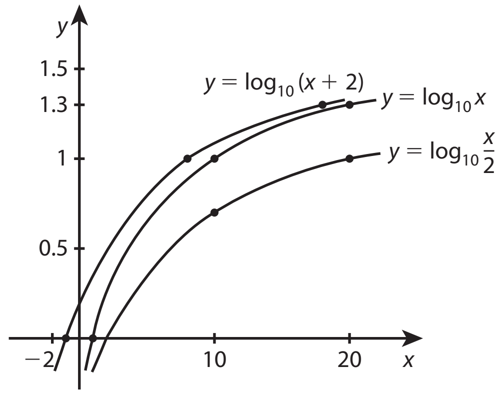


6.

$x =$	0	1	10	20
$y = \log_{10} x =$	undef.	0	1	1.3

$x =$	0	2	10	20
$\frac{x}{2} =$	0	1	5	10
$y = \log_{10} \frac{x}{2} =$	undef.	0	0.7	1

$x =$	-2	-1	8	18
$x + 2 =$	0	1	10	20
$y = \log_{10} (x + 2) =$	u.	0	1	1.3



7. (i)

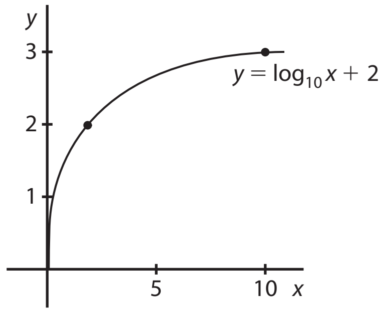
$$3^{x+2} = y + 5$$
$$\Rightarrow \quad \log 3^{x+2} = \log (y + 5)$$
$$\Rightarrow \quad (x + 2) \log 3 = \log (y + 5)$$
$$\Rightarrow \quad x + 2 = \frac{\log (y + 5)}{\log 3}$$
$$\Rightarrow \quad x = \frac{\log (y + 5)}{\log 3} - 2 \quad \text{or} \quad \log_3 (y + 5) - 2$$

(ii) $x = \dfrac{\log (y + 5)}{\log 3} - 2$

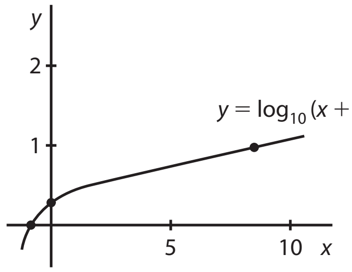
when $y = 30 \quad \Rightarrow x = \dfrac{\log (30 + 5)}{\log 3} - 2 = \dfrac{\log 35}{\log 3} - 2$

$$= \frac{1.544068}{0.47712} - 2$$
$$= 3.2362 - 2$$
$$x = 1.2362 = 1.236$$

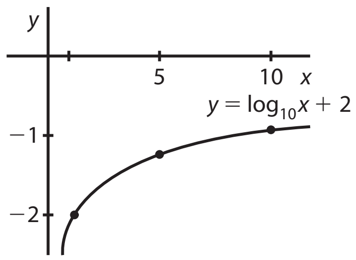
8. (i) Graph of $y = \log_{10} x + 2$



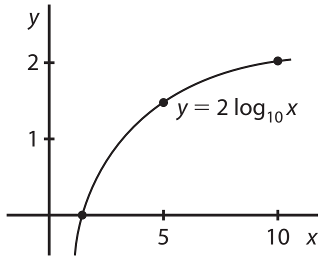
(ii) Graph of $y = \log_{10}(x + 2)$



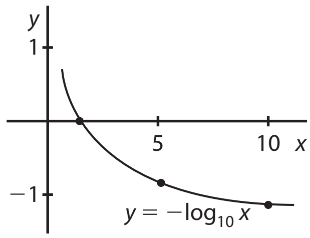
(iii) Graph of $y = \log_{10} x - 2$



(iv) Graph of $y = 2 \log_{10} x$

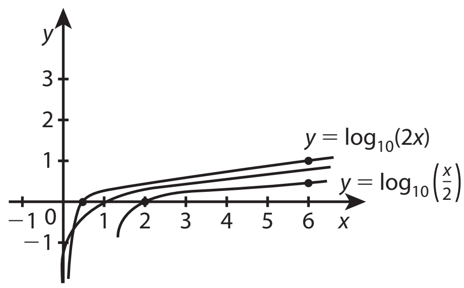


(v) $y = -\log_{10} x$



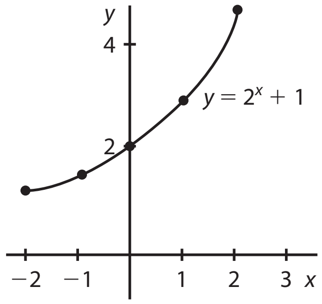
9. Graphs of (i) $y = \log_{10}(2x)$

(ii) $y = \log_{10}\left(\dfrac{x}{2}\right)$

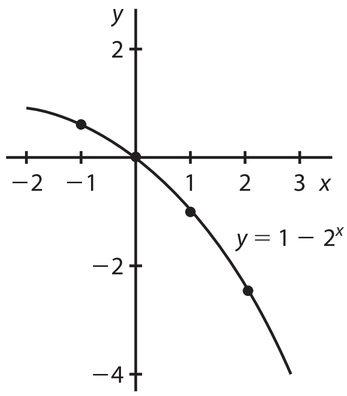


10. Graphs for:

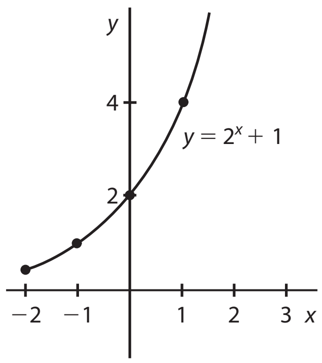
(i) $y = 2^x + 1$



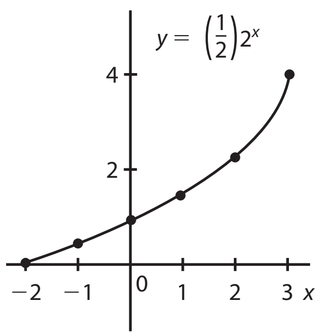
(ii) $y = 1 - 2^x$



(iii) $y = 2^{x+1}$



(iv) $y = \left(\frac{1}{2}\right) . 2^x$



Exercise 12.11

1. (i) (a) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^1 = 5000(1 + 0.006)$

$= 5000(1.006) = €5030$

(b) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^2 = 5000(1.006)^2 = €5060.18$

(c) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^3 = 5000(1.006)^3 = €5090.54$

(ii) $5000(1.006)^t$

(iii) $5000(1.006)^t = 10\,000$

$\Rightarrow \quad (1.006)^t = \frac{10\,000}{5000} = 2$

$\Rightarrow \quad \ln(1.006)^t = \ln 2$

$\Rightarrow \quad t\ln(1.006) = \ln 2$

$\Rightarrow t = \frac{\ln 2}{\ln 1.006} = 115.87 = 116$ months

2. $y = Ae^{bt}$

$t = 0 \Rightarrow 100 = Ae^{b(0)} = Ae^0 = A\,.\,1$

$\Rightarrow \quad A = 100$

Hence, $y = 100e^{bt}$

$t = 6 \quad \Rightarrow 100e^{b(6)} = 450$

$\Rightarrow \quad e^{6b} = \frac{450}{100} = 4.5$

$\Rightarrow \quad \ln e^{6b} = \ln 4.5$

$\Rightarrow \quad 6b\ln e = 6b(1) = \ln 4.5$

$\Rightarrow \quad b = \frac{\ln 4.5}{6} = 0.2507 = 0.25$

3. $T = 15 + 30 \times 10^{-0.02t}$

(i) $t = 0 \Rightarrow T = 15 + 30 \times 10^{-0.02(0)}$

$= 15 + 30 \times 10^0 = 15 + 30(1) = 45°\text{C}$

(ii) $T = 35°\text{C} \Rightarrow 15 + 30 \times 10^{-0.02t} = 35$

$\Rightarrow \quad 30 \times 10^{-0.02t} = 35 - 15 = 20$

$\Rightarrow \quad 10^{-0.02t} = \frac{20}{30} = \frac{2}{3}$

$\Rightarrow \quad \log_{10} 10^{-0.02t} = \log_{10}\frac{2}{3}$

$\Rightarrow \quad -0.02t(\log_{10} 10) = -0.02t(1) = \log\frac{2}{3}$

$\Rightarrow \quad t = \frac{\log\frac{2}{3}}{-0.02}$

$= \frac{-0.17609}{-0.02}$

$= 8.8$ minutes

(iii) As t gets larger, $30 \times 10^{-0.02t}$ gets smaller.

Hence, smallest value for $30 \times 10^{-0.02t} = 0$.

$\Rightarrow$ Room temperature $= 15°\text{C}$

4. $L = 10\log_{10}\left(\frac{I}{I_o}\right) = 10\log_{10}\left(\frac{I}{1\times10^{-12}}\right)$

(i) $L = 100 \quad \Rightarrow 100 = 10[\log_{10} I - \log_{10}(1\times10^{-12})]$

$\Rightarrow 10 = \log_{10} I + 12$

$\Rightarrow \log_{10} I = -2$

$I = 10^{-2} = 0.01 \text{ Wm}^{-2}$

$L = 110 \quad \Rightarrow 110 = 10[\log_{10} I - \log_{10}(1\times10^{-12})]$

$11 = \log_{10} I + 12$

$\Rightarrow \log_{10} I = -1 \Rightarrow I = 10^{-1} = 0.1 \text{ Wm}^{-2}$

$\Rightarrow$ Range is between 0.01 Wm^{-2} and 0.1 Wm–^{2}

(ii) $I = 10 \text{ Wm}^{-2} \quad \Rightarrow \quad L = 10\log_{10}\left(\frac{10}{1\times10^{-12}}\right)$

$= 10\log_{10}10^{13}$

$= 10\,.\,13\log_{10}10$

$= 10\,.\,13(1) = 130$ dB

5. $A = 10^M$ and $E \cong 10^{1.5M+4.8}$

$= 10^{1.5M}\cdot 10^{4.8}$

$= (10^M)^{1.5}\cdot 10^{4.8}$

$= A^{1.5}\cdot 10^{4.8} = 10^a A^b$

Hence, $a = 4.8$ and $b = 1.5$

6. Value = €100 in 2000

(i) t years later $\Rightarrow$ Value $= 100\left(1 + \frac{4.5}{100}\right)^t$

$= 100(1.045)^t$

(ii) 10 years later $\Rightarrow t = 10 \Rightarrow$ Value $= 100(1.045)^{10}$

$= 155.2969 =$ €155.30

(iii) 5 years earlier $\Rightarrow t = -5 \Rightarrow$ Value $= 100(1.045)^{-5}$

$= 80.2451 =$ €80.25

7. $W = 0.6\times1.15^t$

(i) $t = 0 \quad \Rightarrow \quad W = 0.6\times1.15^0$

$= 0.6\times1 = 0.6$ kg

(ii) $1.15 = 1 + 0.15 = 1 + \frac{15}{100}$

$\Rightarrow$ growth constant = 15%

(iii) $W = 2(0.6) = 1.2 \Rightarrow = 0.6\times1.15^t = 1.2$

$\Rightarrow 1.15^t = \frac{1.2}{0.6} = 2$

$\Rightarrow \ln 1.15^t = \ln 2$

$\Rightarrow t\ln 1.15 = \ln 2$

$\Rightarrow t = \frac{\ln 2}{\ln 1.15} = 4.959 = 5$ months

8. $M = M_0e^{-kt}$

(i) $M = 10$ when $t = 0 \quad \Rightarrow \quad M_0e^{-k(0)} = 10$

$\Rightarrow \quad M_0e^0 = 10$

$\Rightarrow \quad M_0(1) = M_0 = 10$

Hence, $M = 10e^{-kt}$

$M = 5$ when $t = 140$ $\Rightarrow$ $10 \,.\, e^{-k(140)} = 5$

$\Rightarrow$ $e^{-140k} = \frac{5}{10} = 0.5$

$\Rightarrow$ $\ln e^{-140k} = \ln 0.5$

$\Rightarrow$ $-140k(\ln e) = \ln 0.5$

$\Rightarrow$ $-140k(1) = -0.693147$

$\Rightarrow$ $k = \frac{-0.693147}{-140}$

$= 0.00495$

(ii) $M = 10e^{-0.00495t}$

$t = 70$ $\Rightarrow$ $M = 10\, e^{-0.00495(70)}$

$= 10\, e^{-0.3465}$

$= 10(0.7071)$

$= 7.071 = 7g$

(iii) $M = 2g$ $\Rightarrow$ $10e^{-0.00495t} = 2$

$\Rightarrow$ $e^{-0.00495t} = \frac{2}{10} = 0.2$

$\Rightarrow$ $\ln e^{-0.00495t} = \ln 0.2$

$\Rightarrow$ $-0.00495t(\ln e) = \ln 0.2$

$-0.00495t(1) = -1.609438$

$\Rightarrow$ $t = \frac{-1.609438}{-0.00495} = 325.1 = 325$ days

Exercise 12.12(A)

1. Proof:

(i) $n = 1?$ $\Rightarrow 2(1) = 1(1 + 1) \Rightarrow 2 = 2$, true $n = 1$

(ii) Assume true for $n = k$.

$\Rightarrow 2 + 4 + 6 + 8 + \ldots 2k = k(k + 1)$

(iii) Also true for $n = k + 1?$

$\Rightarrow$ $2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$

$\Rightarrow$ $2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1)(k + 2)$

$\Rightarrow$ $2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1)[(k + 1) + 1]$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

2. Proof:

(i) $n = 1?$ $\Rightarrow 1 = \frac{1}{2}[3(1) - 1]$ $\Rightarrow 1 = \frac{1}{2}(2) = 1$, true $n = 1$,

(ii) Assume true for $n = k$.

$\Rightarrow 1 + 4 + 7 + 10 + \ldots(3k - 2) = \frac{k}{2}(3k - 1)$

(iii) Also true for $n = k + 1?$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \frac{k}{2}(3k - 1) + (3k + 1)$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \frac{k(3k - 1) + 2(3k + 1)}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \dots (3k - 2) + (3k + 1) = \dfrac{3k^2 - k + 6k + 2}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \dots (3k - 2) + (3k + 1) = \dfrac{3k^2 + 5k + 2}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \dots (3k - 2) + (3k + 1) = \dfrac{(k + 1)(3k + 2)}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \dots (3k - 2) + (3k + 1) = \dfrac{(k + 1)[3(k + 1) - 1]}{2}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.

(v) Therefore, it is true for all positive integer values of n.

3. Proof:

(i) $n = 1$? $\Rightarrow 1 . 2 = \frac{1}{3}(1 + 1)(1 + 2)$ $\Rightarrow 2 = \frac{1}{3}(2)(3)$ $\Rightarrow 2 = 2$, true $n = 1$.

(ii) Assume true for $n = k$.

$\Rightarrow 1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \dots k(k + 2) = \frac{k}{3}(k + 1)(k + 2)$

(iii) Also true for $n = k + 1$?

$\Rightarrow 1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \dots k(k + 1) + (k + 1)(k + 2) = \frac{k}{3}(k + 1)(k + 2) + (k + 1)(k + 2)$

$\Rightarrow 1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \dots k(k + 1) + (k + 1)(k + 2) = \dfrac{k(k + 1)(k + 2) + 3(k + 1)(k + 2)}{3}$

$\Rightarrow 1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \dots k(k + 1) + (k + 1)(k + 2) = \dfrac{(k + 1)(k + 2)(k + 3)}{3}$

$\Rightarrow 1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \dots k(k + 1) + (k + 1)(k + 2) = \dfrac{(k + 1)[(k + 1) + 1][(k + 1) + 2]}{3}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.

(v) Therefore, it is true for all positive integer values of n.

4. Proof:

(i) $n = 1$? $\Rightarrow \dfrac{1}{2 . 3} = \dfrac{1}{2(1 + 2)}$ $\Rightarrow \dfrac{1}{2 . 3} = \dfrac{1}{2 . 3}$ $\Rightarrow$ true $n = 1$,

(ii) Assume true for $n = k$.

$\Rightarrow \dfrac{1}{2 . 3} + \dfrac{1}{3 . 4} + \dfrac{1}{4 . 5} + \dfrac{1}{5 . 6} + \dots \dfrac{1}{(k + 1)(k + 2)} + \dfrac{k}{2(k + 2)}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow \frac{1}{2 . 3} + \frac{1}{3 . 4} + \frac{1}{4 . 5} + \frac{1}{5 . 6} + \dots \frac{1}{(k + 1)(k + 2)} + \frac{1}{(k + 2)(k + 3)} = \frac{k}{2(k + 2)} + \frac{1}{(k + 2)(k + 3)}$$

$$= \frac{k(k + 3) + 2(1)}{2(k + 2)(k + 3)}$$

$$= \frac{k^2 + 3k + 2}{2(k + 2)(k + 3)}$$

$$= \frac{(k + 1)(k + 2)}{2(k + 2)(k + 3)}$$

$$= \frac{k + 1}{2(k + 2)} = \frac{k + 1}{2[(k + 1) + 2]}$$

$\therefore$ It is true for $n = k + 1$

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.
(v) Therefore, it is true for all positive integer values of n.

5. Proof:

(i) $n = 1$? $\Rightarrow \frac{1}{4\,.\,5} = \frac{1}{4(1+4)} \quad \Rightarrow \frac{1}{4\,.\,5} = \frac{1}{4\,.\,5}$, true $n = 1$

(ii) Assume true for $n = k$.

$$\Rightarrow \frac{1}{4\,.\,5} + \frac{1}{5\,.\,6} + \frac{1}{6\,.\,7} + \dots \frac{1}{(k+3)(k+4)} = \frac{k}{4(k+4)}$$

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow \quad \frac{1}{4\,.\,5} + \frac{1}{5\,.\,6} + \frac{1}{6\,.\,7} + \dots \frac{1}{(k+3)(k+4)} + \frac{1}{(k+4)(k+5)} &= \frac{k}{4(k+4)} + \frac{1}{(k+4)(k+5)}\\ &= \frac{k(k+5)+4(1)}{4(k+4)(k+5)}\\ &= \frac{k^2+5k+4}{4(k+4)(k+5)}\\ &= \frac{(k+1)(k+4)}{4(k+4)(k+5)}\\ &= \frac{k+1}{4(k+5)} = \frac{k+1}{4[(k+1)+4]}\end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.
(v) Therefore, it is true for all positive integer values of n.

6. Proof:

(i) $n = 1$? $\Rightarrow 1^3 = \frac{(1)^2}{4}(1+1)^2 \quad \Rightarrow 1 = \frac{1}{4} \quad \Rightarrow 1 = 1$, true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 1^3 + 2^3 + 3^3 + \dots k^3 = \frac{k^2}{4}(k+1)^2$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow \quad 1^3 + 2^3 + 3^3 + \dots k^3 + (k+1)^3 &= \frac{k^2}{4}(k+1)^2 + (k+1)^3\\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\ &= \frac{(k+1)^2[k^2+4(k+1)]}{4}\\ &= \frac{(k+1)^2(k+2)(k+2)}{4}\\ &= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2[(k+1)+1]^2}{4}\end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.
(v) Therefore, it is true for all positive integer values of n.

7. $\displaystyle\sum_{n=1}^{n} n(n+2) = 1\,.\,3 + 2\,.\,4 + 3\,.\,5 + \dots n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Proof:

(i) $n = 1$? $\Rightarrow 1\,.\,3 = \frac{1(1+1)(2+7)}{6} \quad \Rightarrow 3 = \frac{2\,.\,9}{6} \quad \Rightarrow 3 = 3$, true $n = 1$.

(ii) Assume true for $n = k \Rightarrow 1\,.\,3 + 2\,.\,4 + 3\,.\,5 + \dots k(k+2) = \dfrac{k(k+1)(2k+7)}{6}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow \quad 1\,.\,3 + 2\,.\,4 + 3\,.\,5 + \dots k(k+2) + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$
$$= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}$$
$$= \frac{(k+1)[2k^2 + 7k + 6k + 18]}{6}$$
$$= \frac{(k+1)[2k^2 + 13k + 18]}{6}$$
$$= \frac{(k+1)(k+2)(2k+9)}{6} = \frac{(k+1)[(k+1)+1][2(k+1)+7]}{6}$$

$\therefore$ It is true for $n = k + 1$

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.

(v) Therefore, it is true for all positive integer values of n.

8. Proof:

(i) $n = 1$? $\quad \Rightarrow x = \dfrac{x(x^1 - 1)}{x - 1} \quad \Rightarrow x = x,\quad$ true $n = 1$.

(ii) Assume true for $n = k \Rightarrow x + x^2 + x^3 + x^4 + \dots + x^k = \dfrac{x(x^k - 1)}{x - 1}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow x + x^2 + x^3 + \dots + x^k + x^{k+1} = \frac{x(x^k - 1)}{x - 1} + x^{k+1}$$
$$= \frac{x(x^k - 1) + x^{k+1}(x - 1)}{x - 1}$$
$$= \frac{x\,.\,x^k - x + x\,.\,x^{k+1} - x^{k+1}}{x - 1}$$
$$= \frac{x^{k+1} + x\,.\,x^{k+1} - x - x^{k+1}}{x - 1}$$
$$= \frac{x(x^{k+1} - 1)}{x - 1}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.

(v) Therefore, it is true for all positive integer values of n.

Exercise 12.12(B)

1. Proof:

(i) True for $n = 1$? $\quad \Rightarrow 5$ is a factor of $6^1 - 1 = 5$, $\quad$ true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 5$ is a factor of $6^k - 1$, $\quad k \in N$.

(iii) Also true for $n = k + 1$?

$\Rightarrow 6^{k+1} - 1 = 6^k\,.\,6^1 - 1$
$= 6^k(5 + 1) - 1$
$= 5\,.\,6^k + (6^k - 1)$

Since $5\,.\,6^k$ is divisible by 5 and $(6^k - 1)$ is assumed divisible by 5,
$\therefore 5\,.\,6^k + (6^k - 1)$ is divisible by 5.
$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \dots$ etc.

(v) Therefore, it is true for all positive integer values of n.

2. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $5^1 - 1 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $5^k - 1$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 5^{k+1} - 1 &= 5^k \,.\, 5^1 - 1 \\ &= 5^k(4 + 1) - 1 \\ &= 4 \,.\, 5^k + (5^k - 1) \end{aligned}$$

Since $4 \,.\, 5^k$ is divisible by 4 and $(5^k - 1)$ is assumed divisible by 4,

$\therefore 4 \,.\, 5^k + (5^k - 1)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

3. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $9^1 - 5^1 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $9^k - 5^k$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 9^{k+1} - 5^{k+1} &= 9^k \,.\, 9^1 - 5^k \,.\, 5^1 \\ &= 9^k(8 + 1) - 5^k(4 + 1) \\ &= 8 \,.\, 9^k + 9^k - 4 \,.\, 5^k - 5^k \\ &= 8 \,.\, 9^k - 4 \,.\, 5^k + (9^k - 5^k) \end{aligned}$$

Since $8 \,.\, 9^k - 4 \,.\, 5^k$ is divisible by 4 and $9^k - 5^k$ is assumed divisible by 4,

$\therefore 8 \,.\, 9^k - 4 \,.\, 5^k + (9^k - 5^k)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

4. Proof:

(i) True for $n = 1$? $\Rightarrow$ 8 is a factor of $3^{2(1)} - 1 = 9 - 1 = 8$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 8 is a factor of $3^{2k} - 1$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k} \,.\, 3^2 - 1 \\ &= 3^{2k} \,.\, 9 - 1 \\ &= 3^{2k}(8 + 1) - 1 \\ &= 8 \,.\, 3^{2k} + (3^{2k} - 1) \end{aligned}$$

Since $8 \,.\, 3^{2k}$ is divisible by 8 and $(3^{2k} - 1)$ is assumed divisible by 8,

$\therefore 8 \,.\, 3^{2k} + (3^{2k} - 1)$ is divisible by 8.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

5. Proof:

(i) True for $n = 1$? $\Rightarrow$ 5 is a factor of $7^1 - 2^1 = 5$, true $n = 1$

(ii) Assume true for $n = k$ $\Rightarrow$ 5 is a factor of $7^k - 2^k$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 7^{k+1} - 2^{k+1} &= 7^k \,.\, 7^1 - 2^k \,.\, 2^1 \\ &= 7^k(5 + 2) - 2 \,.\, 2^k \\ &= 5 \,.\, 7^k + 2 \,.\, 7^k - 2 \,.\, 2^k \\ &= 5 \,.\, 7^k + 2(7^k - 2^k) \end{aligned}$$

Since 5.7^k is divisible by 5 and $(7^k - 2^k)$ is assumed divisible by 5,

$\therefore 5 \,.\, 7^k + 2(7^k - 2^k)$ is divisible by 5.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

6. Proof:

(i) True for $n = 1$? $\Rightarrow$ 8 is a factor of $7^{2(1)+1} + 1 = 7^3 + 1 = 344 = 8 \,.\, 43$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow 7^{2k+1} + 1$ is divisible by 8.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 7^{2(k+1)+1} + 1 &= 7^{2k+2+1} + 1 \\ &= 7^{2k+1} \,.\, 7^2 + 1 \\ &= 7^{2k+1} \,.\, 49 + 1 \\ &= 7^{2k+1}(48 + 1) + 1 \\ &= 48 \,.\, 7^{2k+1} + (7^{2k+1} + 1) \end{aligned}$$

Since $48 \,.\, 7^{2k+1}$ is divisible by 8 and $(7^{2k+1} + 1)$ is assumed divisible by 8,

$\therefore 48 \,.\, 7^{2k+1} + (7^{2k+1} + 1)$ is divisible by 8.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

7. Proof:

(i) True for $n = 1$? $\Rightarrow$ 7 is a factor of $2^{3(1)-1} + 3 = 2^2 + 3 = 7$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 7 is a factor of $2^{3k-1} + 3$, $k \in N$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 2^{3(k+1)-1} + 3 &= 2^{3k+3-1} + 3 \\ &= 2^{3k-1} \,.\, 2^3 + 3 \\ &= 2^{3k-1} \,.\, 8 + 3 \\ &= 2^{3k-1}(7 + 1) + 3 \\ &= 7 \,.\, 2^{3k-1} + (2^{3k-1} + 3) \end{aligned}$$

Since $7 \,.\, 2^{3k-1}$ is divisible by 7 and $2^{3k-1} + 3$ is assumed divisible by 7,

$\therefore 7 \,.\, 2^{3k-1} + (2^{3k-1} + 3)$ is divisible by 7.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it must now be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

8. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $5^1 - 4(1) + 3 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $5^k - 4k + 3$, $k \in N$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 5^{k+1} - 4(k + 1) + 3 &= 5^k \,.\, 5^1 - 4k - 4 + 3 \\ &= 5^k(4 + 1) - 4k - 4 + 3 \\ &= 4 \,.\, 5^k + 5^k - 4k - 4 + 3 \\ &= 4 \,.\, 5^k - 4 + (5^k - 4k + 3) \end{aligned}$$

Since $4 \,.\, 5^k - 4$ is divisible by 4 and $(5^k - 4k + 3)$ is assumed divisible by 4,

$\therefore 4 \,.\, 5^k - 4 + (5^k - 4k + 3)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it must now be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

9. Proof:
(i) True for $n = 1$? $\Rightarrow$ 6 is a factor of $7^1 + 4^1 + 1 = 12$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow$ 6 is a factor of $7^k + 4^k + 1$, $k \in N$.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 7^{k+1} + 4^{k+1} + 1 &= 7^k . 7^1 + 4^k . 4^1 + 1\\ &= 7^k(6 + 1) + 4^k(3 + 1) + 1\\ &= 6 . 7^k + 7^k + 3 . 4^k + 4^k + 1\\ &= 6 . 7^k + 3 . 4^k + (7^k + 4^k + 1)\end{aligned}$$

4^k is an even number $\Rightarrow 3 . 4^k$ is divisible by 6 and
$7^k + 4^k + 1$ is assumed divisible by 6;
$\therefore 6 . 7^k + 3 . 4^k + (7^k + 4^k + 1)$ is divisible by 6.
$\therefore$ It is true for $n = k + 1$.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

10. Proof:
(i) True for $n = 1$? $\Rightarrow 1(1+1)[2(1) + 1] = (1)(2)(3) = 6$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow$ 3 is a factor of $k(k + 1)(2k + 1) = 2k^3 + 3k^2 + k$, $k \in N$.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow (k + 1)[(k + 1) + 1][2(k + 1) + 1] &= (k + 1)(k + 2)(2k + 3)\\ &= (k + 1)(2k^2 + 7k + 6)\\ &= 2k^3 + 9k^2 + 13k + 6\\ &= 2k^3 + 6k^2 + 3k^2 + 12k + k + 6\\ &= 6k^2 + 12k + 6 + (2k^3 + 3k^2 + k)\end{aligned}$$

Since $6k^2 + 12k + 6$ is divisible by 3, and $(2k^3 + 3k^2 + k)$ is assumed divisible by 3,
$\therefore 6k^2 + 12k + 6 + (2k^3 + 3k^2 + k)$ is divisible by 3.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

11. Prove $n^3 - n$ is divisible by 3 for $n \in N$.
Proof:
(i) True for $n = 1$? $\Rightarrow$ 3 is a factor of $1^3 - 1 = 1 - 1 = 0$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow k^3 - k$ is divisible by 3.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow (k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= 3k^2 + 3k + (k^3 - k)\end{aligned}$$

Since $3k^2 + 3k$ is divisible by 3, and $(k^3 - k)$ is assumed divisible by 3,
$\therefore 3k^2 + 3k + (k^3 - k)$ is divisible by 3.
$\therefore$ It is true for $n = k + 1$.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

12. Prove $13^n - 6^{n-2}$ is divisible by 7 for $n \in N$.

Proof:

(i) True for $n = 2$? $\Rightarrow$ 7 is a factor of $13^2 - 6^{2-2} = 169 - 1 = 168 = 7.24$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 13^k - 6^{k-2}$ is divisible by 7.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 13^{k+1} - 6^{(k+1)-2} &= 13^k . 13^1 - 6^{k-2} . 6^1 \\ &= 13^k(7 + 6) - 6^{k-2} . 6 \\ &= 7 . 13^k + 6 . 13^k - 6 . 6^{k-2} \\ &= 7 . 13^k + 6(13^k - 6^{k-2}) \end{aligned}$$

Since $7 . 13^k$ is divisible by 7, and $(13^k - 6^{k-2})$ is assumed divisible by 7,

$\therefore 7 . 13^k + 6(13^k - 6^{k-2})$ is divisible by 7.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n excluding 1.

Exercise 12.12(C)

1. Proof:

(i) True for $n = 3$? $\Rightarrow 2^3 > 2(3) + 1$ $\Rightarrow 8 > 7$ $\Rightarrow$ true $n = 3$.

(ii) Assume true for $n = k$ $\Rightarrow 2^k > 2k + 1,\ k \in N,\ k \geqslant 3$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 2^{k+1} &= 2 . 2^k \\ &> 2(2k + 1) && \text{since } 2^k > 2k + 1 \text{ (assumed)} \\ &= 4k + 2 \\ &= 2k + 2k + 2 \\ &> 2k + 3 && \text{since } 2k + 2 > 3, \text{ for } k \geqslant 3 \\ &= 2(k + 1) + 1 \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 3$, it now must be true for $n = 3 + 1 = 4$.

And if it is true for $n = 4$, it is true for $n = 4 + 1 = 5$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 3$, $n \in N$.

2. Proof:

(i) True for $n = 2$? $\Rightarrow 3^2 > 2^2$ $\Rightarrow 9 > 4$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 3^k > k^2,\ k \in N$, and $k \geqslant 2$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 3^{k+1} &= 3^1 . 3^k \\ &> 3 . k^2 && \text{since } 3^k > k^2 \text{ (assumed)} \\ &= k^2 + k^2 + k^2 \\ &> k^2 + 2k + 1 && \text{since } 2k^2 > 2k + 1, \text{ for } k \geqslant 2 \\ &= (k + 1)(k + 1) \\ &= (k + 1)^2 \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.

And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

3. Proof:

(i) True for $n = 2$? $\Rightarrow 3^2 > 2(2) + 2$ $\Rightarrow 9 > 6$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 3^k > 2k + 2, k \in N, k \geqslant 2$.

(iii) Also true for $n = k + 1$?

$\Rightarrow 3^{k+1} = 3 \, . \, 3^k$

$> 3(2k + 2)$ since $3^k > 2k + 2$ (assumed)

$= 6k + 6$

$= 2k + 4k + 2 + 4$

$= 2k + 4 + 4k + 2$

$> 2k + 4$ since $4k + 2 > 0$, for $k \geqslant 2$

$= 2(k + 1) + 2$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.
And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

4. Proof:

(i) True for $n = 3$? $\Rightarrow 3! > 2^{3-1}$ $\Rightarrow 6 > 4$ $\Rightarrow$ true $n = 3$.

(ii) Assume true for $n = k$ $\Rightarrow k! > 2^{k-1}$, $k \in N$, $k \geqslant 3$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (k + 1)! = (k + 1)k!$

$> (k + 1) \, . \, 2^{k-1}$ since $k! > 2^{k-1}$ (assumed)

$> (1 + 1) \, . \, 2^{k-1}$ since $k + 1 > 2$, for $k \geqslant 3$

$= 2 \, . \, 2^{k-1}$

$= 2^k$

$= 2^{(k+1)-1}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 3$, it now must be true for $n = 3 + 1 = 4$.
And if it is true for $n = 4$, it is true for $n = 4 + 1 = 5$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 3$, $n \in N$.

5. Proof:

(i) True for $n = 2$? $\Rightarrow (2 + 1)! > 2^2$ $\Rightarrow 6 > 4$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow (k + 1)! > 2^k$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (k + 1 + 1)! = (k + 2)!$

$= (k + 2)(k + 1)!$

$> (k + 2) \, . \, 2^k$ since $(k + 1)! > 2^k$ (assumed)

$> 2 \, . \, 2^k$ since $(k + 2) > 2$, for $k \geqslant 2$

$= 2 \, . \, 2^{k+1}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.
And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

6. Proof:

(i) True for $n = 1$? $\Rightarrow (1 + 2x)^1 = 1 + 2(1)x$ $\Rightarrow 1 + 2x = 1 + 2x$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow (1 + 2x)^k \geqslant 1 + 2kx$ for $x > 0$, $k \in N$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (1 + 2x)^{k+1} = (1 + 2x)(1 + 2x)^k$

$\geqslant (1 + 2x)(1 + 2kx)$ since $(1 + 2x)^k \geqslant 1 + 2kx$ (assumed)

$= 1 + 2kx + 2x + 4kx^2$

$\geqslant 1 + 2kx + 2x$ since $4kx^2 \geqslant 0$, for $x > 0, k \geqslant 1$

$= 1 + 2(k+1)x$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 1$, $n \in N$.

7. Prove $(1 + ax)^n \geqslant 1 + anx$ for $a > 0$, $x > 0$, $n \in N$.

Proof:

(i) True for $n = 1$? $\Rightarrow (1 + ax)^1 \geqslant 1 + a(1)x$ $\Rightarrow 1 + ax \geqslant 1 + ax$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow (1 + ax)^k \geqslant 1 + akx$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (1 + ax)^{k+1} = (1 + ax)(1 + ax)^k$

$\geqslant (1 + ax)(1 + akx)$ since $(1 + ax)^k \geqslant 1 + akx$ (assumed)

$= 1 + akx + ax + a^2kx^2$

$\geqslant 1 + akx + ax$ since $a^2kx^2 \geqslant 0$, for $a > 0, x > 0$

$= 1 + a(k + 1)x$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 1$, $n \in N$.

Revision Exercise12 (Core)

1. $-1 \leqslant \dfrac{2x + 4}{3} \leqslant 2, x \in R$

$\Rightarrow -3 \leqslant 2x + 4 \leqslant 6$

$\Rightarrow -7 \leqslant 2x \leqslant 2$

$\Rightarrow -3.5 \leqslant x \leqslant 1$

2. (a) (i) $10^{3.5} = 3162.278 = 3162$

(ii) $\log_{10} 4.5 = 0.6532 = 0.65$

(iii) $t = 0.04 \Rightarrow 10^{3t} = 10^{3(0.04)} = 10^{0.12} = 1.318 = 1.32$

(iv) $n = 100 \Rightarrow \log 5n = \log 5(100) = \log 500 = 2.69897 = 2.7$

(b) (i) $e^{3.4} = 29.964 = 30$

(ii) $\ln 589 = 6.378 = 6.38$

(iii) $t = 40 \Rightarrow e^{-0.02t-4} = e^{-0.02(40)-4} = e^{-0.8-4} = e^{-4.8}$

$= 0.0082297 = 0.00823$

(iv) $k = 3.7 \Rightarrow \ln\left(\dfrac{10}{k}\right) = \ln\left(\dfrac{10}{3.7}\right) = 0.994 = 0.99$

3. (i) $f(x) = 3 \times 4^x$

Point $(a, 6)$ $\Rightarrow$ $f(a) = 3 \times 4^a = 6$

$\Rightarrow$ $4^a = 2$

$\Rightarrow$ $(2^2)^a = 2$

$\Rightarrow$ $2^{2a} = 2^1$

$\Rightarrow$ $2a = 1 \Rightarrow a = \dfrac{1}{2}$

(ii) Point $\left(-\frac{1}{2}, b\right) \Rightarrow f\left(-\frac{1}{2}\right) = 3 \times 4^{-\frac{1}{2}} = b$

$$\Rightarrow 3 \times \frac{1}{2} = b$$

$$\Rightarrow b = \frac{3}{2}$$

4. $x - 8 = -3$ or $x - 8 = 3$

$\Rightarrow x = 5$ or $x = 11$

5. (i)

$$5^{2n} \times 25^{2n-1} = 625$$

$$\Rightarrow 5^{2n} \times (5^2)^{2n-1} = 5^4$$

$$\Rightarrow 5^{2n} \times 5^{4n-2} = 5^4$$

$$\Rightarrow 5^{2n+4n-2} = 5^{6n-2} = 5^4$$

$$\Rightarrow 6n - 2 = 4$$

$$\Rightarrow 6n = 6 \Rightarrow n = 1$$

(ii)

$$27^{n-2} = 9^{3n+2}$$

$$\Rightarrow (3^3)^{n-2} = (3^2)^{3n+2}$$

$$\Rightarrow 3^{3n-6} = 3^{6n+4}$$

$$\Rightarrow 3n - 6 = 6n + 4$$

$$\Rightarrow -3n = 10 \Rightarrow n = -\frac{10}{3}$$

6. $y = a2^x + b$

(i) Point (0, 2.5) $\Rightarrow$ $2.5 = a\,.\,2^0 + b = a\,.\,1 + b \Rightarrow$ $a + b = 2.5$

Point (2, 4) $\Rightarrow$ $4 = a\,.\,2^2 + b = 4a + b$ $\Rightarrow$ $4a + b = 4$

(ii)

$$\begin{array}{rl} & 4a + b = 4 \\ & \underline{a + b = 2.5} \\ \Rightarrow & 3a \quad = 1.5 \end{array} \quad \Rightarrow a = \frac{1.5}{3} = 0.5$$

and $0.5 + b = 2.5$ $\Rightarrow b = 2.5 - 0.5 = 2$

7. (i) Curve C = ln $(x) \Rightarrow$ Point $(1, 0) \Rightarrow$ ln $1 = 0$, true

(ii) Curve B = ln $(x + 1) \Rightarrow$ Point $(0, 0) \Rightarrow$ ln$(0 + 1)$ = ln $1 = 0$, true

(iii) Curve A = ln $(x) + 1 \Rightarrow$ Point $(1, 1) \Rightarrow$ ln $1 + 1 = 0 + 1 = 1$, true

8. Solve $\ln(x - 1) + \ln(x + 2) = \ln(6x - 8)$

$$\Rightarrow \ln(x - 1)(x + 2) = \ln(6x - 8)$$

$$\Rightarrow (x - 1)(x + 2) = 6x - 8$$

$$\Rightarrow x^2 + x - 2 = 6x - 8$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x - 2)(x - 3) = 0$$

$$\Rightarrow x = 2, x = 3$$

9. $y = Ae^{bt}$

$y = 6$ when $t = 1$ $\Rightarrow 6 = Ae^{b(1)}$ $\Rightarrow Ae^b = 6 \Rightarrow e^b = \frac{6}{A}$

$y = 8$ when $t = 2$ $\Rightarrow 8 = Ae^{b(2)}$ $\Rightarrow Ae^{2b} = 8$

$$\Rightarrow A(e^b)^2 = 8$$

$$\Rightarrow A\left(\frac{6}{A}\right)^2 = 8$$

$$\Rightarrow A \cdot \frac{36}{A^2} = \frac{36}{A} = 8$$

$$\Rightarrow 8A = 36 \Rightarrow A = \frac{9}{2}$$

$$e^b = \frac{6}{\frac{9}{2}} = \frac{4}{3}$$

$\Rightarrow \ln e^b = \ln \frac{4}{3}$

$\Rightarrow b(\ln e) = b = \ln \frac{4}{3}$

10. $y = a \log_2 (x - b)$

$(5, 2) \Rightarrow 2 = a \log_2 (5 - b)$

$\Rightarrow 2 = \log_2 (5 - b)^a \Rightarrow (5 - b)^a = 2^2 = 4$

$\Rightarrow 5 - b = 4^{\frac{1}{a}}$

$\Rightarrow b = 5 - 4^{\frac{1}{a}}$

$(7, 4) \Rightarrow 4 = a \log_2(7 - b)$

$\Rightarrow 4 = \log_2(7 - b)^a \Rightarrow (7 - b)^a = 2^4 = 16$

$\Rightarrow 7 - b = 16^{\frac{1}{a}} = (4^2)^{\frac{1}{a}} = 4^{\frac{2}{a}}$

$\Rightarrow b = 7 - 4^{\frac{2}{a}}$

Hence, $5 - 4^{\frac{1}{a}} = 7 - 4^{\frac{2}{a}}$

$\Rightarrow 4^{\frac{2}{a}} - 4^{\frac{1}{a}} = 2$

$\Rightarrow (4^{\frac{1}{a}})^2 - 4^{\frac{1}{a}} - 2 = 0$

Let $y = 4^{\frac{1}{a}} \Rightarrow \quad y^2 - y - 2 = 0$

$\Rightarrow (y - 2)(y + 1) = 0$

$\Rightarrow y = 2, y = -1$

$\Rightarrow 4^{\frac{1}{a}} = 2$ or $4^{\frac{1}{a}} = -1$ (Not valid)

$\Rightarrow 4 = 2^a$

$\Rightarrow 2^2 = 2^a \Rightarrow a = 2$

$\Rightarrow b = 5 - 4^{\frac{1}{2}} = 5 - 2 = 3$

11. Solve $32^{x-1} = 28$

$\Rightarrow \ln 32^{x-1} = \ln 28$

$\Rightarrow (x - 1) \ln 32 = \ln 28$

$\Rightarrow x - 1 = \frac{\ln 28}{\ln 32} = 0.96147$

$\Rightarrow x = 1.96147$

$\Rightarrow x = 1.96$

12. Proof:

(i) $n = 1$? $\quad \Rightarrow 3 = \frac{3(1)}{2}(1 + 1) \quad \Rightarrow 3 = 3$, true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 3 + 6 + 9 + ... + 3k = \frac{3k}{2}(k + 1)$.

(iii) Also true for $n = k + 1$?

$\Rightarrow 3 + 6 + 9 + ... + 3k + (3k + 3) = \frac{3k}{2}(k + 1) + 3(k + 1)$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \frac{3k(k + 1) + 2 \,.\, 3(k + 1)}{2}$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \frac{3}{2}(k + 1)(k + 2)$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \frac{3}{2}(k + 1)[(k + 1) + 1]$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ ... etc.

(v) Therefore, it is true for all positive integer values of n.

13. Proof:

(i) True for $n = 1$? $\Rightarrow$ 7 is a factor of $8^1 + 6 = 14$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ $8^k + 6$ is divisible by 7.

(iii) Also true for $n = k + 1$?

$$\Rightarrow 8^{k+1} + 6 = 8^k \,.\, 8^1 + 6$$
$$= 8^k(7 + 1) + 6$$
$$= 7 \,.\, 8^k + (8^k + 6)$$

Since $7 \,.\, 8^k$ is divisible by 7, and $(8^k + 6)$ is assumed divisible by 7,

$\therefore 7 \,.\, 8^k + (8^k + 6)$ is divisible by 7.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ ... etc.

(v) Therefore, it is true for all positive integer values of n.

14. Prove by induction that $n^2 > 4n + 3$ for $n \geqslant 5, n \in N$.

Proof:

(i) True for $n = 5$? $\Rightarrow 5^2 > 4(5) + 3$ $\Rightarrow 25 > 23$ $\Rightarrow$ true $n = 5$.

(ii) Assume true for $n = k$ $\Rightarrow k^2 > 4k + 3$ for $k \geqslant 5, k \in N$.

(iii) Also true for $n = k + 1$?

$$\Rightarrow (k + 1)^2 = k^2 + 2k + 1$$
$$> 4k + 3 + 2k + 1 \qquad \text{since } k^2 > 4k + 3 \text{ (assumed)}$$
$$= 4k + 4 + 2k$$
$$> 4(k + 1) + 3 \qquad \text{since } 2k > 3, \text{ for } k \geq 5$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 5$, it now must be true for $n = 5 + 1 = 6$.

And if it is true for $n = 6$, it is true for $n = 6 + 1 = 7$ ... etc.

(v) Therefore, it is true for all values of n, $n \geqslant 5, n \in N$.

Revision Exercise12 (Advanced)

1. (i) $3x + 4 < x^2 - 6$

$x^2 - 3x - 10 > 0$

Let $x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

$x = 5$ or $x = -2$

Solution: $x < -2$ or $x < 5$

(ii) $x^2 - 6 < 9 - 2x$

$x^2 + 2x - 15 < 0$

Let $x^2 + 2x - 15 = 0$

$(x + 5)(x - 3) = 0$

$x = -5$ or $x = 3$

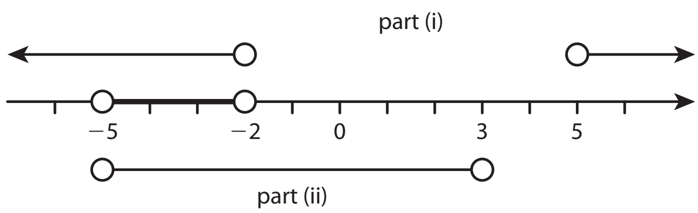
Solution: $-5 < x < 3$

(iii) The solution of

$$3x + 4 < x^2 - 6 < 9 - 2x$$

is then the intersection of the above solution sets, i.e.

$$-5 < x < -2.$$



2. $M = 30 \times 2^{-0.001t}$

(i) $t = 0 \Rightarrow M = 30 \times 2^{-0.001(0)} = 30 \times 2^0 = 30 \times 1 = 30\,\text{g}$

(ii) $M = 10$

$$\begin{aligned}
&\Rightarrow & 30 \times 2^{-0.001t} &= 10 \\
&\Rightarrow & 2^{-0.001t} &= \frac{10}{30} = \frac{1}{3} \\
&\Rightarrow & \log 2^{-0.001t} &= \log \frac{1}{3} \\
&\Rightarrow & -0.001t \,.\, \log 2 &= \log \frac{1}{3} \\
&\Rightarrow & -0.001t &= \frac{\log \frac{1}{3}}{\log 2} = -1.5849625 \\
&\Rightarrow & t &= \frac{-1.5849625}{-0.001} = 1584.9625 \\
&\Rightarrow & t &= 1585 \text{ years}
\end{aligned}$$

(iii) 1% of 30 = 0.3

$$\begin{aligned}
&\Rightarrow & 30 \times 2^{-0.001t} &= 0.3 \\
&\Rightarrow & 2^{-0.001t} &= \frac{0.3}{30} = 0.01 \\
&\Rightarrow & \log 2^{-0.001t} &= \log 0.01 \\
&\Rightarrow & -0.001t \,.\, \log 2 &= \log 0.01 \\
&\Rightarrow & -0.001t &= \frac{\log 0.01}{\log 2} = -6.643856 \\
&\Rightarrow & t &= \frac{-6.643856}{-0.001} = 6643.856 = 6644 \text{ years}
\end{aligned}$$

3. $I = I_0 \times 10^{0.1S}$

(i) $S = 30 \Rightarrow I = I_0 \times 10^{0.1(30)}$

$= I_0 \times 10^3 = 1000\,I_0 \Rightarrow$ Answer $= 1000$

(ii) $S = 28 \Rightarrow I = I_0 \times 10^{0.1(28)} = I_0 \times 10^{2.8}$

$S = 15 \Rightarrow I = I_0 \times 10^{0.1(15)} = I_0 \times 10^{1.5}$

$\Rightarrow$ No. of times $= \dfrac{I_0 \times 10^{2.8}}{I_0 \times 10^{1.5}} = 10^{1.3} = 19.95 = 20$ Times

4. Solve $\log_5 x - 1 = 6 \log_x 5$.

$\Rightarrow \log_5 x - 1 = 6 \dfrac{\log_5 5}{\log_5 x} = 6 \,.\, \dfrac{1}{\log_5 x} = \dfrac{6}{\log_5 x}$

Let $y = \log_5 x$

$$\begin{aligned}
&\Rightarrow & y - 1 &= \frac{6}{y} \\
&\Rightarrow & y^2 - y &= 6 \\
&\Rightarrow & y^2 - y - 6 &= 0 \\
&\Rightarrow & (y-3)(y+2) &= 0 \\
&\Rightarrow & y = 3, y &= -2
\end{aligned}$$

Hence, $\log_5 x = 3$ or $\log_5 x = -2$

$\Rightarrow \quad x = 5^3 = 125 \quad$ or $\quad x = 5^{-2} = \dfrac{1}{25}$

5. Solve $(0.7)^x \geqslant 0.3$

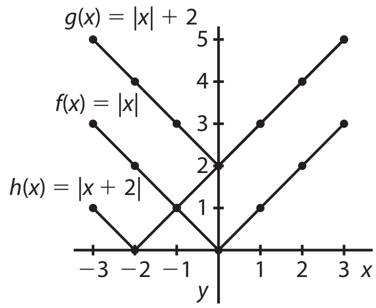
$$\begin{aligned}
&\Rightarrow & \log(0.7)^x &\geqslant \log(0.3) \\
&\Rightarrow & x \,.\, \log(0.7) &\geqslant \log(0.3) \\
&\Rightarrow & x(-0.1549) &\geqslant -0.5228787 \\
&\Rightarrow & x(0.1549) &\leqslant 0.5228787 \\
&\Rightarrow & x &\leqslant \frac{0.5228787}{0.1549} = 3.3755 \\
&\Rightarrow & x &\leqslant 3.38
\end{aligned}$$

6. (i) (ii) (iii)

$x =$	-3	-2	-1	0	1	2	3
$f(x) = \lvert x \rvert =$	3	2	1	0	1	2	3
$g(x) = \lvert x \rvert + 2$	5	4	3	2	3	4	5
$x + 2 =$	-1	0	1	2	3	4	5
$h(x) = \lvert x + 2 \rvert$	1	0	1	2	3	4	5

(iv) $f(x) \cap h(x) \quad \Rightarrow \quad x = -1$

(v) $g(x) > h(x) \quad \Rightarrow \quad -3 \leq x < 0$



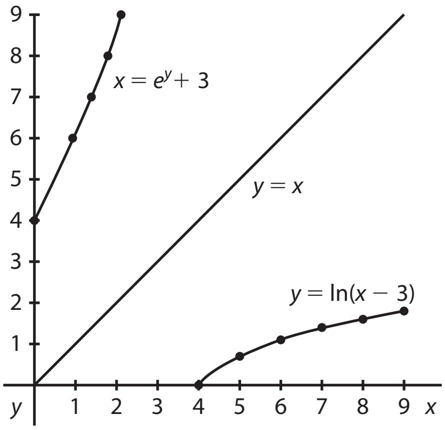
7.

$x =$	4	5	6	7	8	9
$x - 3 =$	1	2	3	4	5	6
$\ln(x - 3) =$	0	0.7	1.1	1.4	1.6	1.8

$\ln(x - 3) = \log_e(x - 3) = y$

$\Rightarrow e^y = x - 3$

$\Rightarrow x = e^y + 3$

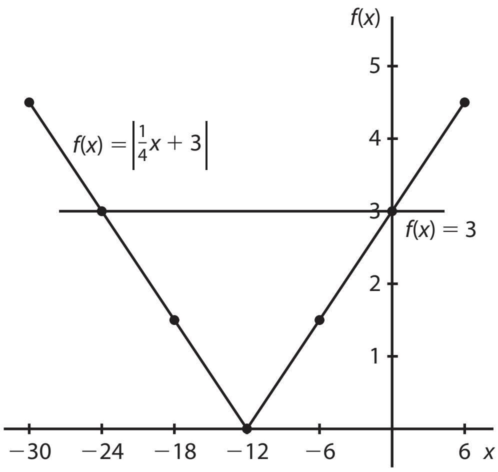


8.

$x = 1$	-30	-24	-18	-12	-6	0	6
$\frac{1}{4}x =$	-7.5	-6	-4.5	-3	-1.5	0	1.5
$\frac{1}{4}x + 3 =$	-4.5	-3	-1.5	0	1.5	3	4.5
$f(x) = \left\lvert \frac{1}{4}x + 3 \right\rvert =$	4.5	3	1.5	0	1.5	3	4.5

Solve $\left\lvert \frac{1}{4}x + 3 \right\rvert \geqslant 3$

$\Rightarrow -24 \geqslant x \geqslant 0$



9. $\dfrac{x^{\frac{3}{2}} - x^{-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}$

$$= \frac{x^{-\frac{1}{2}}(x^2 - 1)}{x^{-\frac{1}{2}}(x - 1)} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1$$

10. Proof:

(i) True for $n = 1$? $\Rightarrow \dfrac{1}{(1 + r)^1} \leqslant \dfrac{1}{1 + (1)r} \Rightarrow \dfrac{1}{1 + r} \leqslant \dfrac{1}{1 + r} \Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k \Rightarrow \dfrac{1}{(1 + r)^k} \leqslant \dfrac{1}{1 + kr}$ for $k \geqslant 1,\ r > 0,\ k \in N$.

(iii) Also true for $n = k + 1$?

$$\Rightarrow \frac{1}{(1 + r)^{k+1}} = \frac{1}{(1 + r)^k(1 + r)} \leqslant \frac{1}{(1 + kr)(1 + r)} \qquad \text{since } \frac{1}{(1 + r)^k} \leqslant \frac{1}{1 + kr} \text{ (assumed)}$$

$$= \frac{1}{1 + kr + r + kr^2}$$

$$\leqslant \frac{1}{1+kr+r} \qquad \text{since } kr^2 > 0 \text{ for } k \geqslant 1,\ r > 0$$

$$= \frac{1}{1+(k+1)r}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

11. Prove $\dfrac{4x}{(x+1)^2} \leqslant 1$ for all $x \in R,\ x \neq -1$.

Proof: $\dfrac{4x}{(x+1)^2} \cdot (x+1)^2 \leqslant 1(x+1)^2$

$\Rightarrow \quad 4x \leqslant x^2 + 2x + 1$

$\Rightarrow \quad -x^2 + 2x - 1 \leqslant 0$

$\Rightarrow \quad x^2 - 2x + 1 \leqslant 0$

$\Rightarrow \quad (x-1)^2 \leqslant 0$ true for all $x \in R,\ x \neq -1$

Hence, $\dfrac{4x}{(x+1)^2} \leqslant 1$.

12. $(1 + 2k)x^2 - 10x + (k - 2) = 0$

(i) Real Roots $\Rightarrow b^2 - 4ac \geqslant 0$

$\Rightarrow (-10)^2 - 4(1 + 2k)(k - 2) \geqslant 0$

$\Rightarrow 100 - 4(2k^2 - 3k - 2) \geqslant 0$

$\Rightarrow 100 - 8k^2 + 12k + 8 \geqslant 0$

$\Rightarrow -8k^2 + 12k + 108 \geqslant 0$

$\Rightarrow 8k^2 - 12k - 108 \leqslant 0$

$\Rightarrow 2k^2 - 3k - 27 \leqslant 0$

Factors: $(k + 3)(2k - 9) = 0$

Roots: $k = -3,\ k = 4\frac{1}{2}$

Hence, $-3 \leqslant k \leqslant 4\frac{1}{2}$.

(ii) Sum of roots $> 5 \Rightarrow \dfrac{10}{1+2k} > 5$

$\Rightarrow \dfrac{10}{1+2k}(1 + 2k)^2 > 5(1 + 2k)^2$

$\Rightarrow 10(1 + 2k) > 5(1 + 2k)^2$

$\Rightarrow 2(1 + 2k) > (1 + 2k)^2$

$\Rightarrow 2 + 4k > 1 + 4k + 4k^2$

$\Rightarrow -4k^2 + 1 > 0$

$\Rightarrow 4k^2 - 1 < 0$

Factors: $(2k + 1)(2k - 1) = 0$

Roots: $k = -\frac{1}{2},\ k = \frac{1}{2}$

Hence, $-\frac{1}{2} < k < \frac{1}{2}$.

13. Proof:

(i) True for $n = 1$? $\Rightarrow 1 = (1 - 1)2^1 + 1 \Rightarrow 1 = 0 + 1 \Rightarrow 1 = 1 \Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k \Rightarrow 1 + 2\,.\,2 + 3\,.\,2^2 + \cdots\cdots k\,.\,2^{k-1} = (k - 1)\,.\,2^k + 1$.

(iii) Also true for $n = k + 1$?

$\Rightarrow\ 1 + 2\,.\,2 + 3\,.\,2^2 + \cdots\cdots + k\,.\,2^{k-1} + (k+1)\,.\,2^{(k+1)-1}$

$= (k-1)2^k + 1 + (k+1)2^{(k+1)-1}$

$= (k-1)2^k + 1 + (k+1)2^k$

$= k\,.\,2^k - 2^k + 1 + k\,.\,2^k + 2^k$

$= 2k\,.\,2^k + 1$

$= k\,.\,2^{k+1} + 1 = [(k+1) - 1]\,.\,2^{k+1} + 1$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it is now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \cdots$ etc.

(v) Therefore, it is true for all positive integer values of n.

14. $u_n = (n - 20)\,.\,2^n$

$\Rightarrow\ u_{n+1} = (n + 1 - 20)\,.\,2^{n+1} = (n - 19)\,.\,2\,.\,2^n$

$\Rightarrow\ u_{n+2} = (n + 2 - 20)\,.\,2^{n+2} = (n - 18)\,.\,2^2\,.\,2^n = (n - 18)\,.\,4\,.\,2^n$

Hence, $u_{n+2} - 4u_{n+1} + 4u_n$

$= (n - 18)\,.\,4\,.\,2^n - 4(n - 19)\,.\,2\,.\,2^n + 4\,.\,(n - 20)2^n$

$= 2^n[4n - 72 - 8n + 152 + 4n - 80]$

$= 2^n[8n - 8n + 152 - 152] = 0$

15. $2\log y = \log 2 + \log x$ and $2^y = 4^x$ for $y > 0$

$\Rightarrow\ \log y^2 = \log 2x \qquad \Rightarrow\ 2^y = (2^2)^x$

$\Rightarrow\ y^2 = 2x \qquad \Rightarrow\ 2^y = 2^{2x}$

$\Rightarrow\ y^2 = y \qquad \Rightarrow\ y = 2x$

$\Rightarrow\ y^2 - y = 0$

$\Rightarrow\ y(y - 1) = 0$

$\Rightarrow y = 0$ (Not Valid) or $y = 1 \Rightarrow 2x = 1 \Rightarrow x = \dfrac{1}{2}$

16. (i) An exponential function : $P = 40\,000(1.03)^n$

(ii) $n = 12 \Rightarrow P = 40\,000(1.03)^{12}$

$= 40\,000(1.42576)$

$= 57\,030.43 = 57\,030$

(iii) $n = 0 \Rightarrow P = 40\,000(1.03)^0$

$= 40\,000(1) = 40\,000$

(iv) $P = 80\,000 \Rightarrow 40\,000(1.03)^n = 80\,000$

$\Rightarrow\ (1.03)^n = \dfrac{80\,000}{40\,000} = 2$

$\Rightarrow\ \log(1.03)^n = \log 2$

$\Rightarrow\ n\log(1.03) = \log 2$

$\Rightarrow\ n = \dfrac{\log 2}{\log 1.03} = 23.449 = 23.5$ years

17. (i) $P = Ae^{kt}$, where $A = 8000$, $P = 15\,000$, $t = 8$; find k.

$\Rightarrow\ 15\,000 = 8000e^{k(8)}$

$\Rightarrow\ e^{8k} = \dfrac{15\,000}{8000} = 1.875$

$\Rightarrow\ \ln e^{8k} = \ln 1.875$

$\Rightarrow\ 8k(\ln e) = 8k(1) = 8k = 0.62860865$

$\Rightarrow k = \dfrac{0.62860865}{8} = 0.078576$

Hence, $P = Ae^{kt}$, where $k = 0.078576$, t = number of years and $A = 8000$.

(ii) $t = 10 \Rightarrow P = 8000\, e^{0.078576(10)}$
$= 8000\, e^{0.78576} = 8000(2.1940738)$
$= 17\,552.59 = 17\,553$

(iii) $P = 30\,000 \Rightarrow 15\,000\, e^{0.078576t} = 30\,000$
$\Rightarrow \quad e^{0.078576t} = \dfrac{30\,000}{15\,000} = 2$
$\Rightarrow \quad \ln e^{0.078576t} = \ln 2$
$\Rightarrow \quad 0.078576t \ln e = 0.69314718$
$\Rightarrow \quad 0.078576t(1) = 0.69314718$
$\Rightarrow \quad t = \dfrac{0.69314718}{0.078576} = 8.82 = 9$ years
Ans $= 2016$

Revision Exercise 12 (Extended-Response Questions)

1. $N = 5000e^{-0.15t}$

(i) $t = 0 \quad \Rightarrow N = 5000e^{-0.15(0)} = 5000e^0 = 5000(1) = 5000$
$t = 5 \quad \Rightarrow N = 5000e^{-0.15(5)} = 5000e^{-0.75}$
$= 5000(0.47236655)$
$= 2361.83 = 2362$

$\Rightarrow$ Claim is justified.

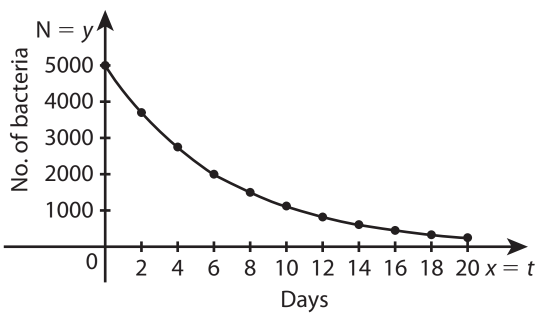
(ii) $t = 10 \Rightarrow N = 5000e^{-0.15(10)}$
$= 5000e^{-1.5}$
$= 5000(0.23313)$
$= 1115.65$

(iii) $t = 0 \quad \Rightarrow N = 5000e^{-0.15(0)} = 5000 \,.\, e^0 = 5000 \,.\, 1 = 5000$

(iv) $N = 100 \Rightarrow 5000e^{-0.15t} = 100$
$\Rightarrow \quad e^{-0.15t} = \dfrac{100}{5000} = 0.02$
$\Rightarrow \quad \ln e^{-0.15t} = \ln(0.02)$
$\Rightarrow \quad -0.15t(\ln e) = \ln(0.02)$
$\Rightarrow \quad -0.15t(1) = -0.15t = -3.912$
$\Rightarrow \quad t = \dfrac{-3.912}{-0.15} = 26.08 = 26.1$ days

(v)

$t =$	0	2	4	6	8	10	12	14	16	18	20
$N = 5000^{-0.15t}$	5000	3704	2744	2033	1506	1116	826	612	454	336	249



2. (i) $A = 0.02(0.92)^{\frac{x}{10}}$

(ii) Length $= \dfrac{5}{3} = 1.66667$
$\Rightarrow \quad A = 0.02(0.92)^{\frac{1.66667}{10}} = 0.02(0.92)^{0.166667}$
$= 0.01972 = 0.0197$

(iii) $S = (0.92)^{10-3x}$

$W = S \times A$

$= 0.02(0.92)^{\frac{x}{10}} . (0.92)^{10-3x}$

$= 0.02(0.92)^{0.1x+10-3x}$

$= 0.02(0.92)^{10-2.9x}$

(iv) $W < (0.02)(0.92)^{2.5}$

$\Rightarrow 0.02(0.92)^{10-2.9x} < (0.02)(0.92)^{2.5}$

$\Rightarrow \quad 10 - 2.9x < 2.5$

$\Rightarrow \quad -2.9x < 2.5 - 10 = -7.5$

$\Rightarrow \quad 2.9x > 7.5$

$$\Rightarrow \quad x > \frac{7.5}{2.9} = 2.586$$

$\Rightarrow \quad x > 2.59$

3. (i) $A = (0.83)^n I$; $\quad B = (0.66)(0.89)^n I$

(ii) $(0.83)^n I = (0.66)(0.89)^n I$

$\Rightarrow \quad \log(0.83)^n = \log(0.66)(0.89)^n$

$\Rightarrow \quad n\log(0.83) = \log(0.66) + \log(0.89)^n$

$\Rightarrow \quad n\log(0.83) = \log(0.66) + n\log(0.89)$

$\Rightarrow \quad n[\log(0.83) - \log(0.89)] = \log(0.66)$

$\Rightarrow \quad n[-0.0809219 + 0.05061] = -0.180456$

$\Rightarrow \quad n(-0.0303119) = -0.180456$

$$\Rightarrow \quad n = \frac{-0.180456}{-0.0303119} = 5.9533$$

$\Rightarrow \quad n = 6$ stations

4. (i) $P_g = A\left(1 + \frac{11}{100}\right)^t = A(1.11)^t$

(ii) $P_r = 10A\left(1 - \frac{5}{100}\right)^t = 10A(0.95)^t$

(iii) $A(1.11)^t = 10A(0.95)^t$

$$\Rightarrow \quad \frac{(1.11)^t}{(0.95)^t} = 10$$

$$\Rightarrow \quad \left(\frac{1.11}{0.95}\right)^t = 10$$

$\Rightarrow \quad (1.168421)^t = 10$

$\Rightarrow \quad \log(1.168421)^t = \log 10$

$\Rightarrow \quad t\log(1.168421) = 1$

$$\Rightarrow \quad t = \frac{1}{\log(1.168421)} = 14.793 = 14.8 \text{ years}$$

(iv) $A(1.11)^t = 100A(0.95)^t$ .... [Note: $P_g = 10P_r$ when proportions are reversed.]

$$\Rightarrow \quad \left(\frac{1.11}{0.95}\right)^t = 100$$

$\Rightarrow \quad (1.168421)^t = 100$

$\Rightarrow \quad \log(1.168421)^t = \log 100$

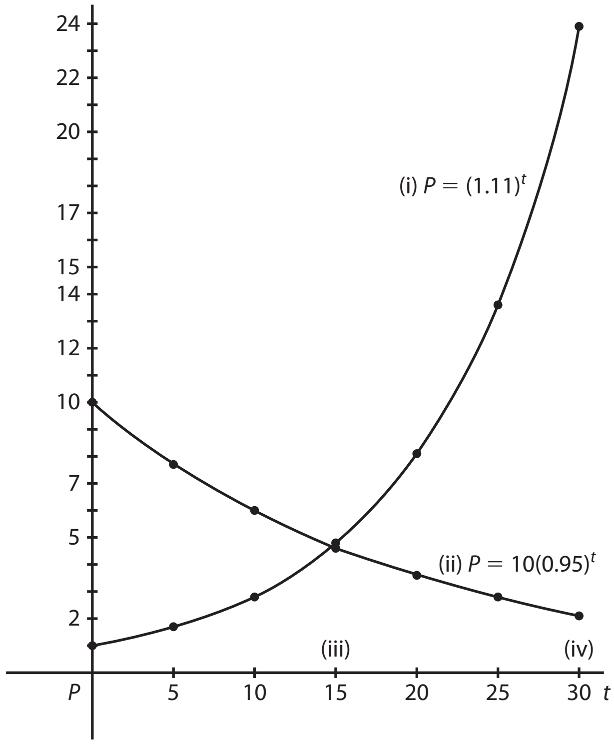
$\Rightarrow \quad t\log(1.168421) = 2$

$$\Rightarrow \quad t = \frac{2}{\log(1.168421)} = 29.586$$

$\Rightarrow \quad t = 29.6$ years

(v)

$t =$	0	5	10	15	20	25	30
$P = 10(0.95)^t$	10	7.7	6.0	4.6	3.6	2.8	2.1
$P = (1.11)^t$	1	1.7	2.8	4.8	8.1	13.6	23.9



5. $n = A(1 - e^{-bt})$

(i) Growth; as t increases, n also increases.

(ii) $t = 2$ when $n = 10\,000 \Rightarrow 10\,000 = A(1 - e^{-2b})$

$$\Rightarrow \frac{10{,}000}{A} = 1 - e^{-2b}$$

$$\Rightarrow \quad e^{-2b} = 1 - \frac{10\,000}{A}$$

$t = 4$ when $n = 15\,000 \Rightarrow 15\,000 = A(1 - e^{-4b})$

$$\Rightarrow \frac{15\,000}{A} = 1 - e^{-4b}$$

$$\Rightarrow \quad e^{-4b} = 1 - \frac{15\,000}{A}$$

Hence, $2e^{-4b} - 3e^{-2b} + 1$

$$= 2\left(1 - \frac{15\,000}{A}\right) - 3\left(1 - \frac{10\,000}{A}\right) + 1$$

$$= 2 - \frac{30\,000}{A} - 3 + \frac{30\,000}{A} + 1 = 0$$

(iii) $a = e^{-2b} \Rightarrow e^{-4b} = (e^{-2b})^2 = a^2$

Hence, $\quad 2e^{-4b} - 3e^{-2b} + 1 = 0$

becomes $\quad 2a^2 - 3a + 1 = 0$

(iv) Factors: $\quad (a - 1)(2a - 1) = 0$

Roots: $\quad a = 1, a = \frac{1}{2}$

(v) $e^{-2b} = 1$ or $e^{-2b} = \frac{1}{2}$

$\Rightarrow e^{-2b} = e^0 \qquad \Rightarrow \qquad \ln e^{-2b} = \ln \frac{1}{2}$

$\Rightarrow -2b = 0 \qquad \Rightarrow \qquad -2b(\ln e) = \ln 1 - \ln 2$

$\Rightarrow \quad b = 0$ (Not valid) $\qquad \Rightarrow \qquad -2b = -\ln 2$

$\Rightarrow \qquad b = \frac{1}{2} \ln 2$

(vi) $n = A(1 - e^{-t\cdot\frac{1}{2}\ln 2})$

$t = 2$ when $n = 10\,000 \Rightarrow 10\,000 = A(1 - e^{-2\cdot\frac{1}{2}\ln 2})$

$\Rightarrow 10\,000 = A(1 - e^{-\ln 2})$

$\Rightarrow 10\,000 = A(1 - 0.5)$

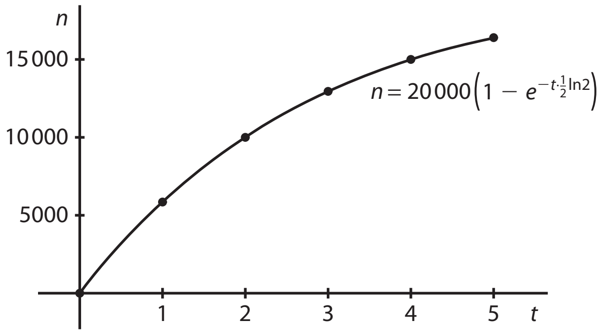
$\Rightarrow 10\,000 = A(0.5)$

$\Rightarrow 20\,000 = A(1)$

$\Rightarrow A = 20\,000$

(vii) $n = 20\,000(1 - e^{-t\cdot\frac{1}{2}\ln 2})$

$t =$	0	1	2	3	4	5
$n =$	0	5858	10 000	12 929	15 000	16 464



(viii) $n = 18\,000 \Rightarrow 20\,000(1 - e^{-t\cdot\frac{1}{2}\ln 2}) = 18\,000$

$\Rightarrow (1 - e^{-t\cdot\frac{1}{2}\ln 2}) = \dfrac{18\,000}{20\,000} = 0.9$

$\Rightarrow -e^{-t\cdot\frac{1}{2}\ln 2} = 0.9 - 1 = -0.1$

$\Rightarrow e^{-t\cdot\frac{1}{2}\ln 2} = 0.1$

$\Rightarrow \ln e^{-t\cdot\frac{1}{2}\ln 2} = \ln 0.1$

$\Rightarrow -t\cdot\dfrac{1}{2}\ln 2(\ln e) = -2.3$

$\Rightarrow t = \dfrac{2.3}{\frac{1}{2}\ln 2}$

$\Rightarrow t = \dfrac{2.3}{0.34657} = 6.636 = 6.64$ hours