Chapter 1

Exercise 1.1

1. (i) $\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$

(ii) $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$

(iii) $\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$

(iv) $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

2. (i) $\sqrt{18} + \sqrt{50} = \sqrt{9 \times 2} + \sqrt{25 \times 2}$

$= 3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

(ii) $\sqrt{48} + \sqrt{147} = \sqrt{16 \times 3} + \sqrt{49 \times 3}$

$= 4\sqrt{3} + 7\sqrt{3} = 11\sqrt{3}$

3. (i) $\{-8, -4\} \in Z \backslash N$

(ii) $\left\{\frac{2}{3}, -\frac{4}{7}\right\} \in Q \backslash Z$

(iii) $\{\sqrt{2}, \pi\} \in R \backslash Q$

4. (i) Z is the set of integers; whole positive and negative numbers, including zero.

(ii) $Q \backslash Z$ is the set of rational numbers that are not integers (positive and negative fractions).

(iii) $Q \backslash N$ is the set of rational numbers not including natural numbers (the set of all fractions excluding whole positive numbers).

(iv) $R \backslash Z$ is the set of real numbers not including the integers (irrational numbers and positive and negative fractions).

(v) $R \backslash Q$ is the set of irrational numbers.

5. (i) $\sqrt{125} - \sqrt{20} = \sqrt{25 \times 5} - \sqrt{4 \times 5}$

$= 5\sqrt{5} - 2\sqrt{5} = 3\sqrt{5}$

(ii) $\sqrt{32} - \sqrt{18} - \sqrt{8} = \sqrt{16 \times 2} - \sqrt{9 \times 2} - \sqrt{4 \times 2}$

$= 4\sqrt{2} - 3\sqrt{2} - 2\sqrt{2} = -\sqrt{2}$

(iii) $3\sqrt{8} + 5\sqrt{2} = 3\sqrt{4 \times 2} + 5\sqrt{2}$

$= 6\sqrt{2} + 5\sqrt{2} = 11\sqrt{2}$

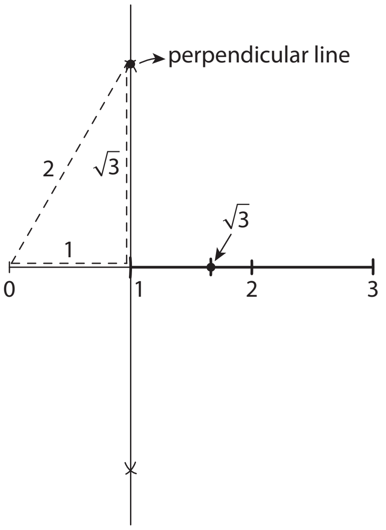
(iv) $4\sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - \sqrt{288}$

$= 4\sqrt{9 \times 2} - 2\sqrt{9 \times 3} + 3\sqrt{3} - \sqrt{144 \times 2}$

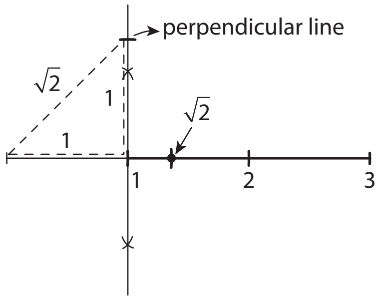
$= \cancel{12\sqrt{2}} - 6\sqrt{3} + 3\sqrt{3} - \cancel{12\sqrt{2}}$

$= 3\sqrt{3}$

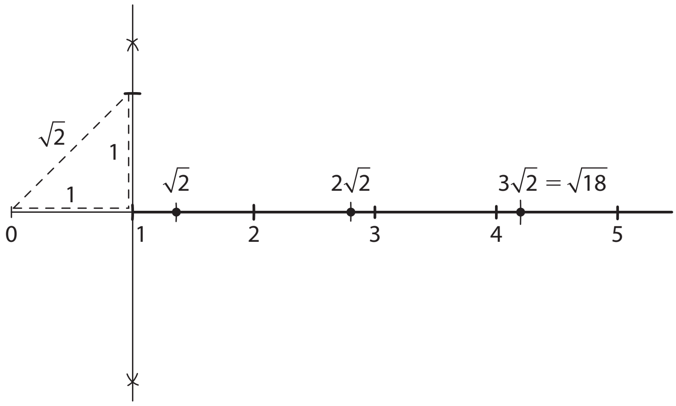
6. (i) $\sqrt{3}$



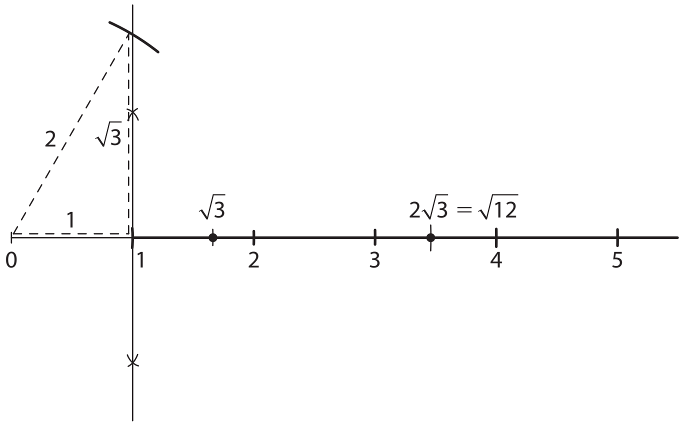
(ii) $\sqrt{2}$



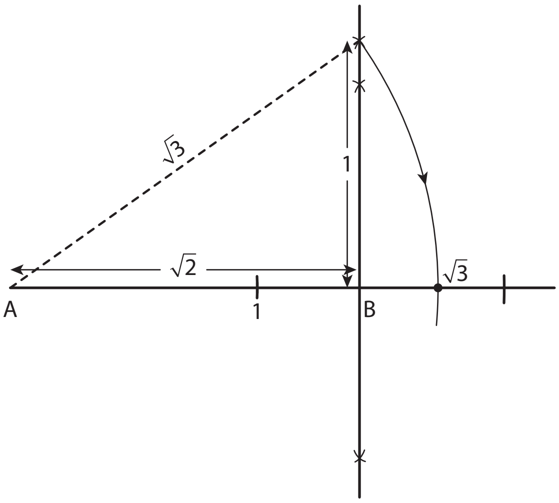
7. $\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$



8. $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$



9.
- Given line segment with 1 and $\sqrt{2}$ marked
- Draw a perpendicular line at B.
- Mark 1 unit along perpendicular.
  ∴ Hypotenuse $= \sqrt{3}$
- Draw an arc, centre A, length $\sqrt{3}$ to line segment.



10. Perimeter $= \sqrt{45} + \sqrt{80} + \sqrt{20}$
$= \sqrt{9 \times 5} + \sqrt{16 \times 5} + \sqrt{4 \times 5}$
$= 3\sqrt{5} + 4\sqrt{5} + 2\sqrt{5}$
$= 9\sqrt{5}$

11. The irrational numbers are $\sqrt{3}$, π, e, $\sqrt[5]{2}$

12. $a^2 = 1^2 + 1^2 = 2$
$\Rightarrow a = \sqrt{2}$

$b^2 = 1^2 + (\sqrt{2})^2 = 3$
$\Rightarrow b = \sqrt{3}$

$c^2 = 1^2 + (\sqrt{3})^2 = 4$

$\Rightarrow c = \sqrt{4} = 2$

$d^2 = 1^2 + 2^2 = 5$

$\Rightarrow d = \sqrt{5}$

$e^2 = 1^2 + \left(\sqrt{5}\right)^2 = 6$

$\Rightarrow e = \sqrt{6}$

$f^2 = 1^2 + (\sqrt{6})^2 = 7$

$\Rightarrow f = \sqrt{7}$

$g^2 = 1^2 + (\sqrt{7})^2 = 8$

$\Rightarrow g = \sqrt{8} = 2\sqrt{2}$

The length of "*c*" is a rational number.

13. Since the length of the carpet equals the sum of the height and width of the stairs, we need to find the opposite and adjacent sides of the triangle.

$\tan 45° = \dfrac{h}{2\sqrt{2}}$

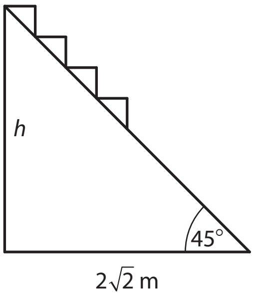
$\Rightarrow h = 2\sqrt{2} \tan 45°$

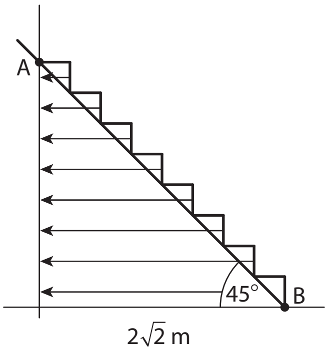
$= 2\sqrt{2}(1)$

$= 2\sqrt{2}$ m

$\therefore$ The length of carpet needed $= \left(2\sqrt{2} + 2\sqrt{2}\right)$ m

$= 4\sqrt{2}$ m





Angle of stairs increased to 60°

$\tan 60° = \dfrac{h}{2\sqrt{2}}$

$\Rightarrow h = 2\sqrt{2} \tan 60° \quad [\tan 60° = \sqrt{3}]$

$= 2\sqrt{2} . \sqrt{3}$

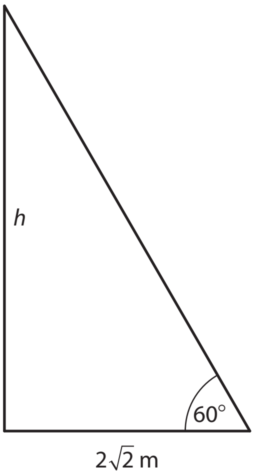
$= 2\sqrt{6}$

$\therefore$ The length of carpet needed $= \left(2\sqrt{6} + 2\sqrt{2}\right)$ m

$\therefore$ The <u>extra</u> length of carpet needed $= 2\sqrt{6} + 2\sqrt{2} - 4\sqrt{2}$

$= 2\sqrt{6} - 2\sqrt{2}$

$= 2\left(\sqrt{6} - \sqrt{2}\right)$



14. (i) $\sqrt{3 - x}$ is rational if $x = -1$

(ii) Any value of x for which $3 - x$ is a perfect square.

Exercise 1.2

1. (i) $\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = 2i$

(ii) $\sqrt{-36} = \sqrt{36} \times \sqrt{-1} = 6i$

(iii) $\sqrt{-27} = \sqrt{27} \times \sqrt{-1} = \sqrt{9 \times 3} \times i = 3\sqrt{3}i$

(iv) $\sqrt{-20} = \sqrt{20} \times \sqrt{-1} = \sqrt{4 \times 5} \times i = 2\sqrt{5}i$

2. (i) $x^2 + 9 = 0$

$x^2 = -9$

$x = \sqrt{-9} = \sqrt{9} \times \sqrt{-1} = \pm 3i$

(ii) $x^2 + 12 = 0$

$x^2 = -12$

$x = \sqrt{-12} = \sqrt{12} \times \sqrt{-1} = \sqrt{4 \times 3} \times i$

$= \pm 2\sqrt{3}i$

3. (i) $(3 + 2i) + (5 - i) = 3 + 5 + 2i - i$
$= 8 + i$

(ii) $(7 - 2i) + (3 - 4i) = 7 + 3 - 2i - 4i$
$= 10 - 6i$

(iii) $(-3 + 4i) + (6 - 4i) = -3 + 6 + 4i - 4i$
$= 3 + 0i$

(iv) $(-3 - i) + (-2 + 6i) = -3 - 2 - i + 6i$
$= -5 + 5i$

(v) $(5 - 3i) + (-5 + 6i) = 5 - 5 - 3i + 6i$
$= 0 + 3i$

(vi) $(1 + i) + (2 - 3i) = 1 + 2 + i - 3i$
$= 3 - 2i$

4. (i) $(2 + 6i) - (1 + 4i) = 2 - 1 + 6i - 4i$
$= 1 + 2i$

(ii) $(3 - 5i) - (2 + 4i) = 3 - 2 - 5i - 4i$
$= 1 - 9i$

(iii) $(4 - 7i) - (-1 + 3i) = 4 + 1 - 7i - 3i$
$= 5 - 10i$

(iv) $3 - (1 + 4i) = 3 - 1 - 4i$
$= 2 - 4i$

(v) $(3 - 6i) - 4i = 3 - 6i - 4i$
$= 3 - 10i$

(vi) $(-3 - 2i) - (4 - 7i) = -3 - 2i - 4 + 7i$
$= -7 + 5i$

5. (i) $(3 + 2i)(2 + 3i) = 6 + 9i + 4i + 6i^2$
$= 6 + 9i + 4i - 6$
$= 0 + 13i$

(ii) $(4 + i)(3 - 5i) = 12 - 20i + 3i - 5i^2$
$= 12 - 20i + 3i + 5$
$= 17 - 17i$

(iii) $(5 - 2i)(3 - 5i) = 15 - 25i - 6i + 10i^2$
$= 15 - 25i - 6i - 10$
$= 5 - 31i$

(iv) $(3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2$
$= 9 - 12i + 12i + 16$
$= 25 + 0i$

(v) $(5 - i)(5 + i) = 25 + 5i - 5i - i^2$
$= 25 + 5i - 5i + 1$
$= 26 + 0i$

(vi) $(3 - 2i)^2 = 9 + 2(3)(-2i) + (-2i)^2$
$= 9 - 12i + 4i^2$
$= 9 - 12i - 4$
$= 5 - 12i$

6. $z_1 = 2 + 4i, \quad z_2 = 3 - i$ and $z_3 = 4 - 2i$

(i) $3z_1 = 3(2 + 4i) = 6 + 12i$

(ii) $z_2 + z_3 = (3 - i) + (4 - 2i)$
$= 7 - 3i$

(iii) $2z_1 + z_2 = 2(2 + 4i) + (3 - i)$
$= 4 + 8i + 3 - i$
$= 7 + 7i$

(iv) $-3z_2 = -3(3 - i)$

$= -9 + 3i$

(v) $z_1 . z_2 = (2 + 4i) . (3 - i)$

$= 6 - 2i + 12i - 4i^2$

$= 6 + 10i + 4$

$= 10 + 10i$

(vi) $z_2 . z_3 = (3 - i)(4 - 2i)$

$= 12 - 6i - 4i + 2i^2$

$= 12 - 10i - 2$

$= 10 - 10i$

(vii) $i(z_3) = i(4 - 2i)$

$= 4i - 2i^2$

$= 4i + 2$

$= 2 + 4i$

(viii) $z_2(z_1 - z_2) = (3 - i)[(2 + 4i) - (3 - i)]$

$= (3 - i)[-1 + 5i]$

$= -3 + 15i + i - 5i^2$

$= -3 + 16i + 5$

$= 2 + 16i$

7. (i) $x^2 - 2x + 17 = 0.$

$\Rightarrow$ $a = 1, b = -2, c = 17$

$$\therefore \quad x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(17)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-64}}{2}$$

$$= \frac{2 \pm \sqrt{64 \times (-1)}}{2} = \frac{2 \pm 8i}{2} = 1 \pm 4i$$

(ii) $x^2 - 4x + 13 = 0.$

$\Rightarrow$ $a = 1, b = -4, c = 13$

$$\therefore \quad x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36 \times (-1)}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

(iii) $x^2 - 10x + 26 = 0.$

$\Rightarrow$ $a = 1, b = -10, c = 26$

$$\therefore \quad x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(26)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{-4}}{2}$$

$$= \frac{10 \pm \sqrt{4 \times (-1)}}{2} = \frac{10 \pm 2i}{2} = 5 \pm i$$

(iv) $x^2 - 8x + 52 = 0.$

$\Rightarrow$ $a = 1, b = -8, c = 52$

$$\therefore \quad x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(52)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{-144}}{2}$$

$$= \frac{8 \pm \sqrt{144 \times (-1)}}{2} = \frac{8 \pm 12i}{2} = 4 \pm 6i$$

8. $2z^2 - 8z + 9 = 0.$

$\Rightarrow \quad a = 2, b = -8, c = 9$

$$\therefore \quad z = \frac{8 \pm \sqrt{(-8)^2 - 4(2)(9)}}{2(2)}$$

$$= \frac{8 \pm \sqrt{-8}}{4}$$

$$= \frac{8 \pm \sqrt{8 \times (-1)}}{4} = \frac{8 \pm 2\sqrt{2}i}{4} = 2 \pm \frac{\sqrt{2}}{2}i$$

9. $i = i$

$i \times i = i^2 = -1$

$i \times i \times i = i^3 = i(-1) = -i$

$i \times i \times i \times i = i^4 = i(-i) = -i^2 = -(-1) = 1$

$i \times i \times i \times i \times i = i^5 = i(1) = i$

$i \times i \times i \times i \times i \times i = i^6 = i(i) = -1$

The pattern consists of four elements ($i, -1, -i, 1$) repeated.
Given i^n, divide n by 4 and find the remainder R (if there is one),

i.e. $\frac{n}{4} = k + \text{remainder} \qquad (k \in N)$

$\therefore \quad n = 4k + \text{remainder}.$

Since $i^4 = i^8 = i^{16} \ldots \text{etc} = 1$

$\therefore \quad i^{4k} = 1$

$\therefore \quad i^{4k+\text{Remainder}} = i^{4k} . i^R = i^R$

$\therefore \quad$ if $R = 0$: $i^n = i^0 = 1$

if $R = 1$: $i^n = i^1 = i$

if $R = 2$: $i^n = i^2 = -1$

if $R = 3$: $i^n = i^3 = -i$

Given $i^{32} = i^{4\times 8+0} = i^0 = 1$

$i^{29} = i^{4\times 7+1} = i^1 = i$

10. (i) $i^{30} = i^{4\times 7+2} = i^2 = -1$

(ii) $i^{11} = i^{4\times 2+3} = i^3 = -i$

(iii) $i^{19} = i^{4\times 4+3} = i^3 = -i$

(iv) $i^{21} = i^{5\times 4+1} = i^1 = i$

(v) $i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

11. (i) $i^{16} + i^{10} + i^6 - i^{12}$

$= i^{4\times 4+0} + i^{4\times 2+2} + i^{4\times 1+2} - i^{4\times 3+0}$

$= i^0 + i^2 + i^2 - i^0$

$= \cancel{1} - \cancel{1} - 1 - 1$

$= -2$

(ii) $i^3 - i^{11} + i^{17} - i^{29}$

$= i^3 - i^{4\times 2+3} + i^{4\times 4+1} - i^{4\times 7+1}$

$= \cancel{i^3} - \cancel{i^3} + \cancel{i} - \cancel{i}$

$= 0$

12. (i) $i^2 . i^6 . i^5 = i^{13} = i^{4\times 3+1} = i^1 = i$

(ii) $3i^3 . 2i^5 . 4i^2 = 24i^{10} = 24i^{4\times 2+2} = 24i^2$

$= -24$

(iii) $(2i^7)^3 = 8i^{21} = 8i^{4\times 5+1} = 8i$

13. $4i^3 + 7i^9 = 4(-i) + 7i^{4\times 2+1}$

$= -4i + 7i$

$= +3i$

Exercise 1.3

1. (i) $z = 3 + 4i \Rightarrow \bar{z} = 3 - 4i$

(ii) $z = 2 - 6i \Rightarrow \bar{z} = 2 + 6i$

(iii) $z = -5 - 2i \Rightarrow \bar{z} = -5 + 2i$

(iv) $z = -8 + 3i \Rightarrow \bar{z} = -8 - 3i$

2. (i) $z = 2 + 5i \Rightarrow \bar{z} = 2 - 5i$

(ii) $z = -3 - 4i \Rightarrow \bar{z} = -3 + 4i$

(iii) $z = 1 + 7i \Rightarrow \bar{z} = 1 - 7i$

(iv) $z = -5 + i \Rightarrow \bar{z} = -5 - i$

3. (i)
$$\begin{aligned}\frac{2+3i}{4-i} &= \frac{2+3i}{4-i} \times \frac{4+i}{4+i}\\ &= \frac{8+2i+12i+3i^2}{16+\cancel{4}i-\cancel{4}i-i^2}\\ &= \frac{5+14i}{17} = \frac{5}{17} + \frac{14}{17}i\end{aligned}$$

(ii)
$$\begin{aligned}\frac{4+3i}{5+i} &= \frac{4+3i}{5+i} \times \frac{5-i}{5-i}\\ &= \frac{20-4i+15i-3i^2}{25-\cancel{5}i+\cancel{5}i-i^2}\\ &= \frac{23+11i}{26} = \frac{23}{26} + \frac{11}{26}i\end{aligned}$$

(iii)
$$\begin{aligned}\frac{8-i}{2+3i} &= \frac{8-i}{2+3i} \cdot \frac{2-3i}{2-3i}\\ &= \frac{16-24i-2i+3i^2}{4-\cancel{6}i+\cancel{6}i-9i^2}\\ &= \frac{13-26i}{13} = 1 - 2i\end{aligned}$$

(iv)
$$\begin{aligned}\frac{2+5i}{-3+2i} &= \frac{2+5i}{-3+2i} \cdot \frac{-3-2i}{-3-2i}\\ &= \frac{-6-4i-15i-10i^2}{9+\cancel{6}i-\cancel{6}i-4i^2}\\ &= \frac{4-19i}{13} = \frac{4}{13} - \frac{19}{13}i\end{aligned}$$

4. $z = 2 + 6i \Rightarrow \bar{z} = 2 - 6i$

(i)
$$\begin{aligned}z.\bar{z} &= (2+6i)(2-6i)\\ &= 4 - \cancel{12i} + \cancel{12i} - 36i^2\\ &= 4 + 36 = 40 + 0i\end{aligned}$$

(ii)
$$\begin{aligned}z + \bar{z} &= (2+6i) + (2-6i)\\ &= 4 + 0i\end{aligned}$$

(iii)
$$\begin{aligned}z - \bar{z} &= (2+6i) - (2-6i)\\ &= 2 + 6i - 2 + 6i\\ &= 0 + 12i\end{aligned}$$

(iv)
$$\begin{aligned}z^2 &= (2+6i)^2\\ &= 4 + 2(2)(6i) + (6i)^2\\ &= 4 + 24i - 36\\ &= -32 + 24i\end{aligned}$$

5. (i)
$$\begin{aligned}\frac{(3+4i)+(2+i)}{4-i} &= \frac{5+5i}{4-i} \cdot \frac{4+i}{4+i}\\ &= \frac{20+5i+20i+5i^2}{16+\cancel{4}i-\cancel{4}i-i^2}\\ &= \frac{15+25i}{17}\\ &= \frac{15}{17} + \frac{25}{17}i\end{aligned}$$

(ii)
$$\begin{aligned}\frac{(2-6i)-(3+2i)}{2+2i} &= \frac{-1-8i}{2+2i} \cdot \frac{2-2i}{2-2i}\\ &= \frac{-2+2i-16i+16i^2}{4-\cancel{4}i+\cancel{4}i-4i^2}\\ &= \frac{-18-14i}{8}\\ &= -\frac{9}{4} - \frac{7}{4}i\end{aligned}$$

(iii)
$$\begin{aligned}\frac{3(2+4i)}{5i} &= \frac{6+12i}{5i} \cdot \frac{(-5i)}{(-5i)}\\ &= \frac{-30i-60i^2}{-25i^2}\\ &= \frac{-30i+60}{25}\\ &= \frac{12}{5} - \frac{6}{5}i\end{aligned}$$

(iv) $$\begin{aligned}\frac{(2+i)+(3-2i)}{(4+i)-(3+2i)} &= \frac{5-i}{1-i}\\ &= \frac{5-i}{1-i}\cdot\frac{1+i}{1+i}\\ &= \frac{5+5i-i-i^2}{1+\cancel{i}-\cancel{i}-i^2}\\ &= \frac{6+4i}{2} = 3+2i\end{aligned}$$

(v) $$\begin{aligned}\frac{(3+2i)(1-i)}{2+4i} &= \frac{3-3i+2i-2i^2}{2+4i}\\ &= \frac{5-i}{2+4i}\cdot\frac{2-4i}{2-4i}\\ &= \frac{10-20i-2i+4i^2}{4-\cancel{8i}+\cancel{8i}-16i^2}\\ &= \frac{6-22i}{20}\\ &= \frac{3}{10}-\frac{11}{10}i\end{aligned}$$

(vi) $$\begin{aligned}\frac{(3+i)(2-i)}{(4+i)(2+i)} &= \frac{6-3i+2i-i^2}{8+4i+2i+i^2}\\ &= \frac{7-i}{7+6i}\cdot\frac{7-6i}{7-6i}\\ &= \frac{49-42i-7i+6i^2}{49-\cancel{42i}+\cancel{42i}-36i^2}\\ &= \frac{43-49i}{85}\\ &= \frac{43}{85}-\frac{49}{85}i\end{aligned}$$

6. (i) $x+yi = 4-2i$

$\Rightarrow x = 4$ and $y = -2$

(ii) $$\begin{aligned}x+yi &= (2+i)(3-2i)\\ &= 6-4i+3i-2i^2\\ &= 8-i\end{aligned}$$

$\Rightarrow x = 8$ and $y = -1$

(iii) $$\begin{aligned}x+yi &= \frac{7+i}{2-i}\\ &= \frac{7+i}{2-i}\cdot\frac{2+i}{2+i}\\ &= \frac{14+7i+2i+i^2}{4+\cancel{2i}-\cancel{2i}-i^2}\\ &= \frac{13+9i}{5} = \frac{13}{5}+\frac{9i}{5}\end{aligned}$$

$\Rightarrow$ $x = \frac{13}{5}$ and $y = \frac{9}{5}$

(iv) $$\begin{aligned}x+yi &= (2-3i)^2\\ &= 2^2+2(2)(-3i)+(-3i)^2\\ &= 4-12i+9i^2\\ &= -5-12i\end{aligned}$$

$\Rightarrow$ $x = -5$ and $y = -12$

7. (i) $a+bi+3-2i = 4(-2+5i)$

$\Rightarrow$ $a+bi+3-2i = -8+20i$

$$\begin{aligned}a+bi &= -8+20i-3+2i\\ &= -11+22i\end{aligned}$$

$\Rightarrow$ $a = -11$ and $b = 22$

(ii) $a(1 + 2i) - b(3 + 4i) = 5$

$a + 2ai - 3b - 4bi = 5$

$a - 3b + (2a - 4b)i = 5 + 0i$

$\therefore \quad a - 3b = 5$

and $2a - 4b = 0$

$\therefore \quad \underline{2a - 6b = 10}$ (multiplying 1st line by 2)

$2b = -10$ (subtracting)

$b = -5$

$\therefore \; a - 3(-5) = 0$

$a \qquad = 5 - 15$

$= -10$

$\therefore a = -10, \; b = -5$

8. $z = x + yi$ and $3(z - 1) = i(3 + i)$

$\Rightarrow 3(x + yi - 1) = 3i + i^2$

$\Rightarrow 3x - 3 + 3yi = 3i - 1$

$\therefore 3x - 3 = -1$ and $3y = 3$

$3x \quad = +2 \qquad y = 1$

$$x = \frac{2}{3}$$

9. $z_1 = -3 + 4i, \; z_2 = 1 + 2i$

$z_1 + (p + iq)z_2 = 0$

$\Rightarrow \; (p + iq)z_2 = -z_1$

$$\Rightarrow \quad p + iq = \frac{-z_1}{z_2}$$

$$= \frac{-(-3 + 4i)}{1 + 2i}$$

$$= \frac{3 - 4i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i}$$

$$= \frac{3 - 6i - 4i + 8i^2}{1 - \cancel{2i} + \cancel{2i} - 4i^2}$$

$$= \frac{-5 - 10i}{5}$$

$$= -1 - 2i$$

$\Rightarrow \; p = -1$ and $q = -2$

10. $z = \sqrt{3 + 4i}$

$\Rightarrow \quad a + bi = \sqrt{3 + 4i}$

$\Rightarrow \quad (a + bi)^2 = 3 + 4i$

$\Rightarrow \; a^2 + 2abi + (bi)^2 = 3 + 4i$

$\Rightarrow \quad a^2 - b^2 + 2abi = 3 + 4i$

$\therefore \; a^2 - b^2 = 3$ and $2ab = 4$

$ab = 2$

$$b = \frac{2}{a}$$

$$\therefore \quad a^2 - \left(\frac{2}{a}\right)^2 = 3$$

$$a^2 - \frac{4}{a^2} = 3$$

$$a^4 - 4 = 3a^2$$

$$\therefore \quad a^4 - 3a^2 - 4 = 0$$

$\Rightarrow\ (a^2)^2 - 3(a^2) - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

$\therefore\ a^2 = 4 \quad \text{or} \quad a^2 = -1$

$\Rightarrow\ a = \pm 2 \qquad a = \sqrt{-1} = i$

but $a \in R \Rightarrow a \neq i$

Since $b = \dfrac{2}{a} \Rightarrow$ if $a = +2,\ b = \dfrac{2}{2} = 1$

if $a = -2,\ b = \dfrac{2}{-2} = -1$

$\therefore\ a + bi = (2 + i) \quad \text{or} \quad (-2 - i)$

11. $(x + iy)^2 = 8 - 6i$

$\Rightarrow\ x^2 + 2xyi + (iy)^2 = 8 - 6i$

$x^2 - y^2 + 2xyi = 8 - 6i$

$\therefore\ x^2 - y^2 = 8 \quad \text{and} \quad 2xy = -6$

$xy = -3$

$y = \dfrac{-3}{x}$

$\therefore\ x^2 - \left(\dfrac{-3}{x}\right)^2 = 8$

$x^2 - \dfrac{9}{x^2} = 8$

$x^4 - 9 = 8x^2$

$(x^2)^2 - 8x^2 - 9 = 0$

$\therefore\ (x^2 - 9)(x^2 + 1) = 0$

$\therefore\ x^2 = 9 \quad \text{or} \quad x^2 = -1$

$x = \sqrt{9} = \pm 3 \qquad x = \sqrt{-1} = i$

but $x \in R \Rightarrow x \neq i$

$\therefore$ if $x = +3,\ y = \dfrac{-3}{x} = \dfrac{-3}{3} = -1$

if $x = -3,\ y = \dfrac{-3}{x} = \dfrac{-3}{-3} = 1$

$\therefore\ (x, y) = (3, -1) \quad \text{or} \quad (-3, 1)$

12. (i) $\sqrt{-12 - 16i} = a + bi$

$-12 - 16i = (a + bi)^2 = a^2 + 2abi + (bi)^2$

$-12 - 16i = a^2 - b^2 + 2abi$

$\therefore\ 2ab = -16 \quad \text{and} \quad a^2 - b^2 = -12$

$ab = -8 \qquad a^2 - \left(\dfrac{-8}{a}\right)^2 = -12$

$b = \dfrac{-8}{a} \qquad a^2 - \dfrac{64}{a^2} = -12$

$(a^2)^2 + 12a^2 - 64 = 0$

$(a^2 + 16)(a^2 - 4) = 0$

$\therefore\ a^2 = -16 \quad \text{or} \quad a^2 = 4$

$a = \sqrt{-16} \quad \text{or} \quad a = \pm 2$

$= 4i$

Since $a \in R, a \neq 4i$

$\therefore$ if $a = +2,\ b = \dfrac{-8}{2} = -4$

if $a = -2,\ b = \dfrac{-8}{-2} = 4$

$\therefore\ \sqrt{-12 - 16i} = 2 - 4i \ \text{or}\ -2 + 4i$

(ii) $\sqrt{-15 + 8i} = a + bi$

$$\begin{aligned} -15 + 8i &= (a + bi)^2 \\ &= a^2 + 2abi + (bi)^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

$\therefore \quad 2ab = 8 \quad$ and $\quad a^2 - b^2 = -15$

$ab = 4 \qquad \therefore \quad a^2 - \left(\dfrac{4}{a}\right)^2 = -15$

$b = \dfrac{4}{a} \qquad \therefore \quad a^2 - \dfrac{16}{a^2} = -15$

$\therefore \quad (a^2)^2 + 15a^2 - 16 = 0$

$\therefore \quad (a^2 + 16)(a^2 - 1) = 0$

$\therefore \quad a^2 = -16 \quad$ or $\quad a^2 = 1$

$a = \sqrt{-16} \quad$ or $\quad a = \pm 1$

$a = 4i$

Since $a \in R$, $a \neq 4i$

$\therefore \quad$ if $\quad a = +1, b = \dfrac{4}{1} = 4$

if $\quad a = -1, b = \dfrac{4}{-1} = -4$

$\therefore \quad \sqrt{-15 + 8i} = (1 + 4i) \quad$ or $\quad (-1 - 4i)$

(iii) $\sqrt{9 - 40i} = a + bi$

$$\begin{aligned} 9 - 40i &= (a + bi)^2 \\ &= a^2 + 2abi + (bi)^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

$\therefore \; 2ab = -40 \qquad$ and $\qquad a^2 - b^2 = 9$

$ab = -20 \qquad \therefore a^2 - \left(\dfrac{-20}{a}\right)^2 = 9$

$b = \dfrac{-20}{a} \qquad a^2 - \dfrac{400}{a^2} = 9$

$$\begin{aligned} (a^2)^2 - 400 &= 9a^2 \\ (a^2)^2 - 9a^2 - 400 &= 0 \\ (a^2 - 25)(a^2 + 16) &= 0 \end{aligned}$$

$\therefore \quad a^2 = 25 \quad$ or $\quad a^2 = -16$

$a = \pm 5 \quad$ or $\quad a = \sqrt{-16} = 4i$

Since $a \in R$, $a \neq 4i$

$\therefore$ if $\quad a = +5, \; b = \dfrac{-20}{5} = -4$

if $\quad a = -5, \; b = \dfrac{-20}{-5} = 4$

$\therefore \quad \sqrt{9 - 40i} = (5 - 4i) \quad$ or $\quad (-5 + 4i)$

13. $z_1 = 2 + 3i, \; z_2 = -1 - 5i$

(i) $$\begin{aligned} \overline{z_1 + z_2} &= \overline{(2 + 3i) + (-1 - 5i)} \\ &= \overline{1 - 2i} \\ &= 1 + 2i \end{aligned}$$

(ii) $$\begin{aligned} \overline{z_1 z_2} &= \overline{(2 + 3i)(-1 - 5i)} \\ &= \overline{-2 - 10i - 3i - 15i^2} \\ &= \overline{13 - 13i} \\ &= 13 + 13i \end{aligned}$$

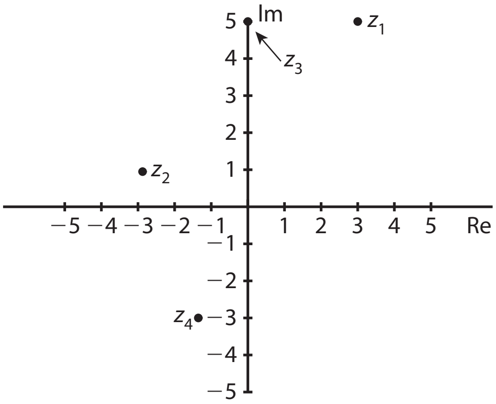
Exercise 1.4

1. (i) $z_1 = 3 + 5i$

(ii) $z_2 = -3 + i$

(iii) $z_3 = 5i$

(iv) $z_4 = -1 - 3i$



2. (i) $z_1 = 2 + i$

(ii) $z_2 = -4 + 3i$

(iii) $\bar{z}_1 = 2 - i$

(iv) $\bar{z}_2 = -4 - 3i$

(v) $z_1 + z_2 = (2 + i) + (-4 + 3i)$

$= -2 + 4i$

(vi) $z_1 - z_2 = (2 + i) - (-4 + 3i)$

$= 6 - 2i$

(vii) $z_1z_2 = (2 + i)(-4 + 3i)$

$= -8 + 6i - 4i + 3i^2$

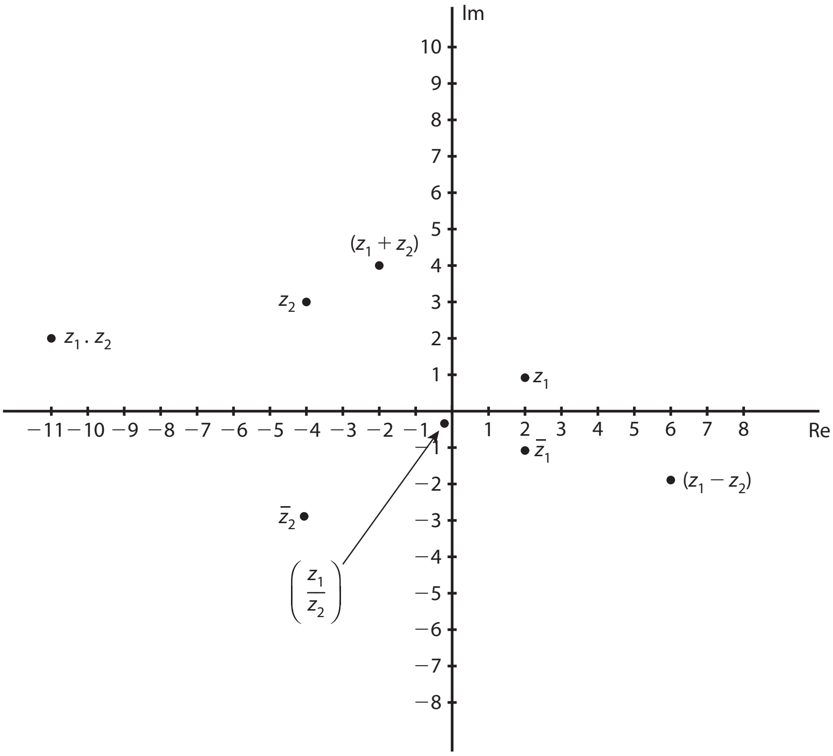
$= -8 + 2i - 3$

$= -11 + 2i$

(viii) $$\frac{z_1}{z_2} = \frac{2+i}{-4+3i} \cdot \frac{-4-3i}{-4-3i}$$

$$= \frac{-8-6i-4i-3i^2}{16+\cancel{12i}-\cancel{12i}-9i^2}$$

$$= \frac{-5-10i}{16+9} = \frac{-5-10i}{25} = -\frac{1}{5} - \frac{2}{5}i$$



3. $z_1 = 3 - i$

$z_2 = 2 + 4i$

(i) $z_1 . \bar{z}_1 = (3 - i)(3 + i)$

$= 9 + 3i - 3i - i^2$

$= 10$

(ii) $z_1 + \bar{z}_1 = (3 - i) + (3 + i)$

$= 6$

(iii) $\frac{1}{z_1} = \frac{1}{3 - i} \cdot \frac{3 + i}{3 + i}$

$= \frac{3 + i}{9 + \cancel{3i} - \cancel{3i} - i^2}$

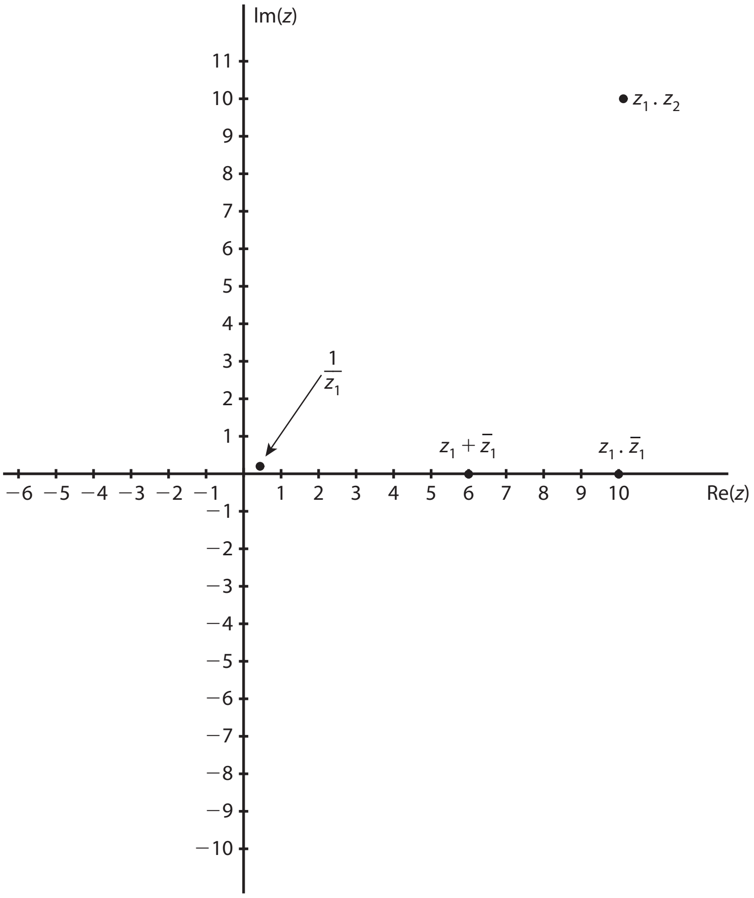
$= \frac{3 + i}{10}$

$= \frac{3}{10} + \frac{1}{10}i$

(iv) $z_1z_2 = (3 - i)(2 + 4i)$

$= 6 + 12i - 2i - 4i^2$

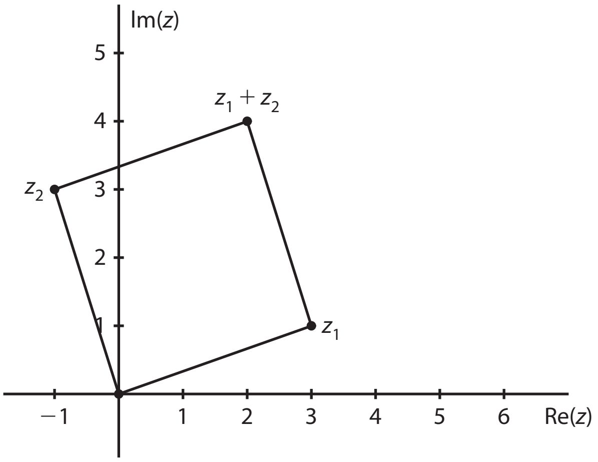
$= 10 + 10i$



4. (a) $z_1 = 3 + i$

$z_2 = -1 + 3i$

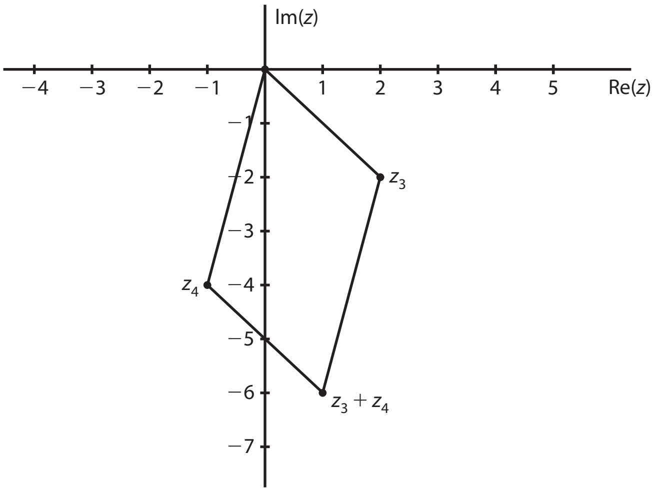
$z_1 + z_2 = 2 + 4i$



(b) $z_3 = 2 - 2i$

$z_4 = -1 - 4i$

$z_3 + z_4 = 1 - 6i$



(c) The points 0, z_1, z_2, $z_1 + z_2$ when joined, form a parallelogram.

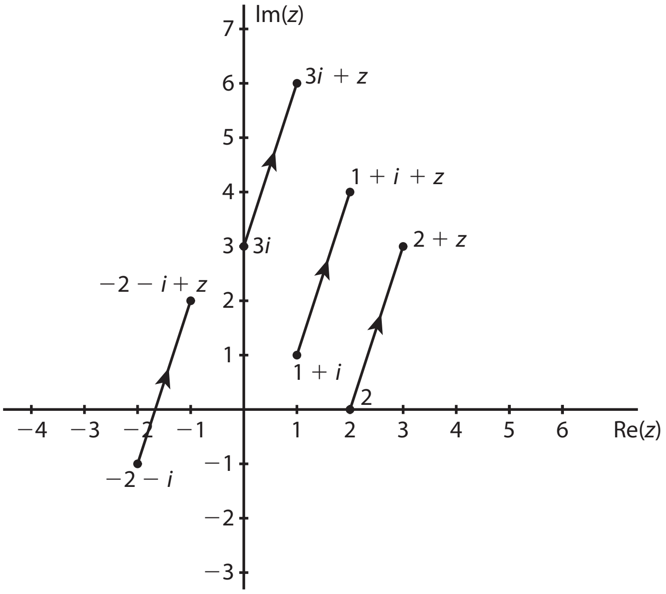
5. $z = 1 + 3i$

(ii) $2 + z = 2 + 1 + 3i$
$= 3 + 3i$

(iv) $3i + z = 3i + 1 + 3i$
$= 1 + 6i$

(vi) $1 + i + z = 1 + i + 1 + 3i$
$= 2 + 4i$

(viii) $-2 - i + z = -2 - i + 1 + 3i$
$= -1 + 2i$



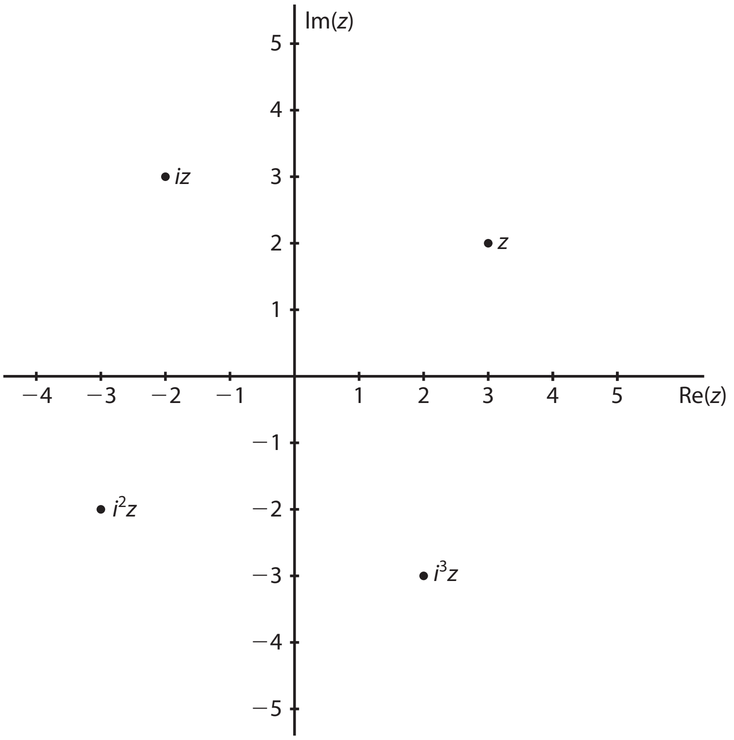
When the same complex number is added to several different complex numbers, they all move the same distance in the same direction, i.e. it causes a translation.

6. $z = 3 + 2i$

(i) $iz = i(3 + 2i)$
$= 3i + 2i^2$
$= -2 + 3i$

(ii) $i^2z = (-1)(3 + 2i)$
$= -3 - 2i$

(iii) $i^3z = (-i)(3 + 2i)$
$= -3i - 2i^2$
$= 2 - 3i$



7. (i) $5 + 2i \Rightarrow |5 + 2i| = \sqrt{5^2 + 2^2}$
$= \sqrt{29}$

(ii) $4 - 2i \Rightarrow |4 - 2i| = \sqrt{4^2 + (-2)^2}$
$= \sqrt{20} = 2\sqrt{5}$

(iii) $-2 - 4i \Rightarrow |-2 - 4i| = \sqrt{(-2)^2 + (-4)^2}$
$= \sqrt{20} = 2\sqrt{5}$

(iv) $-3 + i \Rightarrow |-3 + i| = \sqrt{(-3)^2 + (1)^2}$
$= \sqrt{10}$

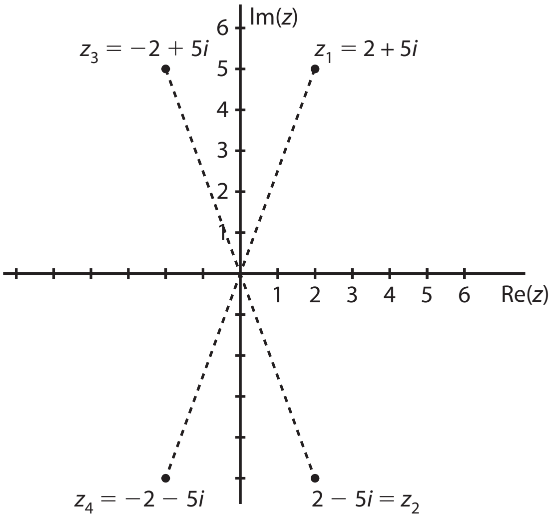
8. $z_1 = 2 + 5i$

$|z_1| = |2 + 5i|$

$|z_2| = |2 - 5i|$

$|z_3| = |-2 + 5i|$

$|z_4| = |-2 - 5i|$



9. (i) $$\left|\frac{3+i}{-2-3i}\right| = \left|\frac{3+i}{-2-3i}\cdot\frac{-2+3i}{-2+3i}\right|$$

$$= \frac{-6+9i+2i+3i^2}{4-\cancel{6i}+\cancel{6i}-9i^2}$$

$$= \left|\frac{-9+7i}{13}\right| = \left|\frac{-9}{13}+\frac{7}{13}i\right|$$

$$= \sqrt{\left(\frac{-9}{13}\right)^2+\left(\frac{7}{13}\right)^2}$$

$$= \sqrt{\frac{130}{169}} = \sqrt{\frac{10}{13}}$$

(ii) $$|(4+2i)(3-i)| = |12-4i+6i-2i^2|$$

$$= |14+2i|$$

$$= \sqrt{(14)^2+2^2}$$

$$= \sqrt{200} = 10\sqrt{2}$$

(iii) $$\left|\frac{1}{3+5i}\right| = \left|\frac{1}{3+5i}\cdot\frac{3-5i}{3-5i}\right|$$

$$= \left|\frac{3-5i}{9-15i+15i-25i^2}\right|$$

$$= \left|\frac{3-5i}{34}\right| = \left|\frac{3}{34}-\frac{5}{34}i\right|$$

$$= \sqrt{\left(\frac{3}{34}\right)^2+\left(\frac{-5}{34}\right)^2}$$

$$= \sqrt{\frac{34}{34^2}} = \frac{\sqrt{34}}{34}$$

10. $z_1 = -2 - 3i$

$z_2 = 3 + i$

$$\Rightarrow \frac{z_1}{z_2} = \frac{-2-3i}{3+i}\cdot\frac{3-i}{3-i}$$

$$= \frac{-6+2i-9i+3i^2}{9-\cancel{3i}+\cancel{3i}-i^2}$$

$$= \frac{-9-7i}{10}$$

$$= -\frac{9}{10}-\frac{7}{10}i$$

$$|z_1| = |-2-3i| = \sqrt{(-2)^2+(-3)^2}$$

$$= \sqrt{13}$$

$$|z_2| = |3+i| = \sqrt{3^2+1^2}$$
$$= \sqrt{10}$$
$$\left|\frac{z_1}{z_2}\right| = \left|-\frac{9}{10} - \frac{7}{10}i\right| = \sqrt{\left(\frac{-9}{10}\right)^2 + \left(\frac{-7}{10}\right)^2}$$
$$= \sqrt{\frac{130}{100}} = \frac{\sqrt{130}}{10}$$
$$\frac{|z_1|}{|z_2|} = \frac{\sqrt{13}}{\sqrt{10}} = \frac{\sqrt{13}}{\sqrt{10}} = \sqrt{\frac{130}{100}} = \frac{\sqrt{130}}{10}$$

$\therefore \quad \dfrac{|z_1|}{|z_2|} = \left|\dfrac{z_1}{z_2}\right|$ is true.

11. $v = 3 + 4i$

$$\frac{1}{v} = \frac{1}{3+4i} \cdot \left(\frac{3-4i}{3-4i}\right)$$
$$= \frac{3-4i}{9 - \cancel{12i} + \cancel{12i} - 16i^2}$$
$$= \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$$

$w = 4 - 3i$

$$\frac{1}{w} = \frac{1}{4-3i} \cdot \left(\frac{4+3i}{4+3i}\right)$$
$$= \frac{4+3i}{16 + \cancel{12i} - \cancel{12i} - 9i^2}$$
$$= \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$
$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{3}{25} - \frac{4}{25}i + \frac{4}{25} + \frac{3}{25}i$$
$$\frac{1}{u} = \frac{7}{25} - \frac{1}{25}i = \frac{7-i}{25}$$
$$u = \frac{25}{7-i} \cdot \left(\frac{7+i}{7+i}\right)$$
$$= \frac{25(7+i)}{49 + \cancel{7i} - \cancel{7i} - i^2}$$
$$= \frac{25(7+i)}{50} = \frac{7}{2} + \frac{1}{2}i$$

12. $z = 4 - 2i$

$2z = 2(4-2i) = 8 - 4i$

$3z = 3(4-2i) = 12 - 6i$

$|z| = |4-2i| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$

$|2z| = |8-4i| = \sqrt{8^2 + (-4)^2} = \sqrt{80} = 4\sqrt{5}$

$|3z| = |12-6i| = \sqrt{12^2 + (-6)^2} = \sqrt{180} = 6\sqrt{5}$

$\therefore \quad |2z| = 2|z|.$

13. Is $|z| = |\bar{z}|$ For all $z \in C$?

Let $z = a + ib$

$\Rightarrow \bar{z} = a - ib$

$\therefore \quad |z| = \sqrt{a^2 + b^2} \quad = \sqrt{a^2 + b^2}$

$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

$\therefore \quad |z| = |\bar{z}|$ for all $z \in C$.

14. $z_1 = s + 8i, \; z_2 = t + 8i$

(i) $|z_1| = |s + 8i| = \sqrt{s^2 + 8^2} = 10$

$\therefore \quad s^2 + 64 = 100$

$s^2 = 36$

$s = \pm 6$

(ii) $|z_2| = |t + 8i| = \sqrt{t^2 + 8^2} = 2|z_1|$

$\therefore \quad \sqrt{t^2 + 8^2} = 2(10) = 20$

$\therefore \quad t^2 + 8^2 = 400$

$t^2 = 336$

$t = \sqrt{336}$

$= \pm 4\sqrt{21}$

15. Modulus of $\dfrac{i}{1-i} = \left|\dfrac{i}{1-i} \cdot \left(\dfrac{1+i}{1+i}\right)\right|$

$$= \left|\frac{i + i^2}{1 + \cancel{i} - \cancel{i} - i^2}\right|$$

$$= \left|\frac{i-1}{2}\right|$$

$$= \left|-\frac{1}{2} + \frac{1}{2}i\right|$$

$$= \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \left(\frac{\sqrt{2}}{2}\right)$$

16. Solutions of $|z - 1||z - 1| = 1$.

Let $z = x + iy \Rightarrow |x + iy - 1|^2 = 1$

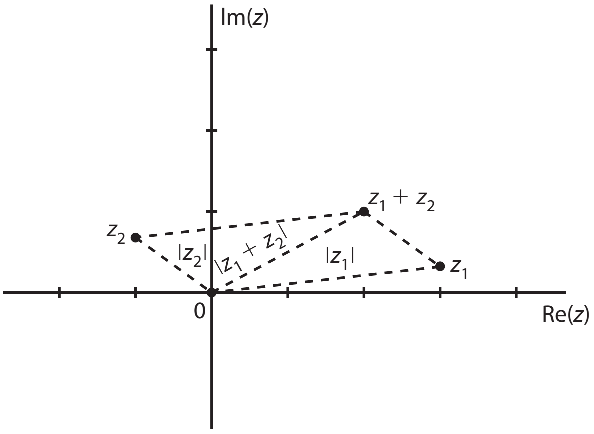
$\Rightarrow |(x - 1) + iy|^2 = 1$

$\therefore \quad \sqrt{(x-1)^2 + y^2} = 1$

$\therefore \quad (x-1)^2 + y^2 = 1 = (1^2)$

which is the equation of a circle centre (1, 0), radius = 1.

17.



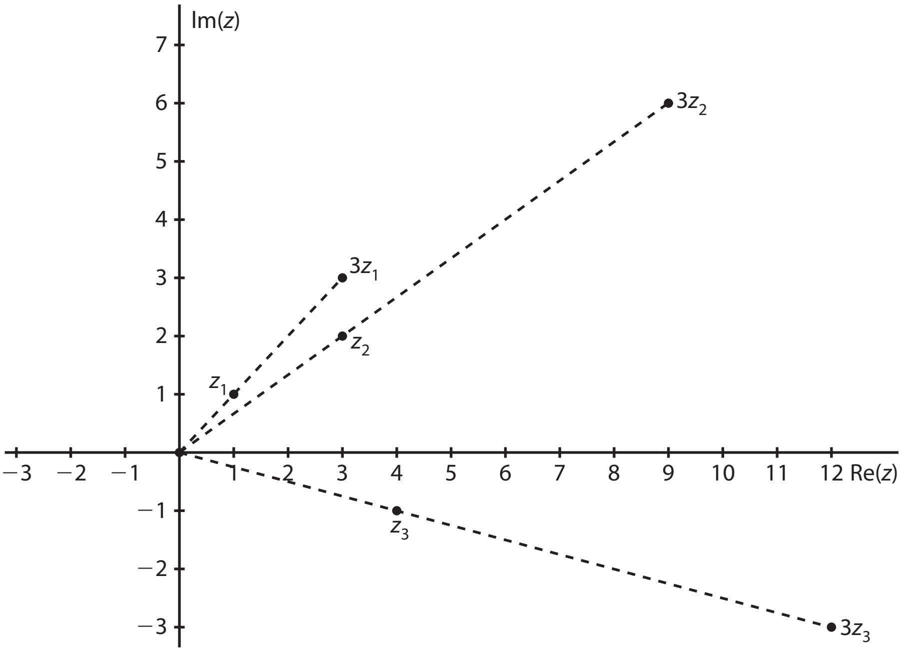
$|z_1 + z_2| = |z_1| + |z_2|$ if 0, z_1 and z_2 are collinear.

Exercise 1.5

1. $z_1 = 1 + i \Rightarrow 3z_1 = 3 + 3i$

$z_2 = 3 + 2i \Rightarrow 3z_2 = 9 + 6i$

$z_3 = 4 - i \Rightarrow 3z_3 = 12 - 3i$

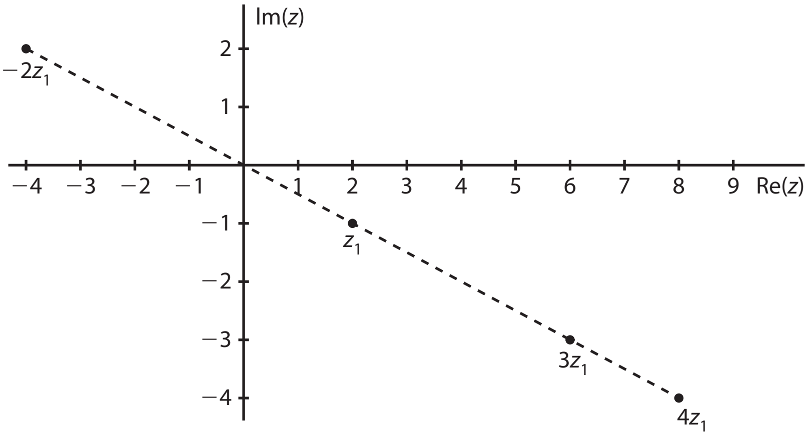


2. $z_1 = 2 - i$

$3z_1 = 3(2 - i) = 6 - 3i$

$4z_1 = 4(2 - i) = 8 - 4i$

$-2z_1 = -2(2 - i) = -4 + 2i$



Multiplication by a real number stretches the number along a line from the origin.

3. A → $1 + 5i$

B → $1 + 2i$

C → $2 + 2i$

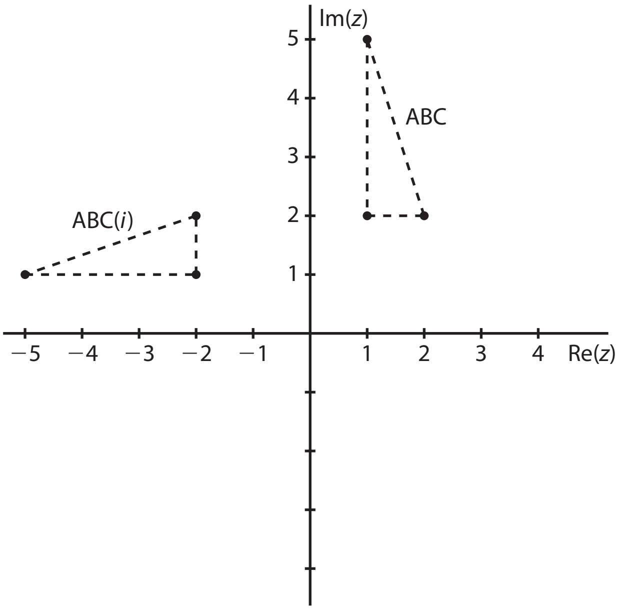
D → $-2 + i$

E → $-1 - 2i$

F → $-2 - 2i$

N → $-1 + i$

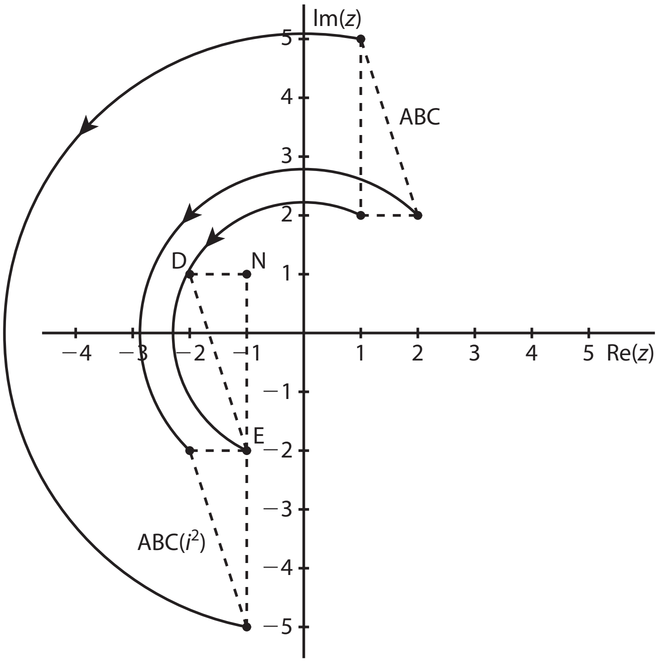
(i) ABC → DEF results from the translation $-3 - 4i$.



(ii) $(ABC)(i) \Rightarrow$

$$A(i) = (1 + 5i)i = i + 5i^2$$
$$= -5 + i$$
$$B(i) = (1 + 2i)i = i + 2i^2$$
$$= -2 + i$$
$$C(i) = (2 + 2i)i = 2i + 2i^2$$
$$= -2 + 2i$$

(iii) ABC → DEN



DNE = ABC(i^2) + 3 ie., a rotation of i^2 followed by a translation of +3.

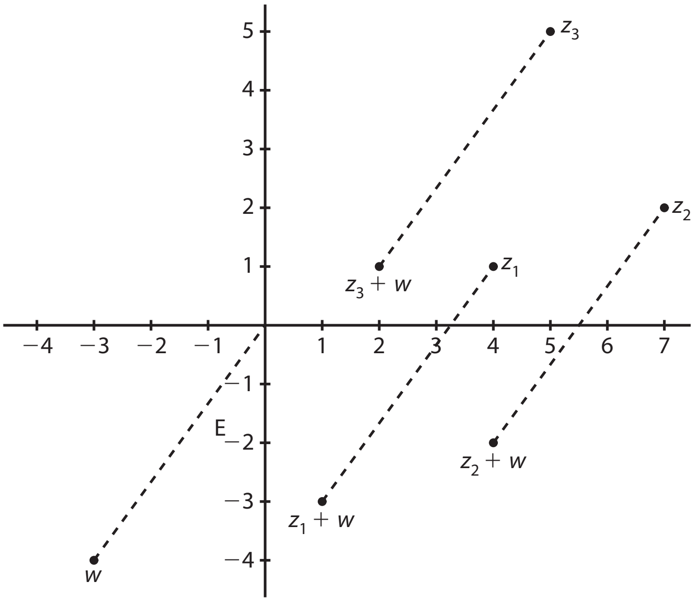
4. $z_1 = 4 + i,\ z_2 = 7 + 2i,\ z_3 = 5 + 5i$

$w = -3 - 4i$

$z_1 + w = (4 + i) + (-3 - 4i) = 1 - 3i$

$z_2 + w = (7 + 2i) + (-3 - 4i) = 4 - 2i$

$z_3 + w = (5 + 5i) + (-3 - 4i) = 2 + i$



A translation.

5. (i) $z_1 = 6 - 2i$

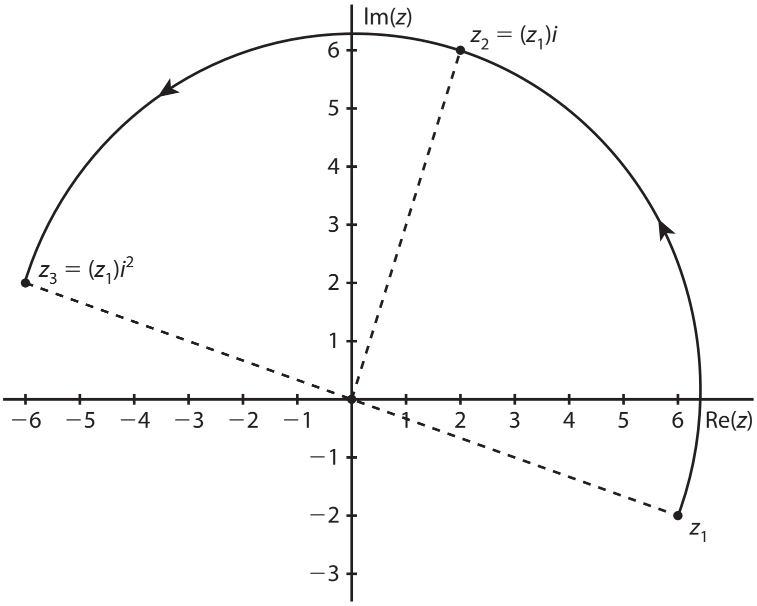
(ii) $(z_1)i = (6 - 2i)i$

$= 6i - 2i^2$

$= 2 + 6i$

(iii) $(z_1)i^2 = (6 - 2i)(-1)$

$= -6 + 2i$



A rotation.

6. (i) $z_2 = 3 + 6i$

$z_1 = 1 + 2i$

$$z_2 = az_1 \Rightarrow \quad 3 + 6i = a(1 + 2i)$$

$$\left(\frac{3 + 6i}{1 + 2i}\right) = a$$

$$\therefore \quad a = 3$$

(ii) $z_3 = -2 + i$

$$z_3 = bz_1 \quad \Rightarrow \quad -2 + i = b(1 + 2i)$$

$$\Rightarrow \quad \frac{-2 + i}{1 + 2i} = b$$

$$\therefore \quad b = \frac{-2 + i}{1 + 2i} \cdot \left(\frac{1 - 2i}{1 - 2i}\right)$$

$$= \frac{-\cancel{2} + 4i + i - \cancel{2i^2}}{1 - \cancel{2i} + \cancel{2i} - 4i^2} = \frac{5i}{5} = i$$

$$\therefore \quad b = i$$

(iii) $z_4 = -1 - 2i$

$$z_4 = cz_1 \quad \Rightarrow \quad -1 - 2i = c(1 + 2i)$$

$$\frac{-1 - 2i}{1 + 2i} = c$$

$$\frac{-(1 + 2i)}{1 + 2i} = c$$

$$\therefore \quad c = -1.$$

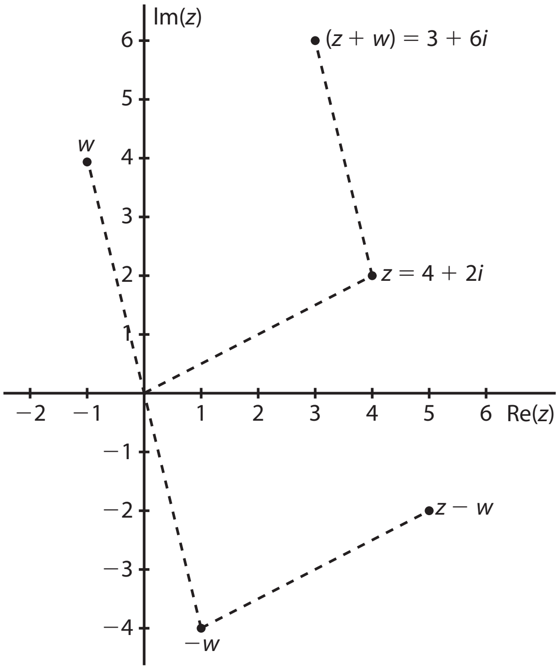
7.

$$\begin{aligned} z + w &= 3 + 6i \\ z \quad &= 4 + 2i \\ \hline \Rightarrow \quad w &= -1 + 4i \end{aligned}$$

$$-w = 1 - 4i$$

$$\Rightarrow z - w = 4 + 2i + (1 - 4i)$$

$$= 5 - 2i$$



8. (i) $z \rightarrow z + k,\ k \in C$ a translation of the plane.

(ii) $z \rightarrow kz,\ k \in R$ a stretching or contraction away from or towards the origin.

(iii) $z \rightarrow kz,\ k \in C$ a combination of stretching and rotation occurs.

9. $z_1 = 2 + i$

$z_2 = 2 + 3i$

$z_3 = 1 + 3i$

$z_4 = 1 + i$

(i) $z \rightarrow 2z \Rightarrow z_1 \rightarrow 2z_1 = 2(2 + i) = 4 + 2i$

$z_2 \rightarrow 2z_2 = 2(2 + 3i) = 4 + 6i$

$z_3 \rightarrow 2z_3 = 2(1 + 3i) = 2 + 6i$

$z_4 \rightarrow 2z_4 = 2(1 + i) = 2 + 2i$

(ii) $z \to (i)z \Rightarrow z_1 \to iz_1 = i(2+i) = 2i + i^2 = -1 + 2i$

$z_2 \to iz_2 = i(2+3i) = 2i + 3i^2 = -3 + 2i$

$z_3 \to iz_3 = i(1+3i) = i + 3i^2 = -3 + i$

$z_4 \to iz_4 = i(1+i) = i + i^2 = -1 + i$

(iii) $z \to (2+i)z \;\Rightarrow\; z_1 \to (2+i)z_1 = (2+i)(2+i) = 4 + 2i + 2i + i^2$

$= 3 + 4i$

$z_2 \to (2+i)z_2 = (2+i)(2+3i) = 4 + 6i + 2i + 3i^2$

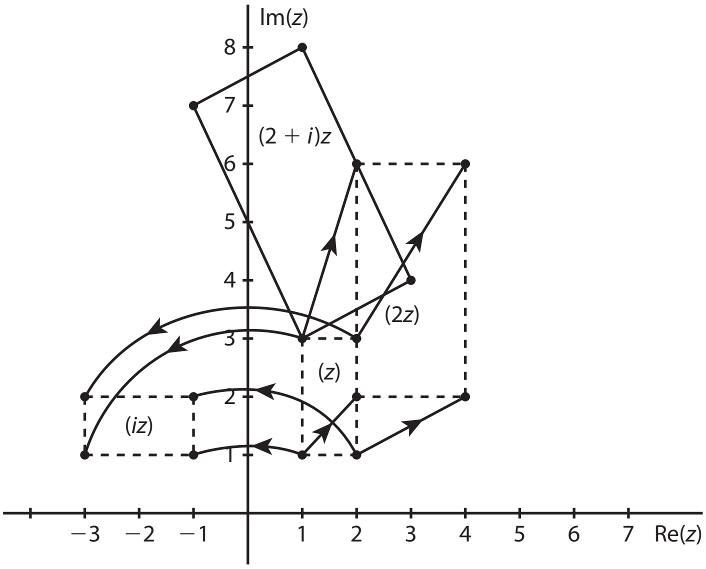
$= 1 + 8i$

$z_3 \to (2+i)z_3 = (2+i)(1+3i) = 2 + 6i + i + 3i^2$

$= 1 + 7i$

$z_4 \to (2+i)z_4 = (2+i)(1+i) = 2 + 2i + i + i^2$

$= 1 + 3i$



10. (i) $z_1 = 1 + 2i$; $1 + 2i \to \frac{1}{2} + i$

$z_2 = -3 + 2i$; $-3 + 2i \to -1\frac{1}{2} + i$

$z_3 = -2 - 2i$; $-2 - 2i \to -1 - i$

$\therefore$ Contraction by a factor of $\frac{1}{2}$.

(ii) $z_1 = 1 + 2i$; $1 + 2i \to 3 + 6i$

$z_2 = -3 + 2i$; $-3 + 2i \to -9 + 6i$

$z_3 = -2 - 2i$; $-2 - 2i \to -6 - 6i$

$\therefore$ Stretching by a factor of 3.

Exercise 1.6

1. $-2 + 4i$ a root of $z^2 + 4z + 20 = 0$

$\Rightarrow (-2+4i)^2 + 4(-2+4i) + 20 = 0$

$\Rightarrow 4 + 2(-2)(4i) + (4i)^2 - 8 + 16i + 20 = 0$

$\Rightarrow 4 - \cancel{16i} - 16 - 8 + \cancel{16i} + 20 = 0$

$-24 + 24 = 0$ True

$\therefore\; -2 + 4i$ is a root of $z^2 + 4z + 20 = 0$

$\Rightarrow$ second root $= -2 - 4i$.

2. (i) $z^2 - 2z + 17 = 0$

$a = 1, \quad b = -2, \quad c = 17$

$$z = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(17)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-64}}{2}$$

$$= \frac{2 \pm 8i}{2} = 1 \pm 4i$$

(ii) $z^2 + 4z + 7 = 0$

$a = 1, \quad b = 4, \quad c = 7$

$$z = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}i}{2} = -2 \pm \sqrt{3}i$$

3. (i) $z_1 = 1 + 3i, \ z_2 = 1 - 3i$

$\Rightarrow$ The sum of the roots $= 1 + \cancel{3i} + 1 - \cancel{3i} = 2$

also, the product of the roots $= (1 + 3i)(1 - 3i)$

$= 1 - \cancel{3i} + \cancel{3i} - (3i)^2$

$= 10$

$\therefore$ equation $= z^2 - 2z + 10 = 0.$

(ii) $z_1 = -2 + i, \ z_2 = -2 - i$

$\Rightarrow$ The sum of the roots $= -2 + \cancel{i} - 2 - \cancel{i} = -4$

also, the product of the roots $= (-2 + i)(-2 - i)$

$= 4 + 2i - 2i - i^2$

$= 5$

$\therefore$ equation $= z^2 + 4z + 5 = 0.$

(iii) $z_1 = 4 + 2i, \ z_2 = 4 - 2i$

$\Rightarrow$ The sum of the roots $= 4 + \cancel{2i} + 4 - \cancel{2i} = 8$

also, the product of the roots $= (4 + 2i)(4 - 2i)$

$= 16 - \cancel{8i} + \cancel{8i} - 4i^2$

$= 20$

$\therefore$ equation $= z^2 - 8z + 20 = 0.$

(iv) $z_1 = +5i, \ z_2 = -5i$

$\Rightarrow$ The sum of the roots $= 5i - 5i = 0$

also, the product of the roots $= (5i)(-5i)$

$= -25i^2$

$= 25$

$\therefore$ equation $= z^2 - (0)z + 25 = 0$

$\Rightarrow \quad z^2 + 25 = 0.$

4. $z = 4 - i$ is a root of $z^2 - 8z + 17 = 0$

$\Rightarrow$ $\bar{z} = 4 + i$ is also a root

$\Rightarrow$ $(4 + i)^2 - 8(4 + i) + 17 = 0$

$16 + 2(4)(i) + i^2 - 32 - 8i + 17 = 0$

$16 + \cancel{8i} - 1 - 32 - \cancel{8i} + 17 = 0$

$+33 - 33 = 0$

$\therefore 4 + i$ is also a root.

5. $-2+2i$ is a root of $z^3+3z^2+4z-8=0$

$\Rightarrow \quad (-2+2i)^3+3(-2+2i)^2+4(-2+2i)-8=0$

$(-2)^3+3(-2)^2(2i)+3(-2)(2i)^2+3[+4+2(-2)(2i)+(2i)^2]-8+8i-8=0$

$-8+\cancel{24i}+24-\cancel{8i}+12-\cancel{24i}-12-8+\cancel{8i}-8=0$

$+36-36=0$

Since the coefficients of $f(z)$ are real $\Rightarrow$ the conjugate $-2-2i$ is also a root.

$z_1=-2+2i$

$z_2=-2-2i$

$\Rightarrow \quad$ sum of roots $=(-2+2i)+(-2-2i)=-4$

also, the product of the roots $=(-2+2i)(-2-2i)=4+\cancel{4i}-\cancel{4i}-4i^2$

$=8$

$\therefore$ the quadratic formed from two roots $=z^2+4z+8=0$

$$\therefore \quad \begin{array}{r|l} & z-1 \\ \cline{2-2} z^2+4z+8 & z^3+3z^2+4z-8 \\ & \underline{z^3+4z^2+8z} \\ & \quad -z^2-4z-8 \\ & \quad \underline{-z^2-4z-8} \end{array}$$

$\therefore z-1$ is a factor $\Rightarrow z=1$ is the third root

6. Root 1 $=2+3i$

Root 2 $=2-3i$

$\Rightarrow \quad$ sum of the roots $=(-2+3i)+(2-3i)=4$

also, the product of the roots $=(2+3i)(2-3i)=4-\cancel{6i}+\cancel{6i}-9i^2$

$=13$

$\therefore$ the quadratic equation formed from roots $=z^2-4z+13=0.$

$$\therefore \quad \begin{array}{r|l} & 2z-1 \\ \cline{2-2} z^2-4z+13 & 2z^3-9z^2+30z-13 \\ & \underline{2z^3-8z^2+26z} \\ & \quad -z^2+4z-13 \\ & \quad \underline{-z^2+4z-13} \end{array}$$

$\therefore \quad 2z-1$ is a factor $\Rightarrow 2z-1=0$

$z=\dfrac{1}{2}$ is a root.

$\therefore \quad$ roots are $2\pm 3i, \dfrac{1}{2}$

7. $f(z)=z^2+(-1+5i)z+14-7i=0.$

$f(1+2i)=(1+2i)^2+(-1+5i)(1+2i)+14-7i=0$

$=1+2(1)(2i)+(2i)^2-1-2i+5i+10i^2+14-7i=0$

$=1+\cancel{4i}-4-1-\cancel{2i}+\cancel{5i}-10+14-\cancel{7i}$

$15-15=0$

$\Rightarrow 1+2i$ is a root.

$f(1-2i)=(1-2i)^2+(-1+5i)(1-2i)+14-7i=0$

$=1+2(1)(-2i)+(-2i)^2-1+2i+5i-10i^2+14-7i=0$

$=\cancel{1}-4i-4-\cancel{1}+\cancel{2i}+\cancel{5i}+10+14-\cancel{7i}=0$

$20-4i\neq 0$

$\Rightarrow 1-2i$ is not a root.

This is not a root because the conjugate of a root is only a root of polynomials with real coefficients.

8. $$\frac{1+2i}{1-2i} = \frac{1+2i}{1-2i}\left(\frac{1+2i}{1+2i}\right)$$
$$= \frac{1+2i+2i+4i^2}{1+\cancel{2i}-\cancel{2i}-4i^2}$$
$$= \frac{-3+4i}{5} = \frac{-3}{5}+\frac{4}{5}i \text{ is a root}$$

$\Rightarrow -\frac{3}{5}-\frac{4}{5}i$ is also a root.

$\therefore$ the sum of the roots $= -\frac{3}{5}+\frac{\cancel{4}}{\cancel{5}}i-\frac{3}{5}-\frac{\cancel{4}}{\cancel{5}}i = \frac{-6}{5}$

also, the product of the roots $= \left(\frac{-3}{5}+\frac{4}{5}i\right)\left(\frac{-3}{5}-\frac{4}{5}i\right)$
$$= \frac{9}{25}+\frac{\cancel{12i}}{\cancel{25}}-\frac{\cancel{12i}}{\cancel{25}}-\frac{16i}{25}$$
$$= \frac{25}{25} = 1$$

$\therefore$ the equation $f(z) = z^2+\frac{6}{5}z+1 = 0$

$\Rightarrow 5z^2+6z+5 = 0.$

$\Rightarrow a = 5,\ b = 6$

9. $z^3-1 = (z-1)(z^2+az+b)$

$\Rightarrow \frac{z^3-1}{z-1} = z^2+az+b.$

$$\begin{array}{rl} & \;\; z^2+z+1 \\ \cline{2-2} z-1 \big| & z^3 \qquad\qquad -1 \\ & \underline{z^3-z^2} \\ & \quad +z^2 \qquad -1 \\ & \quad \underline{+z^2-z} \\ & \qquad\quad +z-1 \\ & \qquad\quad \underline{+z-1} \end{array}$$

$\therefore z^3-1 = (z-1)(z^2+z+1)$

$\Rightarrow a = 1,\ b = 1.$

solving $z^2+z+1 = 0$:

$a = 1,\ b = 1,\ c = 1$

$$\Rightarrow \quad z = \frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$$
$$= \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{3}i}{2}$$

$\therefore$ the roots are $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

10. roots $= -2\pm i$

$\Rightarrow$ sum of roots $= (-2+i)+(-2-i) = -4$

product of roots $= (-2+i)(-2-i) = 4+\cancel{2i}-\cancel{2i}-i^2$
$= 5$

$\therefore$ the quadratic equation $f(z) = z^2+4z+5 = 0.$

$$\begin{array}{rl} & \;\; z-3 \\ \cline{2-2} z^2+4z+5 \big| & z^3+z^2-7z-15 \\ & \underline{z^3+4z^2+5z} \\ & \quad -3z^2-12z-15 \\ & \quad \underline{-3z^2-12z-15} \end{array}$$

$\Rightarrow$ the other two roots are $-2-i, 3$.

11. roots $= -3 + 2i, -3 - 2i$

$$\text{sum of roots} = (-3 + 2i) + (-3 - 2i) = -6$$

also, product of roots $= (-3 + 2i)(-3 - 2i) = 9 + \cancel{6i} - \cancel{6i} - 4i^2 = 13$

$\therefore$ the quadratic equation: $f(z) = z^2 + 6z + 13 = 0.$

Given a third root $z = 2 \quad \Rightarrow (z - 2)$ is a factor

$\therefore$ the cubic equation is: $f(z) = (z - 2)(z^2 + 6z + 13)$

$$= z^3 + 6z^2 + 13z - 2z^2 - 12z - 26$$

$$= z^3 + 4z^2 + z - 26 = 0.$$

12. $z_1 = -1 + i$ is a root $\quad \Rightarrow z_2 = -1 - i$ is also a root

$z_3 = 2.$

$\therefore$ sum of roots $= (-1 + i) + (-1 - i) = -2$

also, product of the roots $= (-1 + i)(-1 - i) = 1 + \cancel{i} - \cancel{i} - i^2 = 2$

$\therefore$ the quadratic equation: $\quad f(z) = z^2 + 2z + 2 = 0$

$\therefore$ the cubic equation is: $f(z) = (z - 2)(z^2 + 2z + 2) = 0$

$$= z^3 + \cancel{2z^2} + 2z - \cancel{2z^2} - 4z - 4$$

$$= z^3 - 2z - 4 = 0$$

13. $\quad z = 1^{\frac{1}{3}}$

$\Rightarrow z^3 = 1$

$\Rightarrow z^3 - 1 = 0$

$\Rightarrow (z - 1)(z^2 + z + 1) = 0$

$$\Rightarrow z_1 = 1 \quad \text{or} \quad z_2 = \frac{-1 + \sqrt{3}i}{2} \quad \text{or} \quad z_3 = \frac{-1 - \sqrt{3}i}{2}$$

$$\therefore \ \alpha = \frac{-1 + \sqrt{3}i}{2}, \ \beta = \frac{-1 - \sqrt{3}i}{2}$$

(i) $$\alpha^2 = \left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{(-1)^2 + 2(-1)(\sqrt{3}i) + (\sqrt{3}i)^2}{4}$$

$$= \frac{1 - 2\sqrt{3}i - 3}{4}$$

$$= \frac{-2 - 2\sqrt{3}i}{4} = \frac{-1 - \sqrt{3}i}{2} = \beta$$

(ii) $$1 + \alpha + \beta = 1 + \left(\frac{-1 + \sqrt{3}i}{2}\right) + \left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$= \frac{\cancel{2} - \cancel{1} + \cancel{\sqrt{3}}i - \cancel{1} - \cancel{\sqrt{3}}i}{2}$$

$$= \frac{0}{2} = 0$$

Exercise 1.7

1. (i) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4(0 + i(1)) = 0 + 4i$

(ii) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(\frac{-\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = -\sqrt{3} + i$

(iii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$$= \frac{-2}{2} + i\cdot\frac{2}{2} = -1 + i$$

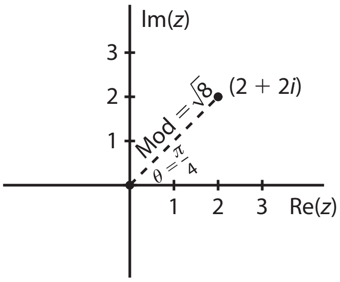
(iv) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$= 1 + \sqrt{3}i$$

2. (i) $z = 2 + 2i$

$$|z| = \sqrt{2^2 + 2^2}$$
$$= \sqrt{8} = 2\sqrt{2}$$

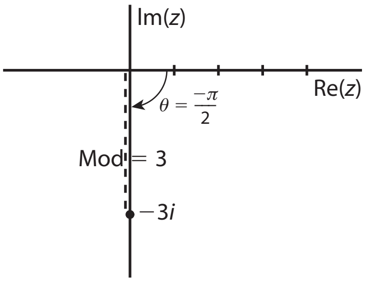
$$\text{arg: } \theta = \tan^{-1}\frac{2}{2}$$
$$= \frac{\pi}{4}\text{ radians}$$



(ii) $z = -3i$

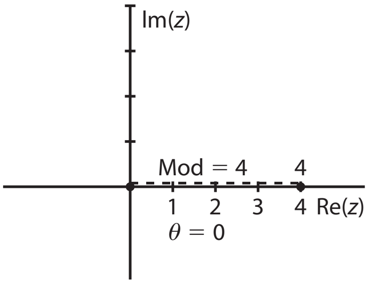
$$|z| = \sqrt{0 + (-3)^2}$$
$$= 3$$

$$\text{arg: } \theta = \tan^{-1}\left(\frac{-3}{0}\right) = \tan^{-1}(\infty)$$
$$= -\frac{\pi}{2}\text{ (rads)}$$



(iii) $z = 4 + 0i$

$$|z| = 4$$
$$\text{arg: } \theta = \tan^{-1}\frac{0}{4}$$
$$= 0\text{ (rads)}$$

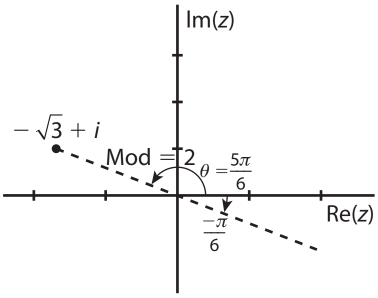


(iv) $z = -\sqrt{3} + i$

$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2}$$
$$= 2$$

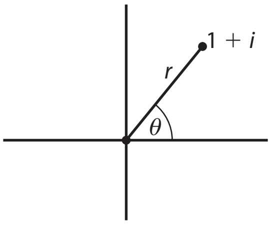
$$\text{arg: } \theta = \tan^{-1}\frac{1}{-\sqrt{3}}$$
$$= \left(-\frac{\pi}{6}\right)\text{ from calculator}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{6}\text{ Radians } \left(\pi - \frac{\pi}{6}\right)$$



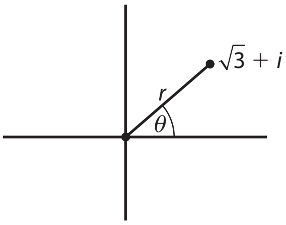
3. (i) $z = 1 + i$

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$
$$z = r(\cos\theta + i\sin\theta)$$
$$\Rightarrow \quad z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$



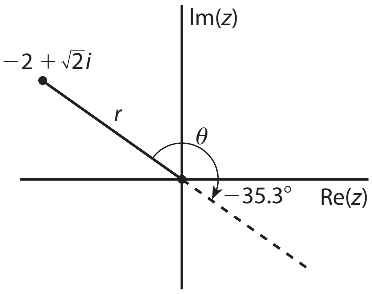
(ii) $z = \sqrt{3} + i$

$$r = |z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$
$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$



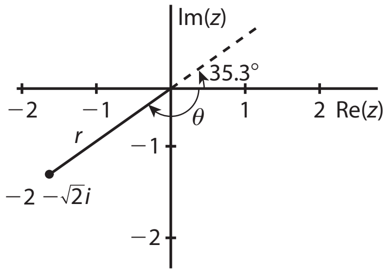
(iii) $z = -2 + \sqrt{2}i$

$$r = |z| = \sqrt{(-2)^2 + (\sqrt{2})^2} = \sqrt{6}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{-2}\right) = -35.3° \quad \text{(from calculator)}$$
$$\Rightarrow \quad \theta = 144.7° \quad (180° - 35.3°)$$
$$\Rightarrow \quad z = r(\cos\theta + i\sin\theta) = \sqrt{6}\,(\cos(144.7°) + i\sin(144.7°))$$



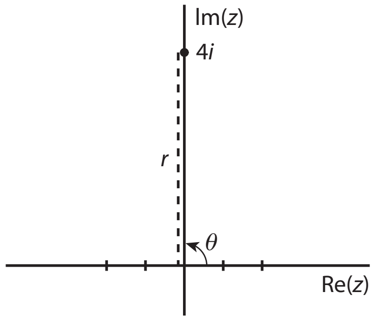
(iv) $z = -2 - \sqrt{2}\,i$

$$r = |z| = \sqrt{(-2)^2 + (-\sqrt{2})^2}$$
$$= \sqrt{6}$$
$$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{-2}\right)$$
$$= 35.3° \text{ (from calculator)}$$
$$= -180° + 35.3° = -144.7°$$
$$z = r(\cos\theta + i\sin\theta) = \sqrt{6}\,(\cos(-144.7°) + i\sin(-144.7°))$$



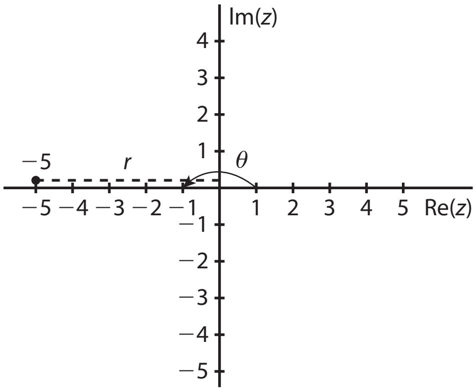
(v) $z = 4i$

$$r = |z| = \sqrt{0^2 + 4^2} = 4$$
$$\theta = \tan^{-1}\left(\frac{4}{0}\right) = \frac{\pi}{2}$$
$$z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$



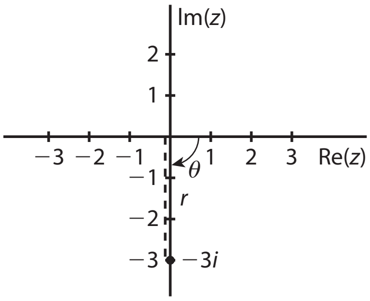
(vi) $z = -5$

$$r = |z| = \sqrt{(-5)^2 + 0} = 5$$
$$\theta = \tan^{-1}\left(\frac{0}{-5}\right) = \tan^{-1} 0$$
$$= 0 \text{ (from calculator)}$$
$$= 180° - 0° = 180° = \pi$$
$$z = r(\cos\theta + i\sin\theta) = 5(\cos\pi + i\sin\pi)$$



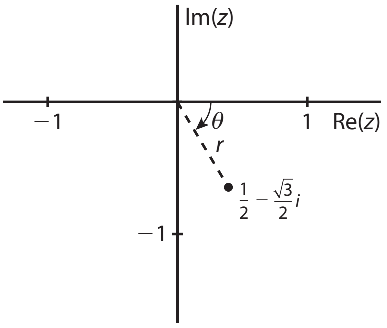
(vii) $z = -3i$

$$r = |z| = \sqrt{0 + (-3)^2} = 3$$
$$\theta = \tan^{-1}\left(\frac{-3}{0}\right) = \tan^{-1}(\infty)$$
$$= -\frac{\pi}{2}$$
$$z = r(\cos\theta + i\sin\theta)$$
$$= 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$



(viii) $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$r = |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{1} = 1$$
$$\theta = \tan^{-1}\left(\frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}(-\sqrt{3})$$
$$= -60° = -\frac{\pi}{3}$$
$$z = r(\cos\theta + i\sin\theta) = 1\,.\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

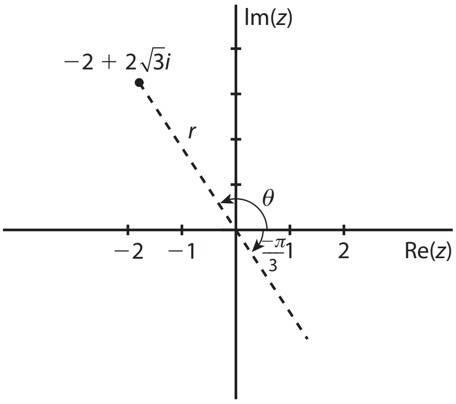


4. (i)
$$\begin{aligned} z &= (1+\sqrt{3}i)^2 \\ &= 1 + 2(1)(\sqrt{3}i) + (\sqrt{3}i)^2 \\ &= 1 + 2\sqrt{3}i - 3 \\ &= -2 + 2\sqrt{3}i \end{aligned}$$

$$r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \\ &= \tan^{-1}(-\sqrt{3}) \\ &= -\frac{\pi}{3} \text{ (from calculator)} \\ &= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

$$\therefore z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

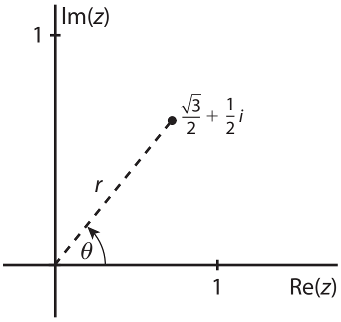


(ii)
$$\begin{aligned} z = \frac{-2}{-\sqrt{3}+i} &= \frac{-2}{-\sqrt{3}+i}\left(\frac{-\sqrt{3}-i}{-\sqrt{3}-i}\right) \\ &= \frac{+2\sqrt{3}+2i}{3+\cancel{\sqrt{3}i}-\cancel{\sqrt{3}i}-i^2} \\ &= \frac{2\sqrt{3}+2i}{4} \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \end{aligned}$$

$$r = |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$$

$$\begin{aligned} \theta = \tan^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= 30° = \frac{\pi}{6} \end{aligned}$$

$$\therefore z = r(\cos\theta + i\sin\theta) = 1\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$



5. $z = 1 + \sqrt{3}i$

(i) $iz = i(1+\sqrt{3}i) = i + \sqrt{3}i^2 = -\sqrt{3} + i$

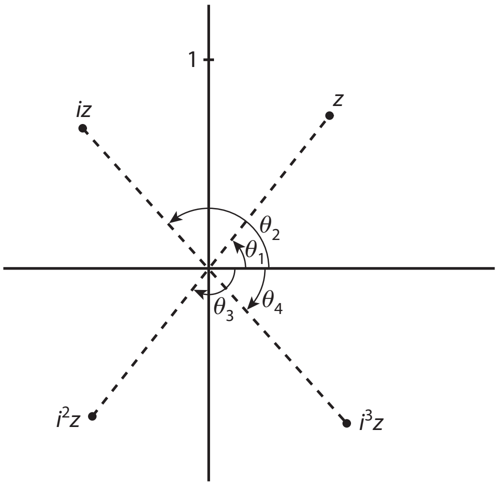
(ii) $i^2z = (-1)(1+\sqrt{3}i) = -1 - \sqrt{3}i$

(iii) $i^3z = (-i)(1+\sqrt{3}i) = -i - \sqrt{3}i^2 = \sqrt{3} - i$

$z_1 = 1 + \sqrt{3}i$:
$$\begin{aligned} \theta_1 &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\ &= 60° = \frac{\pi}{3} \end{aligned}$$

$z_2 = -\sqrt{3} + i$:
$$\begin{aligned} \theta_2 &= \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \\ &= -30° = -\frac{\pi}{6} \text{ (calculator)} \\ &= \pi - \frac{\pi}{6} = \frac{5\pi}{6} \end{aligned}$$

$z_3 = -1 - \sqrt{3}i$:
$$\begin{aligned} \theta_3 &= \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) \\ &= 60° = \frac{\pi}{3} \text{ (calculator)} \\ &= -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}\left[= \frac{4\pi}{3}\right] \end{aligned}$$



$$z_4 = \sqrt{3} - i \;: \qquad \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= -30° = -\frac{\pi}{6}\left[= \frac{11\pi}{6}\right]$$

$$\frac{\pi}{3} \to \frac{5\pi}{6} \qquad : \qquad \frac{5\pi}{6} - \frac{\pi}{3} = \frac{\pi}{2}$$

$$\frac{5\pi}{6} \to \frac{-2\pi}{3} \Rightarrow \frac{5\pi}{6} \to \frac{4\pi}{3} \qquad : \qquad \frac{4\pi}{3} - \frac{5\pi}{6} = \frac{\pi}{2}$$

$$-\frac{2\pi}{3} \to -\frac{\pi}{6} \Rightarrow \frac{4\pi}{3} \to \frac{11\pi}{6} \qquad : \qquad \frac{11\pi}{6} - \frac{4\pi}{3} = \frac{\pi}{2}$$

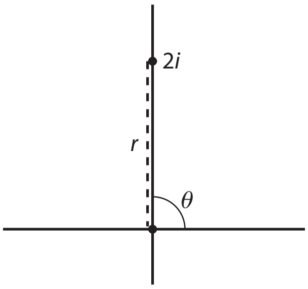
Each time a complex number z is multiplied by i, a rotation of $\frac{\pi}{2}$(90°) rads occurs.

6. (i) $z = 2i$

$r = 2$

$\theta = \frac{\pi}{2}$

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$



(ii) $z = -3 - \sqrt{3}i$

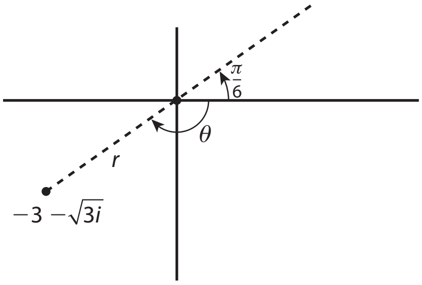
$$r = |z| = \sqrt{(-3)^2 + (-\sqrt{3})^2} = \sqrt{12}$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-3}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= 30°\left(\frac{\pi}{6}\right) \quad \text{(calculator result)}$$

$$= -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

$$z = r(\cos\theta + i\sin\theta) = \sqrt{12}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$



(iii) $$z = \frac{2}{-1+i} = \frac{2}{-1+i}\left(\frac{-1-i}{-1-i}\right)$$

$$= \frac{-2-2i}{1 + \cancel{i} - \cancel{i} - i^2}$$

$$= \frac{-2-2i}{2}$$

$$= -1 - i$$

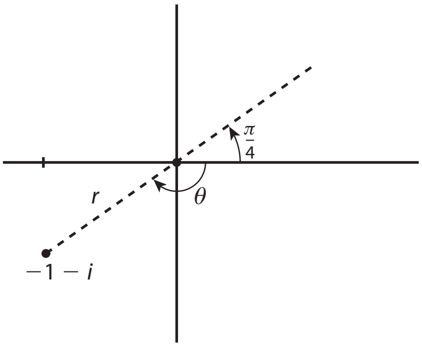
$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1} 1$$

$$= 45°\left(\frac{\pi}{4}\right) \quad \text{(calculator result)}$$

$$= -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$



7. (i) $z^2 - 2z + 2 = 0$

$a = 1,\ b = -2,\ c = 2$

$$z = \frac{+2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

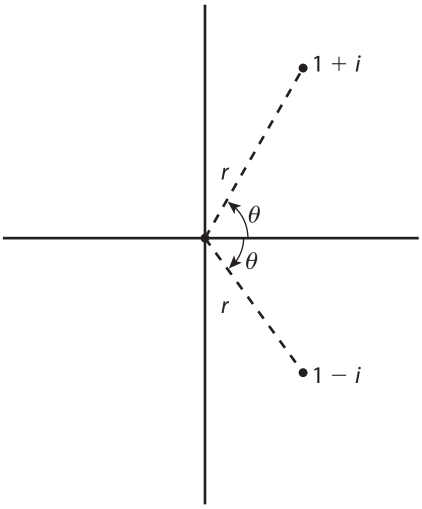
$$= 1 \pm i$$

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45°\left(\frac{\pi}{4}\right) \text{ or } \left(-\frac{\pi}{4}\right)$$

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$



8. $z_2 = t + 8i$

$$\arg(z_2) = \tan^{-1}\left(\frac{8}{t}\right) = \frac{3\pi}{4}$$

$$\Rightarrow \frac{8}{t} = \tan\left(\frac{3\pi}{4}\right) = -1$$

$$\Rightarrow t = -8$$

Exercise 1.8

1. $z_1 = 4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), z_2 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(i) $z_1 z_2 = 4 \times 2\left(\cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)\right)$

$$= 8(\cos\pi + i\sin\pi)$$

(ii) $\frac{z_1}{z_2} = \frac{4}{2}\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$

$$= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

2. $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\Rightarrow z^2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 4\left(\cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)\right)$$

$$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

3. $z_1 = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), z_2 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(i) $|z_1| = 3, \quad \arg(z_1) = \frac{\pi}{2}$

(ii) $|z_2| = 4, \quad \arg(z_2) = \frac{\pi}{3}$

(iii) $|z_1 z_2| = 3 \times 4, \quad \arg(z_1 z_2) = \left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

$= 12 \qquad\qquad = \frac{5\pi}{6}$

(iv) $\left|\frac{z_1}{z_2}\right| = \frac{3}{4}, \quad \arg\left(\frac{z_1}{z_2}\right) = \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$$= \frac{\pi}{6}$$

4. $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \times 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 12\left[\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right]$

$= 12\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

5. $9\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \div 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= \frac{9}{6}\left[\cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)\right]$

$= \frac{3}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

6. $2\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right) \times \frac{1}{3}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right) \times 6\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$

$= 2 \times \frac{1}{3} \times 6 \times \left[\cos\left(\frac{\pi}{9} + \frac{\pi}{9} + \frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9} + \frac{\pi}{9} + \frac{\pi}{9}\right)\right]$

$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}i$

7. $\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^2$

$= \left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)\left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)$

$= \cos\pi + i\sin\pi$

8. (a) $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^3$

$= \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$

$= 8\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right)$

$= 8(\cos\pi + i\sin\pi)$

(b) $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^4$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \cdot 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) . 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \cdot 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

9. $z = 3(\cos\pi + i\sin\pi)$

(i) $\frac{1}{z} = \frac{1}{3(\cos\pi + i\sin\pi)} = \frac{1}{3(\cos\pi + i\sin\pi)} \cdot \left(\frac{\cos\pi - i\sin\pi}{\cos\pi - i\sin\pi}\right)$

$= \frac{\cos\pi - i\sin\pi}{3(\cos^2\pi - \cancel{i\cos\pi\sin\pi} + \cancel{i\cos\pi\sin\pi} - i^2\sin^2\pi)}$

$= \frac{\cos\pi - i\sin\pi}{3(\cos^2\pi + \sin^2\pi)}$

$= \frac{\cos\pi - i\sin\pi}{3} = \frac{1}{3}(\cos\pi - i\sin\pi)$

since $\cos\pi = \cos(-\pi)$

and $\sin\pi = -\sin(-\pi)$

$\therefore \frac{1}{z} = \frac{1}{3}(\cos(-\pi) + \sin(-\pi))$

(ii) $\frac{1}{z} = \frac{1}{3}(-1 + 0) = -\frac{1}{3} + 0 . i$

10.

$$z = -2 + 2\sqrt{3}i$$

$$r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2}$$

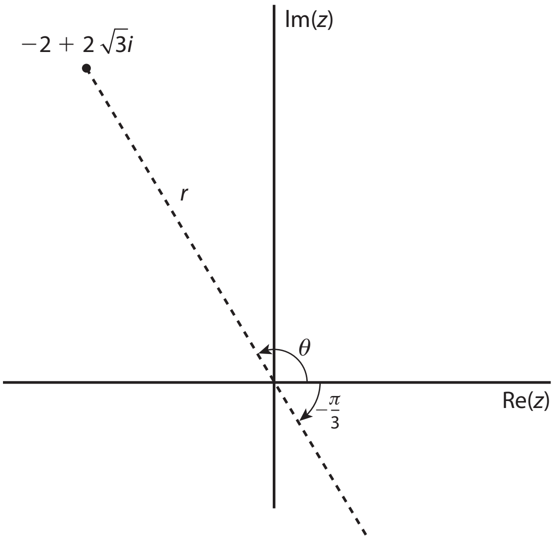
$$= 4$$

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \tan^{-1}(-\sqrt{3})$$

$$= -60°\left(\frac{\pi}{3}\right) \text{ (from calculator)}$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$



(a) (i) $$z^2 = \left[4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^2$$

$$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

(b) (i) $$z^3 = \left[4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^3$$

$$= 64\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)$$

$$= 64(\cos 2\pi + i\sin 2\pi)$$

(a) (ii) $$z^2 = 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 16\left(-\frac{1}{2} + i\left(\frac{-\sqrt{3}}{2}\right)\right)$$

$$= -8 - 8\sqrt{3}i$$

(b) (ii) $$z^3 = 16(\cos 2\pi + i\sin 2\pi) = 64(1 + i(0))$$

$$= 64 + 0i$$

11. $z = \cos\theta + i\sin\theta$

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{\cos\theta + i\sin\theta}\cdot\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - \cancel{i\cos\theta\sin\theta} + \cancel{i\cos\theta\sin\theta} - i^2\sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{1} \qquad [\cos^2\theta + \sin^2\theta = 1]$$

$$= \cos\theta - i\sin\theta = \bar{z}$$

12. $z.\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1$

$$\Rightarrow z = \frac{1}{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

13. $z = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta$$

$$\Rightarrow z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

Exercise 1.9

1. (i) $\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)^4 = \left(\cos\frac{4\pi}{8} + i\sin\frac{4\pi}{8}\right)$

$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 0 + i$

(ii) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7 = \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$

(iii) $\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^8 = \left(\cos\frac{8\pi}{12} + i\sin\frac{8\pi}{12}\right)$

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(iv) $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)$

$= (\cos 2\pi + i\sin 2\pi) = 1 + 0i$

(v) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-6} = \left(\cos\left(\frac{-6\pi}{4}\right) + i\sin\left(\frac{-6\pi}{4}\right)\right)$

$= \left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right) = 0 + i$

(vi) $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^{10} = \left(\cos\frac{20\pi}{5} + i\sin\frac{20\pi}{5}\right)$

$= (\cos 4\pi + i\sin 4\pi) = 1 + 0i$

(vii) $\left(\cos\left(\frac{-\pi}{18}\right) + i\sin\left(\frac{-\pi}{18}\right)\right)^9 = \left(\cos\left(\frac{-9\pi}{18}\right) + i\sin\left(\frac{-9\pi}{18}\right)\right)$

$= \left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

$= 0 - i$

(viii) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{-3} = \left(\cos\left(\frac{-3\pi}{6}\right) + i\sin\left(\frac{-3\pi}{6}\right)\right)$

$= \left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

$= 0 - i$

2. $z = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z^4 = \left[\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^4$

$= (\sqrt{2})^4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -2 - 2\sqrt{3}i$

3. $z = 3\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

$z^5 = \left[3\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right]^5$

$= 3^5\left(\cos\frac{5\pi}{10} + i\sin\frac{5\pi}{10}\right)$

$= 243\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 243(0 + i) = 0 + 243i$

4. (i) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^2=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(ii) $\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^4=\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)$

$$\begin{aligned}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^2=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^4&=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)\\&=\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right)\\&=-\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

5. (i) $z_1=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(ii) $z_2=3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(iii) $\bar{z}_1=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)=2\left(\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right)$

(iv) $\bar{z}_2=3\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)=3\left(\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right)$

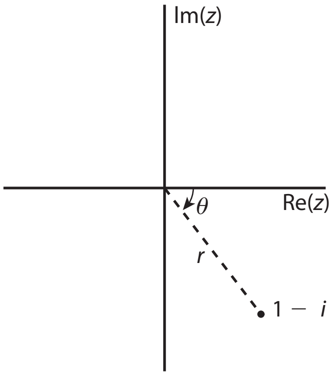
(v) $$\begin{aligned}z_1z_2&=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\times3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\\&=6\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right)\\&=6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\end{aligned}$$

(vi) $$\begin{aligned}\frac{z_1}{z_2}&=\frac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)}\\&=\frac{2}{3}\left(\cos\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\right)\\&=\frac{2}{3}\left(\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right)\end{aligned}$$

6. (i) $$\begin{aligned}z&=1-i\\r&=|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}\\\theta&=\tan^{-1}\left(\frac{-1}{1}\right)=\tan^{-1}(-1)=\frac{-\pi}{4}\end{aligned}$$

$$\Rightarrow z=(1-i)=\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)$$

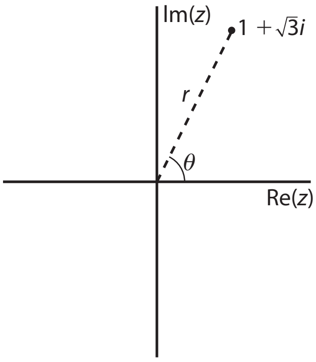
$$\begin{aligned}(1-i)^4&=\left(\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)\right)^4\\&=4(\cos(-\pi)+i\sin(-\pi))\\&=4(-1+0)\\&=-4+0i\end{aligned}$$

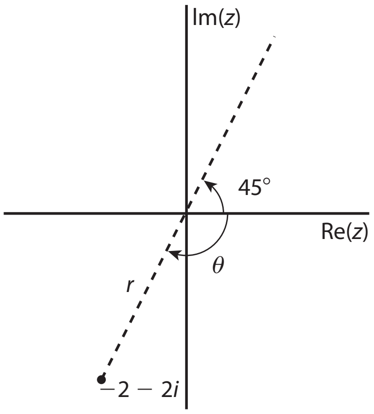


(ii) $$\begin{aligned}z&=1+i\sqrt{3}\\r&=|z|=\sqrt{1^2+\sqrt{3}^2}=2\\\theta&=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}\end{aligned}$$

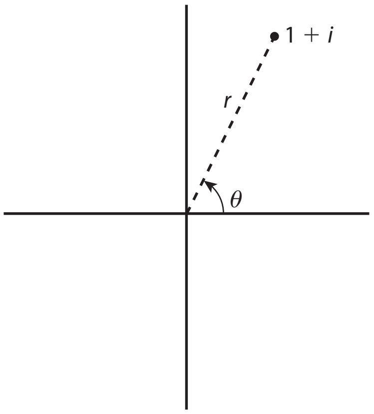
$$\Rightarrow z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\begin{aligned}(1+i\sqrt{3})^3&=\left(2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right)^3\\&=8(\cos\pi+i\sin\pi)\\&=8(-1+i(0))=-8+0i\end{aligned}$$

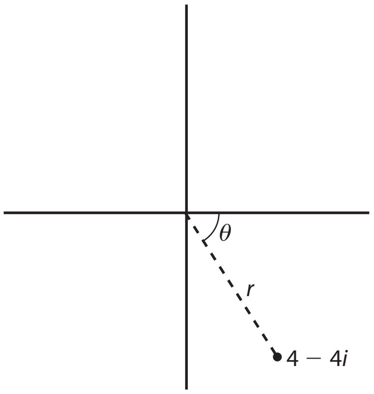




(iii)
$$z = -2 - 2i$$
$$r = |z| = \sqrt{(-2)^2 + (-2)^2}$$
$$= \sqrt{8}$$
$$\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1} 1$$
$$= 45°\left(\frac{\pi}{4}\right) \quad \text{calculator}$$
$$\Rightarrow \quad \theta = -\pi + \frac{\pi}{4}$$
$$= \frac{-3\pi}{4}$$
$$\therefore z = (-2 - 2i) = \sqrt{8}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$$
$$\therefore (-2 - 2i)^4 = \left[\sqrt{8}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)\right]^4$$
$$= (\sqrt{8})^4(\cos(-3\pi) + i\sin(-3\pi))$$
$$= 64(-1 + (0)i)$$
$$= -64 + 0i$$



7. $(1 + i)^4$
$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$
$$z = (1 + i)^4 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4$$
$$= 4(\cos\pi + i\sin\pi)$$
$$= 4(-1 + i(0))$$
$$= -4$$



8. $z = 4 - 4i$
$$r = |z| = \sqrt{4^2 + (-4)^2} = \sqrt{32}$$
$$= 4\sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{-4}{4}\right)$$
$$= \tan^{-1}(-1) = -\frac{\pi}{4}$$
$$\therefore \quad z = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$z = \frac{1}{(4 - 4i)^3} = (4 - 4i)^{-3}$$
$$= \left[4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-3}$$
$$= (4\sqrt{2})^{-3}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$= \frac{1}{128\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= -\frac{1}{256} + \frac{i}{256}$$

9. (i) $(3-\sqrt{3}i)^6$

$$r=|z|=\sqrt{3^2+(-\sqrt{3})^2}=\sqrt{12}$$

$$\theta=\tan^{-1}\left(\frac{-\sqrt{3}}{3}\right)$$

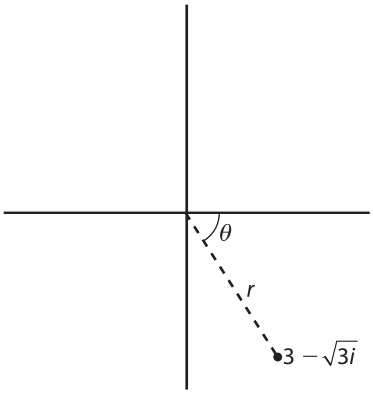
$$=-30°\left(-\frac{\pi}{6}\right)$$

$$z=(3-\sqrt{3}i)^6=\left[\sqrt{12}\left(\cos-\left(\frac{\pi}{6}\right)+i\sin-\left(\frac{\pi}{6}\right)\right)\right]^6$$

$$=(\sqrt{12})^6(\cos(-\pi)+i\sin(-\pi))$$

$$=1728(-1+i\,.\,0)$$

$$=-1728+0i$$



(ii) $(2+2\sqrt{3}i)^6$

$$r=|z|=\sqrt{2^2+(2\sqrt{3})^2}$$

$$=\sqrt{16}=4$$

$$\theta=\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)=\tan^{-1}\sqrt{3}$$

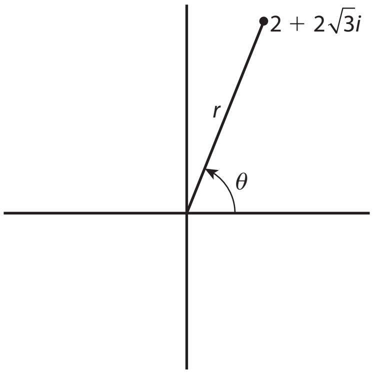
$$=60°\left(\frac{\pi}{3}\right)$$

$$z=(2+2\sqrt{3}i)^6=\left[4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^6$$

$$=4^6(\cos 2\pi+i\sin 2\pi)$$

$$=4096(1+i(0))$$

$$=4096+0i$$



10.

$$\frac{\sqrt{3}+i}{1+\sqrt{3}i}=\frac{\sqrt{3}+i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}$$

$$=\frac{\sqrt{3}-3i+i-\sqrt{3}i^2}{1-\cancel{\sqrt{3}i}+\cancel{\sqrt{3}i}-3i^2}$$

$$=\frac{2\sqrt{3}-2i}{4}=\frac{\sqrt{3}}{2}-\frac{i}{2}$$

$$z=\frac{\sqrt{3}+i}{1+\sqrt{3}i}=\frac{\sqrt{3}}{2}-\frac{i}{2}$$

$$\Rightarrow\quad r=|z|=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}$$

$$=1$$

$$\theta=\tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

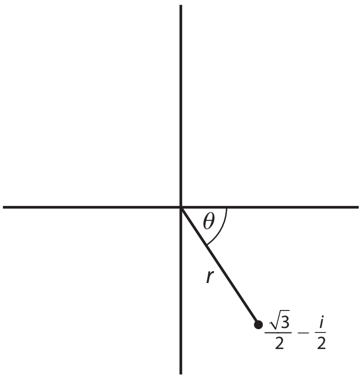
$$=-30°\left(-\frac{\pi}{6}\right)$$

$$z=1\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$z^6=1\left[1\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right]^6$$

$$=1(\cos(-\pi)+i\sin(-\pi))$$

$$=1((-1)+i(0))=1.$$



Exercise 1.10

1. (i) $$\begin{aligned}(\cos \pi - i \sin \pi)^5 &= (\cos(-\pi) + i \sin(-\pi))^5\\ &= \cos(-5\pi) + i \sin(-5\pi)\\ &= -1 + i(0)\end{aligned}$$

(ii) $$\begin{aligned}\left(\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}\right)^{10} &= \left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right)\right)^{10}\\ &= \cos(-2\pi) + i \sin(-2\pi)\\ &= 1 + i(0)\end{aligned}$$

(iii) $$\begin{aligned}\frac{1}{\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^3} &= \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^{-3}\\ &= \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)^{-3}\\ &= \cos \pi + i \sin \pi\\ &= -1 + i(0)\end{aligned}$$

(iv) $$\begin{aligned}\left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}\right)^4 &= \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right)^4\\ &= \cos(-2\pi) + i \sin(-2\pi)\\ &= 1 + i(0)\end{aligned}$$

2. (i) $$\begin{aligned}&(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta\\ \Rightarrow\ &\cos^2 \theta + 2i \cos \theta \sin \theta + (i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta\\ \Rightarrow\ &\cos^2 \theta - \sin^2 \theta + 2i \cos \theta \sin \theta = \cos 2\theta + i \sin 2\theta\\ \Rightarrow\ &\sin 2\theta = 2 \cos \theta \sin \theta\end{aligned}$$

(ii) $$\begin{aligned}&(\cos\theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta\\ \Rightarrow &\cos^3 \theta + 3i \cos^2 \theta \sin\theta + 3 \cos \theta(i \sin \theta)^2 + (i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta\\ &\cos^3 \theta - 3 \cos \theta \sin^2 \theta + 3i \cos^2 \theta \sin \theta - i \sin^3 \theta = \cos 3\theta + i \sin 3\theta\\ \Rightarrow\ &\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta\\ &= \cos^3 \theta - 3 \cos \theta(1 - \cos^2 \theta)\\ &= \cos^3 \theta - 3 \cos \theta + 3\cos^3 \theta\\ &= 4 \cos^3 \theta - 3 \cos \theta\end{aligned}$$

3. $$\begin{aligned}\cos 4\theta + i \sin 4\theta &= (\cos \theta + i \sin \theta)^4\\ &= \cos^4 \theta + 4 \cos^3 \theta(i \sin \theta) + 6 \cos^2 \theta(i \sin \theta)^2 + 4 \cos \theta(i \sin \theta)^3 + (i \sin \theta)^4\\ &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta\\ &= [\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta] + i[4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta]\end{aligned}$$

(i) Putting Re = Re:

$$\begin{aligned}\cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta\\ &= \cos^4 \theta - 6 \cos^2 \theta(1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2\\ &= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta\\ &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1\end{aligned}$$

(ii) Putting Im = Im:

$$\sin 4\theta = 4 \cos^3 \theta \sin\theta - 4 \cos \theta \sin^3 \theta$$

4. $z^3 = 8 = 8 + 0i$

$r = |z| = 8$

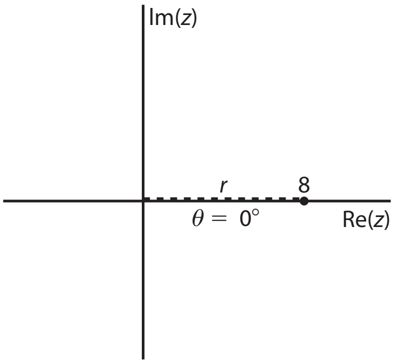
$\theta = 0°(0 \text{ rads})$

$$\begin{aligned}z^3 &= 8(\cos 0 + i \sin 0)\\ &= 8(\cos(0 + 2n\pi) + i \sin(0 + 2n\pi))\end{aligned}$$

$$\begin{aligned}z &= [8 \cos(0 + 2n\pi) + i \sin(0 + 2n\pi)]^{\frac{1}{3}}\\ &= 8^{\frac{1}{3}}\left(\cos\left(\frac{2n\pi}{3}\right) + i \sin\left(\frac{2n\pi}{3}\right)\right)\end{aligned}$$

let $n = 0 : z = 2(\cos 0 + i \sin 0) = 2(1 + 0) = 2$

let $n = 1 : z = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$



$$\text{let } n = 2 : z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}i$$

$\therefore$ the roots are $2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$

5. $z^3 = -8 = -8 + 0i$

$r = |-8| = 8$

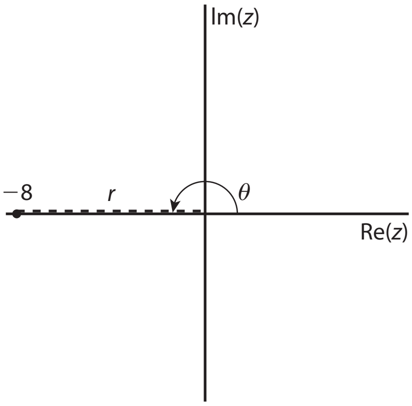
$\theta = 180°(\pi \text{ rad})$

$z^3 = 8(\cos\pi + i\sin\pi)$

$= 8(\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi))$

$\Rightarrow z = [8\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)]^{\frac{1}{3}}$

$$= 8^{\frac{1}{3}}\left[\cos\left(\frac{\pi + 2n\pi}{3}\right) + i\sin\left(\frac{\pi + 2n\pi}{3}\right)\right]$$



$$\text{let } n = 0 : z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 1 + \sqrt{3}i$$

$\text{let } n = 1 : z = 2(\cos\pi + i\sin\pi) = 2(-1 + 0) = -2$

$$\text{let } n = 2 : z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - \sqrt{3}i$$

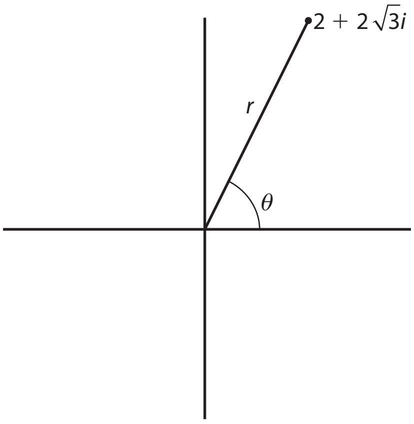
$\therefore$ the roots are $-2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$

6. $z = 2 + 2\sqrt{3}i$

$r = |z| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3}$$

$$= 60°\left(\frac{\pi}{3}\right)$$



$$z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z^2 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z = 4\left(\cos\left(\frac{\pi}{3} + 2n\pi\right) + i\sin\left(\frac{\pi}{3} + 2n\pi\right)\right)^{\frac{1}{2}}$$

$$\Rightarrow z = 4^{\frac{1}{2}}\left[\cos\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right)\right]$$

$$\text{let } n = 0 : z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \sqrt{3} + i$$

$$\text{let } n = 1 : z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

$\therefore$ the roots are $\sqrt{3} + i, -\sqrt{3} - i$.

7. $z = 1$

$|z| = 1$

$\theta = 0 \text{ rad}$

(a) $z = 1(\cos 0 + i\sin 0) = (\cos(0 + 2n\pi) + i\sin(0 + 2n\pi)$

$\Rightarrow z^{\frac{1}{3}} = [1.(\cos(0 + 2n\pi) + i\sin(0 + 2n\pi))]^{\frac{1}{3}}$

$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

$\text{let } n = 0 : z = (\cos 0 + i\sin 0) = 1 + i(0) = 1$

$$\text{let } n = 1 : z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\text{let } n = 2 : z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(b) Thus the sum of the roots is

$$(1) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$

8. $z^3 = 27i = 0 + 27i$

$z = (0 + 27i)^{\frac{1}{3}}$

$r = 27$

$\theta = 90°\left(\frac{\pi}{2}\text{ rad}\right)$

$$\therefore z = \left[27\left(\cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right)\right)\right]^{\frac{1}{3}}$$

$$= 27^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{2} + 2n\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2n\pi}{3}\right)\right)$$

let $n = 0 : z = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= \frac{3\sqrt{3}}{2} + \frac{1}{2}i$

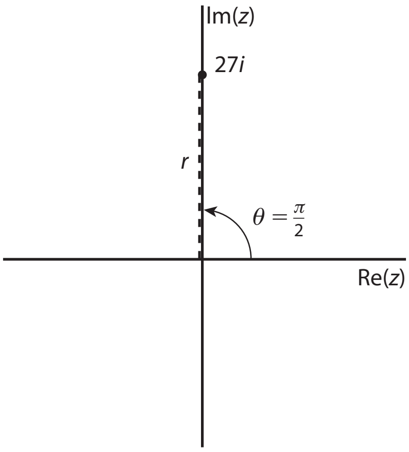
let $n = 1 : z = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 3\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = \frac{-3\sqrt{3}}{2} + \frac{3}{2}i$

let $n = 2 : z = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$= 3(0 + i(-1)) = -3i$

$\therefore$ the cube roots are $-3i, \frac{-3\sqrt{3}}{2} + \frac{3}{2}i, \frac{3\sqrt{3}}{2} + \frac{3}{2}i$



9. (i) $z^2 = 1 + \sqrt{3}i$

$\Rightarrow \quad z = (1 + \sqrt{3}i)^{\frac{1}{2}}$

$r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$$\therefore z = \left[2\cos\left(\frac{\pi}{3} + 2n\pi\right) + i\sin\left(\frac{\pi}{3} + 2n\pi\right)\right]^{\frac{1}{2}}$$

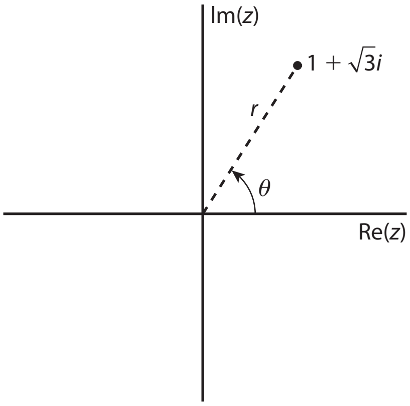
$$= \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right)\right)$$

let $n = 0 : z = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right)$

$= \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

let $n = 1 : z = \sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= \frac{-\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$



(ii) $z^2 = 2 - 2\sqrt{3}i$

$z = (2 - 2\sqrt{3}i)^{\frac{1}{2}}$

$r = |z| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$

$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) = \tan^{-1}(-\sqrt{3})$

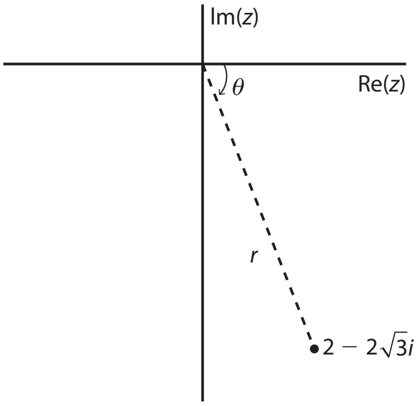
$= -60°\left(-\frac{\pi}{3}\text{ rad}\right)$

$$z = \left[4\left(\cos\left(-\frac{\pi}{3} + 2n\pi\right) + i\sin\left(-\frac{\pi}{6} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$$

$$= 2\left(\cos\left(\frac{\frac{-\pi}{3} + 2n\pi}{2}\right) + i\sin\left(\frac{\frac{-\pi}{6} + 2n\pi}{2}\right)\right)$$

let $n = 0 : z = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$

$= \sqrt{3} - i$



$$\text{let } n = 1: z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -\sqrt{3} + i$$

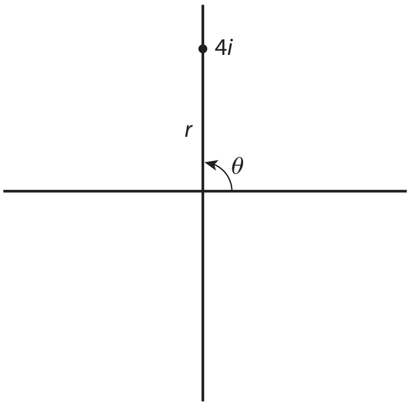
$\therefore$ roots are $\sqrt{3} - i, -\sqrt{3} + i$

(iii) $z^2 = 4i = (0 + 4i)$

$z = (0 + 4i)^{\frac{1}{2}}$

$r = 4$

$\theta = \frac{\pi}{2}$



$$z = \left[4\left(\cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$$
$$= 2\left(\cos\left(\frac{\frac{\pi}{2} + 2n\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2n\pi}{2}\right)\right)$$

$$\text{let } n = 0: z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2} + \sqrt{2}i$$

$$\text{let } n = 1: z = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$
$$= -\sqrt{2} - \sqrt{2}i$$

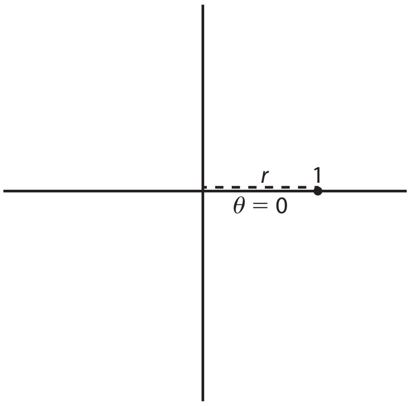
$\therefore$ roots are $\sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i$

10. $z^5 = 1 + 0i$

$z = (1 + 0i)^{\frac{1}{5}}$

$r = 1$

$\theta = 0$



$$\Rightarrow z = [1(\cos(0 + 2n\pi) + i\sin(0 + 2n\pi))]^{\frac{1}{5}}$$
$$= \left(\cos\left(\frac{2n\pi}{5}\right) + i\sin\frac{2n\pi}{5}\right)$$

let $n = 0: z = 2(\cos 0 + i\sin 0) = 1$

let $n = 1: z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = (0.309 + 0.951i)$

let $n = 2: z = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = (-0.809 + 0.588i)$

let $n = 3: z = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = (-0.809 - 0.588i)$

let $n = 4: z = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = (0.309 - 0.951i)$

let $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$\Rightarrow w^2 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$w^3 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$w^2 + w^3 = \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + i\left(\sin\frac{4\pi}{5} + \sin\frac{6\pi}{5}\right)$

Note: $[\sin(180° - \theta) = -\sin(180 + \theta)]$

$\therefore \sin\frac{4\pi}{5} = -\sin\frac{6\pi}{5}$

$\Rightarrow w^2 + w^3 = \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5}$, a real number

Revision Exercise 1 (Core)

1. $\sqrt{80} - \sqrt{20} = \sqrt{16 \times 5} - \sqrt{4 \times 5}$

$= 4\sqrt{5} - 2\sqrt{5} = 2\sqrt{5}$

2. $(x - 1) + yi = y + 4i$

$\Rightarrow x - 1 = y$ and $y = 4$

$\therefore \; x - 1 = 4$

$x = 5, \quad y = 4$

3. $z^2 + 4z + 3 = 0$

$a = 1, \; b = 4, \; c = 3$

$$\Rightarrow \quad z = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

$\therefore \quad z = -1 + 0i, -3 + 0i$

OR $(z + 3)(z + 1) = 0$

$\Rightarrow x = -3$ or $z = -1$

$\Rightarrow z = -3 + 0i$ or $z = -1 + 0i$

4. $z_1 = 5 + i, \quad z_2 = -2 + 3i$

(i) $(z_1)^2 = (5 + i)^2 = 5^2 + 2(5)(i) + (i)^2$

$= 25 + 10i - 1$

$= 24 + 10i$

(ii) $|z_1| = \sqrt{5^2 + 1^2} = \sqrt{26} \Rightarrow |z_1|^2 = 26$

$|z_2| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

$|z_2|^2 = (\sqrt{13})^2 = 13$

$|z_1|^2 = 26 = 2|z_2|^2 = 2 \times 13 = 26$ .... True

5. (i) $z = -1 + i\sqrt{3}$

$z^2 = (-1 + i\sqrt{3})^2 = (-1)^2 + 2(-1)(i\sqrt{3}) + (i\sqrt{3})^2$

$= 1 - 2\sqrt{3}i - 3$

$= -2 - 2\sqrt{3}i$

(ii) $z^2 + pz$ is real.

$\Rightarrow -2 - 2\sqrt{3}i + p\left(-1 + i\sqrt{3}\right)$ is real

$\Rightarrow -2 - 2\sqrt{3}i - p + p\sqrt{3}i$ is real

$\Rightarrow -2 - p + i\left(-2\sqrt{3} + p\sqrt{3}\right)$ is real

$\therefore \quad p = 2$

6. $z = 1 + i$

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1} 1$$

$$= 45°\left(\frac{\pi}{4}\right)$$

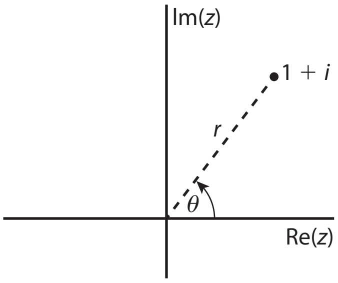
$$\Rightarrow z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow z^4 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4$$

$= 4(\cos\pi + i\sin\pi)$

$= 4(-1 + 0 \,.\, i)$

$= -4 + 0i$



7. $z = -1 + \sqrt{3}i$

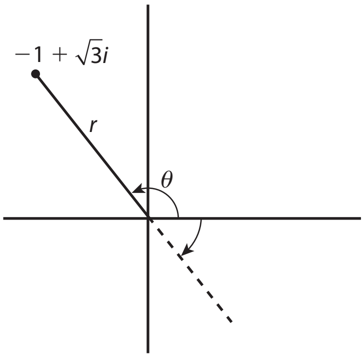
$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}\left(-\sqrt{3}\right)$$

$$= -60°\left(\frac{\pi}{3}\right) \text{ (calculator)}$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\Rightarrow \quad z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$



8. $f(z) = z^2 - 4z + 13$

$f(2 + 3i) = (2 + 3i)^2 - 4(2 + 3i) + 13$

$= 4 + 2(2)(3i) + (3i)^2 - 8 - 12i + 13$

$= \cancel{4} + \cancel{12i} - \cancel{9} - \cancel{8} - \cancel{12i} + \cancel{13}$

$= 0$

$\Rightarrow 2 + 3i$ is a root

$\Rightarrow 2 - 3i$ is the second root since the coefficients of $f(z)$ are real.

9. $z_1 = 2 + 3i \quad \Rightarrow |z_1| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$z_2 = 1 - 4i \quad \Rightarrow |z_2| = \sqrt{1^2 + (-4)^2} = \sqrt{17}$

$z_1 . z_2 = (2 + 3i)(1 - 4i) = 2 - 8i + 3i - 12i^2$

$= 14 - 5i$

$|z_1 . z_2| = \sqrt{14^2 + (-5)^2} = \sqrt{221}$

$|z_1| . |z_2| = \sqrt{13} . \sqrt{17} = \sqrt{221}$

$\Rightarrow |z_1| . |z_2| = |z_1 . z_2|$

10. $\dfrac{5 - 5i}{2 + i} = \dfrac{5 - 5i}{2 + i} \cdot \dfrac{2 - i}{2 - i}$

$= \dfrac{10 - 5i - 10i + 5i^2}{4 - \cancel{2i} + \cancel{2i} - i^2}$

$= \dfrac{5 - 15i}{5} = 1 - 3i$

11. $4i^{13} + 3i^3$

Since $i^2 = -1, i^3 = -i, i^4 = 1.$

$= 4i^{12} . i + 3i^3$

$= 4i - 3i$

$= i$

12. $z_1 = 2 + 3i, z_2 = -1 + 4i$

$\Rightarrow f(z) = z^2 -$ (sum of roots)$z +$ product of roots

$\Rightarrow z_1 + z_2 = (2 + 3i) + (-1 + 4i) = 1 + 7i$

Also, $z_1 . z_2 = (2 + 3i)(-1 + 4i) = -2 + 8i - 3i + 12i^2$

$= -14 + 5i$

$\therefore \quad f(z) = z^2 - (1 + 7i)z - 14 + 5i$

13. $a = 3 + 3i, b = 1 - 2i$

$\Rightarrow a + b = (3 + 3i) + (1 - 2i)$

$= 4 + i$

(i) $a \rightarrow a + b$

$\Rightarrow a + c = a + b$

$\Rightarrow \quad c = b = 1 - 2i$

(ii) $b \rightarrow a + b$

$\Rightarrow b + c = a + b$

$\Rightarrow \quad c = a = 3 + 3i$

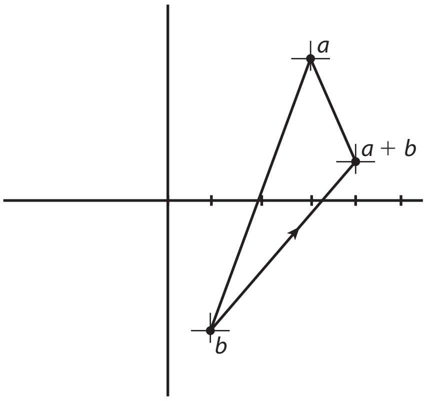
(iii) $a \rightarrow b$

$\Rightarrow \; a + c = b$

$\Rightarrow \quad c = b - a$

$= (1 - 2i) - (3 + 3i)$

$= -2 - 5i$



14. (i) $R \rightarrow S$

$R \rightarrow R'$ rotation of $\left(-\frac{\pi}{2}\right)$

$R' \rightarrow S$ stretch by a factor of $1\frac{1}{2}$

(ii) $S \rightarrow T$

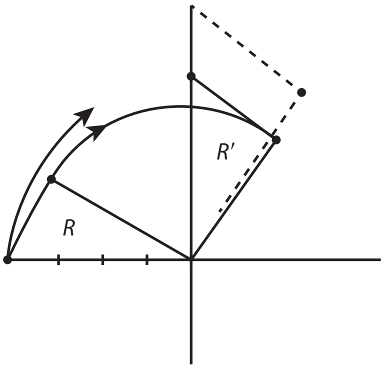
translation $(0, 6) \rightarrow (4, 5)$

i.e $0 + 6i \rightarrow 4 + 5i$

$\therefore \quad (0 + 6i) + c = 4 + 5i$

$\Rightarrow \quad c = 4 + 5i - 6i$

$= 4 - i$



(iii) $z \in R \Rightarrow x + iy \in R$

$\Rightarrow zz_1 = (x + iy)(-1\frac{1}{2}i) \in S$

$z_1 = -1\frac{1}{2}i$

(iv) $z \in R \Rightarrow x + iy \in R$

$\Rightarrow zz_1 + z_3 = (x + iy)(-1\frac{1}{2}i) + 4 - i \in T$

$\Rightarrow \quad z_3 = 4 - i$

Revision Exercise 1 (Advanced)

1. $z = x + iy$

$3(z - 1) = i(z + 1)$

$\Rightarrow \quad 3(x + iy - 1) = i(x + iy + 1)$

$\Rightarrow \quad 3x - 3 + i3y = ix + i^2y + i$

$3x - 3 + i3y = ix - y + i$

$(3x - 3) + i(3y) = -y + i(x + 1)$

$\Rightarrow \quad 3x - 3 = -y$ ...... A

and $\quad 3y = x + 1$ ...... B

$\Rightarrow \quad 9x - 9 = -3y$ 3A

$9x - 9 = -(x + 1)$ substituting B into 3A

$9x + x = -1 + 9$

$10x = 8$

$\Rightarrow \quad x = \frac{8}{10} = \frac{4}{5}$

also, $\quad 3y = x + 1$

$= \frac{4}{5} + 1 = \frac{9}{5}$

$y = \frac{3}{5}$

2. $2 + 3i$ is a root of $2z^3 - 9z^2 + 30z - 13 = 0$

$\Rightarrow$ $2 - 3i$ is a also a root since polynomial has real coefficients

$\Rightarrow$ Sum of roots $= 4$

Products of roots $= (2 + 3i)(2 - 3i) = 4 - \cancel{6i} + \cancel{6i} - 9i^2 = 13$

$\Rightarrow$ forming the quadratic from the first two roots, we get

$f(z) = z^2 - 4z + 13 = 0$

$$\therefore \quad \begin{array}{r|l} & 2z - 1 \\ \hline z^2 - 4z + 13 & 2z^3 - 9z^2 + 30z - 13 \\ & \underline{2z^3 - 8z^2 + 26z} \\ & \quad -z^2 + 4z - 13 \\ & \quad \underline{-z^2 + 4z - 13} \end{array}$$

$\Rightarrow$ third factor $= 2z - 1$

$\Rightarrow$ third factor $\Rightarrow 2z - 1 = 0$

$$\therefore \quad z = \frac{1}{2}$$

$\therefore \quad z = 2 - 3i$ and $z = \frac{1}{2}$ are the other two roots.

3. $z = \sqrt{3} + i$

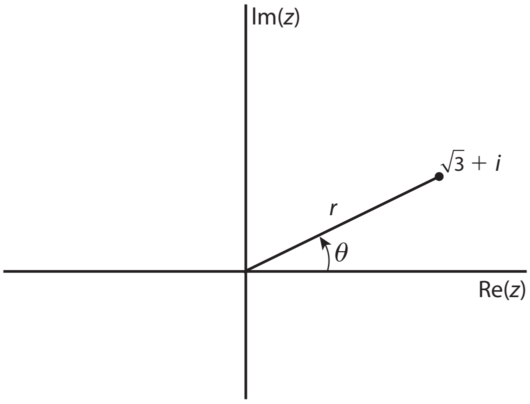
$$\Rightarrow \quad r = |z| = \sqrt{(\sqrt{3})^2 + 1^2}$$
$$= 2$$

also, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$= 30°\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow z^{11} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{11}$$
$$= 2^{11}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$
$$= 2^{11}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = 2^{10}(\sqrt{3} - i)$$



4. $f(z) = z^2 + pz + q$

Sum of roots $= (1 + i) + (4 + 3i) = 5 + 4i = -p$

Products of roots $= (1 + i) + (4 + 3i) = 4 + 3i + 4i - 3$

$= 1 + 7i = q$

$p = -5 - 4i, q = 1 + 7i$

5. $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$w_2 = (w_1)^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$$
$$= \frac{1}{4} + 2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}i}{2}\right) + \frac{3}{4}(-1)$$

$$w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\therefore w_1 + w_2 = -\frac{1}{2} + \cancel{\frac{\sqrt{3}}{2}}i + \left(-\frac{1}{2} - \cancel{\frac{\sqrt{3}}{2}}i\right)$$
$$= -1.$$

6.

$$p = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \bar{p} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$p\,.\,\bar{p} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$= 4\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{3}\right)\right)$$
$$= 4(\cos 0 + i\sin 0) = 4(1 + i0) = 4$$

$\therefore \quad p\bar{p}$ is a real number.

7.
$$\begin{aligned}\frac{(1+2i)^2}{1-i} &= \frac{(1+2i)^2}{1-i}\cdot\frac{1+i}{1+i}\\ &= \frac{(1+4i-4)(1+i)}{1+\cancel{i}-\cancel{i}+1}\\ &= \frac{(-3+4i)(1+i)}{2}\\ &= \frac{-3-3i+4i-4}{2}\\ &= \frac{-7}{2}+\frac{1}{2}i\end{aligned}$$

8.
$$\begin{aligned}\frac{-3+i}{1+ki} &= \frac{-3+i}{1+ki}\cdot\frac{1-i}{1-ki}\\ &= \frac{-3+3ki+i+k}{1-\cancel{ki}+\cancel{ki}+k^2}\\ &= \frac{-3+k}{1+k^2}+\frac{i(3k+1)}{1+k^2}\end{aligned}$$

real part $= -3$

$$\begin{aligned}\Rightarrow\quad & \frac{-3+k}{1+k^2} = -3\\ & -\cancel{3}+k = -\cancel{3}-3k^2\\ \Rightarrow\quad & 3k^2+k \quad = 0\\ & k(3k+1) = 0\\ \Rightarrow\quad & k = 0 \quad \text{or} \quad k = -\frac{1}{3}\\ \Rightarrow\quad & \qquad k = -\frac{1}{3}.\end{aligned}$$

9.
$$\begin{aligned}&\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^2\\ &= \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{2\pi}{12}+i\sin\frac{2\pi}{12}\right)\\ &= \left(\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right)\\ &= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} = 0+i = i\end{aligned}$$

10. $f(z) = z^2-(3+3i)z+5i$

$$\begin{aligned}1+2i \text{ is a root} \Rightarrow f(1+2i) &= (1+2i)^2-(3+3i)(1+2i)+5i\\ &= \cancel{1}+\cancel{4i}-\cancel{4}-\cancel{3}-\cancel{6i}-\cancel{3i}+\cancel{6}+\cancel{5i}\\ &= 0\ldots.\text{True}\end{aligned}$$

$$\begin{aligned}&\text{Sum of roots} = +(3+3i)\\ & z_2+z_1 \qquad\ = 3+3i\\ \Rightarrow\quad & z_2+(1+2i) = 3+3i\\ \Rightarrow\quad & z_2 \qquad\qquad\ = 2+i\end{aligned}$$

11. $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right)$

$= \cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

12. $f(z) = z^3 + z^2 + 4z + \rho$, ρ real

$1 - 3i$ is a root $\Rightarrow f(1-3i) = (1-3i)^3 + (1-3i)^2 + 4(1-3i) + \rho = 0$

$\Rightarrow 1 + 3(-3i) + 3(-3i)^2 + (-3i)^3 + 1 - 6i - 9 + 4 - 12i + \rho = 0$

$\Rightarrow 1 - \cancel{9i} - 27 + \cancel{27i} + 1 - \cancel{6i} - 9 + 4 - \cancel{12i} + \rho = 0$

also, $-30 + \rho = 0$

$\Rightarrow \qquad \rho = 30.$

$\Rightarrow$ 2nd root $= 1 + 3i$

$\Rightarrow$ quadratic formed from roots $= z^2 - (1 - \cancel{3i} + 1 + \cancel{3i})z + (1-3i)(1+3i)$

$= z^2 - 2z + 10.$

$$\begin{array}{r|l} & z + 3 \\ \hline \therefore z^2 - 2z + 10 & z^3 + z^2 \qquad + 4z + 30 \\ & \underline{z^3 - 2z^2 + 10z} \\ & \qquad 3z^2 - 6z + 30 \\ & \qquad \underline{3z^2 - 6z + 30} \end{array}$$

$\therefore \quad z + 3$ is a factor

$\Rightarrow \quad z + 3 = 0$ is a root

$z = -3$

Other roots are -3, $1 + 3i$

13. $x + iy = \sqrt{8 - 6i}$

$x^2 + 2ixy - y^2 = 8 - 6i$

$2xy = -6$ and $x^2 - y^2 = 8$

$y = \frac{-3}{x}$

$\Rightarrow \quad x^2 - \left(\frac{-3}{x}\right)^2 = 8$

$x^2 - \frac{9}{x^2} = 8$

$(x^2)^2 - 8x^2 - 9 = 0$

$(x^2 - 9)(x^2 + 1) = 0$

$\Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$

or $\quad x^2 = -1 \quad \Rightarrow \quad x = \sqrt{-1} = i$

when $\quad x = +3 \quad : \quad y = \frac{-3}{3} = -1$

when $\quad x = -3 \quad : \quad y = \frac{-3}{-3} = 1$

We assume x and y are real

$\Rightarrow \quad (x, y) = (3, -1)$ or $(-3, 1)$

14. $f(z) = z^4 - 2z^3 + 7z^2 - 4z + 10$

ti is a solution of $f(z)$

$\Rightarrow \quad f(ti) = (ti)^4 - 2(ti)^3 + 7(ti)^2 - 4(ti) + 10 = 0$

$\Rightarrow t^4 + 2t^3i - 7t^2 - 4ti + 10 = 0 + 0i$

$\Rightarrow \quad t^4 - 7t^2 + 10 + i(2t^3 - 4t) = 0 + 0i$

$\therefore \quad t^4 - 7t^2 + 10 = 0$ and $2t^3 - 4t = 0$

$\Rightarrow \quad 2t(t^2 - 2) = 0$

$\Rightarrow \quad t = 0$ or $t = \pm\sqrt{2}$

$\therefore \quad (t^2)^2 - 7t^2 + 10 = 0$

$\Rightarrow \quad (t^2 - 5)(t^2 - 2) = 0$

$\Rightarrow \quad t^2 = 5$ or $t^2 = 2$

$t = \pm\sqrt{5} \qquad t = \pm\sqrt{2}$

The only values of t common to both identities are $t = \pm\sqrt{2}$

when $t = +\sqrt{2} \quad \Rightarrow$ solution $= \sqrt{2}i = z_1$

$t = -\sqrt{2} \quad \Rightarrow$ solution $= -\sqrt{2}i = z_2$

quadratic formed from two roots $= z^2 - (z_1 + z_2)z + z_1z_2$

Since $\quad z_1 + z_2 = \sqrt{2}i - \sqrt{2}i = 0$

and $\quad z_1z_2 \quad = (\sqrt{2}i)(-\sqrt{2}i) = 2$

$\Rightarrow f(z) = z^2 - (0)z + 2 = z^2 + 2.$

$$\begin{array}{r|l}
 & z^2 - 2z + 5 \\ \hline
z^2 + 2 & z^4 - 2z^3 + 7z^2 - 4z + 10 \\
 & \underline{z^4 \qquad\quad + 2z^2} \\
 & \qquad -2z^3 + 5z^2 - 4z + 10 \\
 & \qquad \underline{-2z^3 \qquad\quad - 4z} \\
 & \qquad\qquad\quad 5z^2 \qquad + 10 \\
 & \qquad\qquad\quad \underline{5z^2 \qquad + 10}
\end{array}$$

$\therefore f(z) = (z^2 + 2)(z^2 - 2z + 5)$

$\therefore$ solving $z^2 - 2z + 5$

$a = 1, b = -2, c = 5$

$$\Rightarrow \quad z = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\therefore$ Solutions $= \sqrt{2}i, -\sqrt{2}i, 1 + 2i, 1 - 2i.$

15. $z = a + ib$

$\Rightarrow \quad \bar{z} = a - ib$

Given $\quad z\bar{z} - 2iz = 7 - 4i$

$\Rightarrow \quad (a + ib)(a - ib) - 2i(a + ib) = 7 - 4i$

$\Rightarrow \quad a^2 - \cancel{ab}i + \cancel{ab}i + b^2 - 2ai + 2b = 7 - 4i$

$\Rightarrow \quad a^2 + b^2 + 2b = 7$ and $-2a = -4$

$a = 2$

$\Rightarrow \quad 4 + b^2 + 2b = 7$

$\Rightarrow \quad b^2 + 2b - 3 = 0$

$(b + 3)(b - 1) = 0$

$\Rightarrow \quad b = -3$ or $b = 1$

$\Rightarrow \quad z = a + ib = 2 - 3i$ or $2 + i$

16. $(p + iq)^2 = 15 - 8i \quad$ for real p, q

$\Rightarrow \quad p^2 + 2pqi - q^2 = 15 - 8i$

$\Rightarrow \quad 2pq = -8 \quad$ and $\quad p^2 - q^2 = 15$

$$q = \frac{-4}{p}$$

$$\Rightarrow \quad p^2 - \left(\frac{-4}{p}\right)^2 = 15$$

$\Rightarrow \quad (p^2)^2 - 16 = 15p^2$

$\Rightarrow \quad (p^2)^2 - 15p^2 - 16 = 0$

$\quad (p^2 - 16)(p^2 + 1) = 0$

$\Rightarrow \quad p^2 = 16 \Rightarrow p = \pm 4$

also, $\quad p^2 = -1 \Rightarrow p = \sqrt{-1} = i$, but p is real

$\therefore p \neq i$

when $p = +4, q = \dfrac{-4}{4} = -1$

when $p = -4, q = \dfrac{-4}{-4} = 1$

$\Rightarrow (p, q) = (4, -1)$ or $(-4, 1)$

$f(z) = (1 + i)z^2 + (-2 + 3i)z - 3 + 2i = 0.$

$a = 1 + i, b = -2 + 3i, c = -3 + 2i$

$$\Rightarrow \quad z = \frac{2 - 3i \pm \sqrt{(-2 + 3i)^2 - 4(1 + i)(-3 + 2i)}}{2(1 + i)}$$

$$= \frac{2 - 3i \pm \sqrt{4 - 12i - 9 - 4(-3 + 2i - 3i - 2)}}{2(1 + i)}$$

$$= \frac{2 - 3i \pm \sqrt{-5 - 12i + 20 + 4i}}{2(1 + i)}$$

$$= \frac{2 - 3i \pm \sqrt{15 - 8i}}{2 + 2i}$$

but $\sqrt{15 - 8i} = 4 - i \quad$ or $\quad -4 + i$

$$\therefore \quad z = \frac{2 - 3i + 4 - i}{2 + 2i}$$

$$= \frac{6 - 4i}{2 + 2i} = \frac{3 - 2i}{1 + i}$$

$$= \frac{3 - 2i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{3 - 3i - 2i}{1 + 1} = \frac{1}{2} - \frac{5}{2}i$$

$$\text{or} \quad z = \frac{2 - 3i - 4 + i}{2 + 2i} = \frac{-2 - 2i}{2 + 2i} = \frac{-(2 + 2i)}{2 + 2i}$$

$$= -1$$

$$\therefore \quad z = -1, \frac{1}{2} - \frac{5}{2}i$$

Revision Exercise 1 (Extended-Response Questions)

1. $p = 3\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right) \qquad q = 2 - 2\sqrt{3}i$

$$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

(i) $pq = 3\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)(2 - 2\sqrt{3}i)$

$$= 3[\sqrt{3} - 3i + i - \sqrt{3}i^2]$$

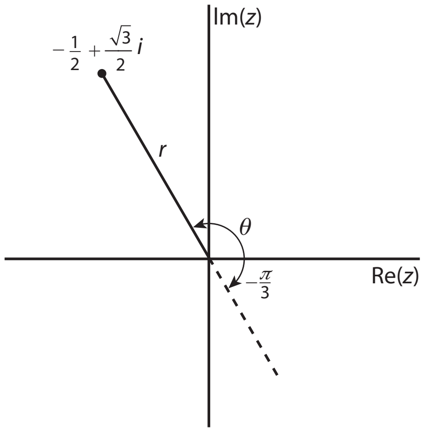
$$= 3[2\sqrt{3} - 2i]$$

$$= 6\sqrt{3} - 6i$$

(ii)
$$
\begin{aligned}
|p| &= \sqrt{9\left(\frac{\sqrt{3}}{2}\right)^2 + 9\left(\frac{1}{2}\right)^2}\\
&= \sqrt{9}\\
&= 3\\
|q| &= \sqrt{2^2 + (2\sqrt{3})^2}\\
&= \sqrt{4 + 12} = 4\\
|pq| &= \sqrt{(6\sqrt{3})^2 + 6^2}\\
&= \sqrt{144}\\
&= 12\\
p + q &= \frac{3\sqrt{3}}{2} + \frac{3}{2}i + 2 - 2\sqrt{3}i\\
&= \frac{3\sqrt{3}}{2} + 2 + i\left(\frac{3}{2} - 2\sqrt{3}\right)\\
|p + q| &= \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\right)^2 + \left(\frac{3}{2} - 2\sqrt{3}\right)^2}\\
&= \sqrt{\frac{27}{4} + \cancel{6\sqrt{3}} + 4 + \frac{9}{4} - \cancel{6\sqrt{3}} + 12}\\
&= \sqrt{25} = 5.
\end{aligned}
$$

2. (i)
$$
\begin{aligned}
z = \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}} &= \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}} \cdot \frac{1 + i\sqrt{3}}{1 + i\sqrt{3}}\\
&= \frac{1 + i\sqrt{3} + i\sqrt{3} - 3}{1 + \cancel{i\sqrt{3}} - \cancel{i\sqrt{3}} + 3}\\
&= \frac{-2 + 2\sqrt{3}i}{4}\\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\end{aligned}
$$

(ii)



(iii)
$$
\begin{aligned}
\Rightarrow \quad r = |z| &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\\
&= \sqrt{1}\\
&= 1\\
\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) &= \tan^{-1}(-\sqrt{3})\\
&= -60°\left(\frac{\pi}{3}\right) \quad \text{(calculator)}
\end{aligned}
$$

$$\Rightarrow \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore \quad z = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

(iv) $$\begin{aligned} z^3 &= \left[1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^3 \\ &= \cos 2\pi + i\sin 2\pi \\ &= \quad 1 \quad\quad + i(0) \qquad = 1. \end{aligned}$$

3. (i) $$\begin{aligned} z &= (1+3i)(p+qi) \\ &= p + qi + 3pi - 3q \qquad (i^2 = -1) \\ &= p - 3q + i(q+3p) \end{aligned}$$

$$\begin{aligned} &\arg z = \frac{\pi}{4} \\ \Rightarrow \quad & \theta = \frac{\pi}{4} \\ \Rightarrow \quad & \tan\theta = 1 \quad = \frac{q+3p}{p-3q} \\ \Rightarrow \quad & p - 3q = q + 3p \\ \Rightarrow \quad & 0 = 4q + 2p \\ \Rightarrow \quad & 0 = 2q + p \end{aligned}$$

(ii) $$\begin{aligned} & |z| = 10\sqrt{2} \\ & \Rightarrow |z| = \sqrt{(p-3q)^2 + (q+3p)^2} \qquad = 10\sqrt{2} \\ & \Rightarrow p^2 - \cancel{6pq} + 9q^2 + q^2 + \cancel{6pq} + 9p^2 = 200 \\ & \qquad\qquad 10p^2 + 10q^2 = 200 \\ & \qquad\qquad p^2 + q^2 = 20 \end{aligned}$$

$$\begin{aligned} \text{also,} \quad & p + 2q = 0 \\ \Rightarrow \quad & p = -2q \\ \Rightarrow \quad & p^2 = 4q^2 \end{aligned}$$

$$\begin{aligned} \therefore \quad 4q^2 + q^2 &= 20 \\ 5q^2 &= 20 \\ q^2 &= 4 \\ q &= \pm 2 \end{aligned}$$

$$\begin{aligned} &\text{when} \quad q = +2, \quad p = -4 \\ &\text{when} \quad q = -2, \quad p = +4 \\ &\text{but} \quad p > 0 \qquad \therefore p = 4, \quad q = -2. \end{aligned}$$

4. $z_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \quad z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(i) $$\begin{aligned} z_1z_2 &= 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdot\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ &= 3\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\right] \\ &= 3\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right] \qquad = [r(\cos\theta + i\sin\theta)] \\ |z_1z_2| &= \sqrt{9\left[\cos^2\left(\frac{5\pi}{12}\right) + \sin^2\left(\frac{5\pi}{12}\right)\right]} \qquad (\cos^2\theta + \sin^2\theta = 1) \\ &= \sqrt{9} \qquad (\text{Note: } r = 3) \\ &= 3 \end{aligned}$$

(ii) $$\arg(z_1z_2) = \frac{5\pi}{12}$$

(iii) $$\begin{aligned} |z_1|^2 &= \left(\sqrt{9\left(\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}\right)}\right)^2 \qquad (\cos^2\theta + \sin^2\theta = 1) \\ &= (\sqrt{9})^2 \qquad (\text{Note: } r = 9) \\ &= 9 \end{aligned}$$

(iv) $|z_2| = r = 1$

$|z_2|^2 = 1$

(v) $\arg\left(z_1^2\right) = \arg\left(3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^2$

$= \arg\left(9\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

(vi) $\arg\left(z_2^2\right) = \arg\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2$

$= \arg\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

(vii) $|zw| = |z||w|$

(a) Let $z = a + ib, \quad w = c + id$

$\Rightarrow \quad zw = (a+ib)(c+id) = ac + iad + ibc - bd \quad (i^2 = -1)$

$= ac - bd + i(ad + bc)$

$\Rightarrow \quad |z| = \sqrt{a^2 + b^2}$

also, $|w| = \sqrt{c^2 + d^2}$

and $|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$

$|zw| = \sqrt{a^2c^2 - \cancel{2acbd} + b^2d^2 + a^2d^2 + \cancel{2adbc} + b^2c^2}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$|z||w| = \sqrt{a^2 + b^2}\,.\sqrt{c^2 + d^2}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} = |zw|$

$|z||w| = |zw|$ is true for all $z, w \in C$.

(b) $\arg z = \tan^{-1}\left(\frac{b}{a}\right)$

$\arg w = \tan^{-1}\left(\frac{d}{c}\right)$

$\arg zw = \tan^{-1}\left(\frac{ad + bc}{ac - bd}\right)$

$\arg z + \arg w = \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right)$

Let $m = \tan^{-1}\left(\frac{b}{a}\right)$ and $n = \tan^{-1}\left(\frac{d}{c}\right)$

$\Rightarrow \tan m = \frac{b}{a}$ and $\tan n = \frac{d}{c}$

Consider $\tan(m + n) = \frac{\tan m + \tan n}{1 - \tan m \tan n} = \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a}\cdot\frac{d}{c}}$

$= \frac{\frac{bc + ad}{ac}}{\frac{ac - bd}{ac}} = \frac{bc + ad}{ac - bd} = \arg zw$

$\therefore \quad \arg(z) + \arg(w) = \arg(zw)$ is true

5. $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z^2 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^2$

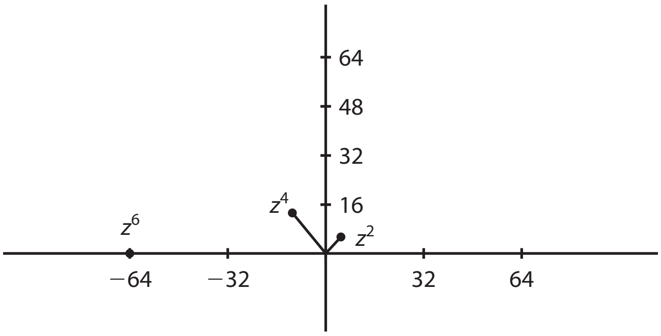
$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \qquad = 2 + 2\sqrt{3}i$

$z^4 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^4$

$= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad = -8 + 8\sqrt{3}i$

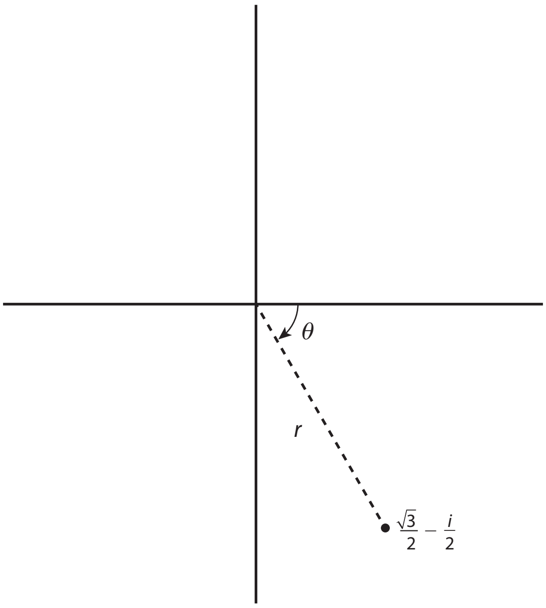
$$z^6 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^6$$
$$= 64(\cos\pi + i\sin\pi) \qquad = -64$$

(i)

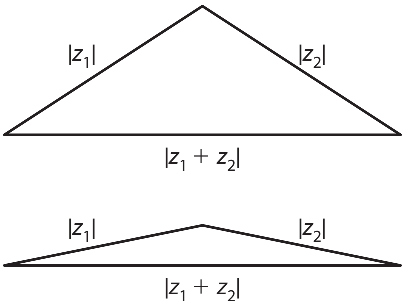


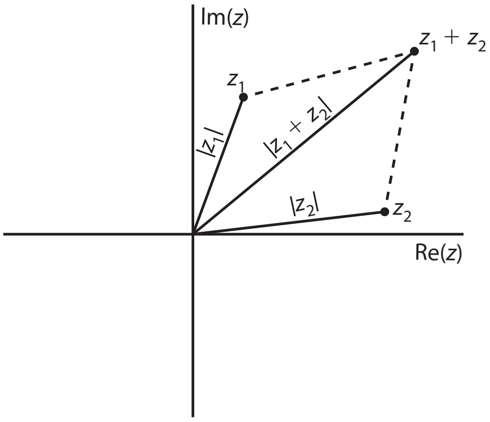
(ii) A rotation of $\frac{\pi}{3}$ and a stretching by a factor of 4.

6. $$\frac{\sqrt{3}+i}{1+\sqrt{3}i} = \frac{\sqrt{3}+i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}$$
$$= \frac{\sqrt{3}-3i+i-\sqrt{3}i^2}{1-\cancel{\sqrt{3}i}+\cancel{\sqrt{3}i}-3i^2}$$
$$= \frac{2\sqrt{3}-2i}{4} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$
$$z = \frac{\sqrt{3}+i}{1+\sqrt{3}i} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$
$$\Rightarrow \quad r = |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$
$$= 1$$
$$\theta = \tan^{-1} = \left(\frac{-\frac{-1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$= -30°\left(-\frac{\pi}{6}\right)$$
$$z = 1\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$z^6 = \left[1\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^6$$
$$= 1(\cos(-\pi) + i\sin(-\pi))$$
$$= 1((-1) + i(0)) = -1.$$



7.





For a triangle to form, $|z_1 + z_2| \leqslant |z_1| + |z_2|$.
If z_1 and z_2 lie on the same line through the origin, then $|z_1 + z_2| = |z_1| + |z_2|$.

8. $zw = 1$

(i) Let $z = a + bi$

$w = c + di$

$\Rightarrow \quad zw = (a + bi)(c + di) = 1$

$= ac + adi + bci - bd = 1 \quad [i^2 = -1]$

$\Rightarrow \quad ac - bd + i(ad + bc) = 1 + 0i$

$\therefore \quad ac - bd = 1$ and $ad + bc = 0.$

(ii) To find a in terms of c and d, eliminate b

$$ac - bd = 1$$

$$\Rightarrow \quad bd = ac - 1$$

$$b = \frac{ac - 1}{d}$$

$\Rightarrow \quad ad + bc = 0$ becomes

$$ad + \left(\frac{ac - 1}{d}\right) \cdot c = 0$$

$$ad^2 + ac^2 - c = 0$$

$$a(d^2 + c^2) - c = 0$$

$$a = \frac{c}{d^2 + c^2}$$

To find b in terms of c and d, eliminate a

$$ac - bd = 1$$

$$ac = 1 + bd$$

$$a = \frac{1 + bd}{c}$$

$ad + bc = 0$ becomes

$$\frac{(1 + bd)d}{c} + bc = 0$$

$$d + bd^2 + bc^2 = 0$$

$$b(d^2 + c^2) = -d$$

$$b = \frac{-d}{d^2 + c^2}$$

(iii) $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}, \quad z = a + bi, \quad \bar{z} = a - bi$

$$\frac{1}{z} = \frac{1}{a + bi} = \frac{1}{a + bi}\left(\frac{a - bi}{a - bi}\right) = \frac{a - bi}{a^2 + b^2}$$

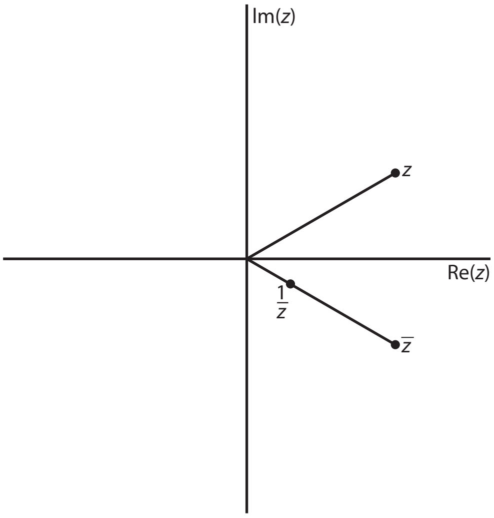
$$= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

$$|z| = \sqrt{a^2 + b^2}$$

$\Rightarrow \quad |z|^2 = a^2 + b^2$

$\Rightarrow \quad \dfrac{\bar{z}}{|z|^2} = \dfrac{a - bi}{a^2 + b^2} = \dfrac{1}{z} \quad \ldots$ True

(iv)



(v) slope of $\bar{z} = \frac{-b}{a}$

slope of $\frac{1}{z} = \frac{\frac{-b}{a^2+b^2}}{\frac{a}{a^2+b^2}} = -\frac{b}{a}$

$\therefore$ $\bar{z}$ and $\frac{1}{z}$ are always collinear with $0 + 0i$.

9. (i) The conjugate of a sum of complex numbers

= the sum of the conjugates

Let $z = a + ib$ and $w = c + id$

$\Rightarrow$ $\bar{z} = a - ib$ and $\bar{w} = c - id$

also, $z + w = (a + c) - i(b + d)$

$\therefore$ $(\overline{z + w}) = (a + c) - i(b + d)$

and $\bar{z} + \bar{w} = a - ib + c - id$

$= (a + c) - i(b + d) = \overline{zw}$ … True

(ii) The conjugate of the difference of complex numbers

= the difference of the conjugates

Let $z = a + ib$ and $w = c + id$

$\Rightarrow$ $\bar{z} = a - ib$ and $\bar{w} = c - id$

also, $z - w = a + ib - (c + id)$

$= a - c + i(b - d)$

$\Rightarrow$ $\overline{z - w} = (a - c) - i(b - d)$

$\bar{z} - \bar{w} = a - ib - (c - id)$

$= a - c - i(b - d) = \overline{z - w}$ … True

(iii) The conjugate of a quotient of complex numbers

= the quotient of the conjugates

Let $z = a + ib$ and $w = c + id$

$\Rightarrow$ $\bar{z} = a - ib$ and $\bar{w} = c - id$

also, $\frac{z}{w} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \left(\frac{c - id}{c - id}\right)$

$= \frac{ac - iad + ibc + bd}{c^2 + d^2}$ $(i^2 = -1)$

$\left(\frac{z}{w}\right) = \frac{ac + bd - i(ad - bc)}{c^2 + d^2}$

$\left(\frac{\bar{z}}{w}\right) = \frac{ac + bd + i(ad - bc)}{c^2 + d^2}$

$\frac{\bar{z}}{\bar{w}} = \frac{a - ib}{c - id} = \frac{a - ib}{c - id} \cdot \frac{c + id}{c + id}$

$\frac{\bar{z}}{\bar{w}} = \frac{ac + iad - ibc + bd}{c^2 + d^2}$

$= \frac{ac + db + i(ad - bc)}{c^2 + d^2} = \left(\frac{\bar{z}}{w}\right)$ … True

(iv) The conjugate of a product of complex numbers

= the product of the conjugates

Let $z = a + ib$ and $w = c + id$

$\Rightarrow$ $\bar{z} = a - ib$ and $\bar{w} = c - id$

also, $zw = (a + ib)(c + id)$

$= ac + iad + ibc - bd$ $(i^2 = -1)$

$zw = ac - bd + i(ad + bc)$

$\Rightarrow$ $\overline{zw} = ac - bd - i(ad + bc)$

$\therefore$ $\bar{z}\bar{w} = (a - ib)(c - id)$

$= ac - iad - ibc - bd$

$= ac - bd - i(ad + bc) = \overline{zw}$ … True

10. (a) $w = -1 + \sqrt{3}i$

(i) $r = |w| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

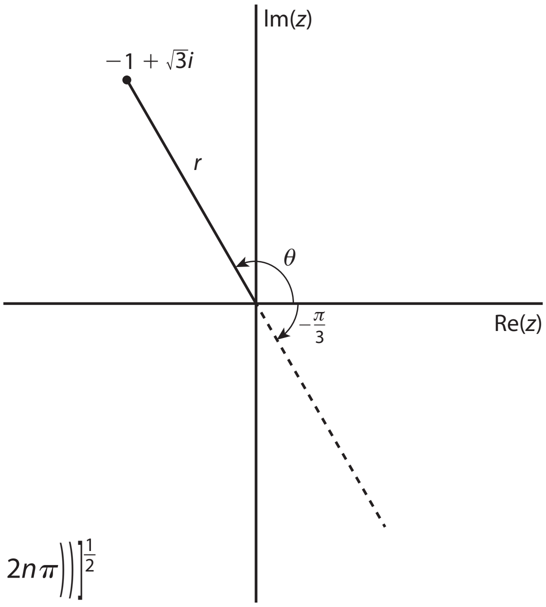
$= 2$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3})$$

$$= -60°\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$



(ii) $z^2 = -1 + \sqrt{3}i$

$$\Rightarrow z = (-1 + \sqrt{3}i)^{\frac{1}{2}}$$

$$= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{\frac{1}{2}}$$

$$= \left[2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}}\left(\cos\left(\frac{\frac{2\pi}{3} + 2n\pi}{2}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2n\pi}{2}\right)\right)$$

let $n = 0$. $= 2^{\frac{1}{2}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2^{\frac{1}{2}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$$

let $n = 1$. $= 2^{\frac{1}{2}}\left(\cos\left(\frac{\frac{2\pi}{3} + 2\pi}{2}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi}{2}\right)\right)$

$$= 2^{\frac{1}{2}}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = 2^{\frac{1}{2}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

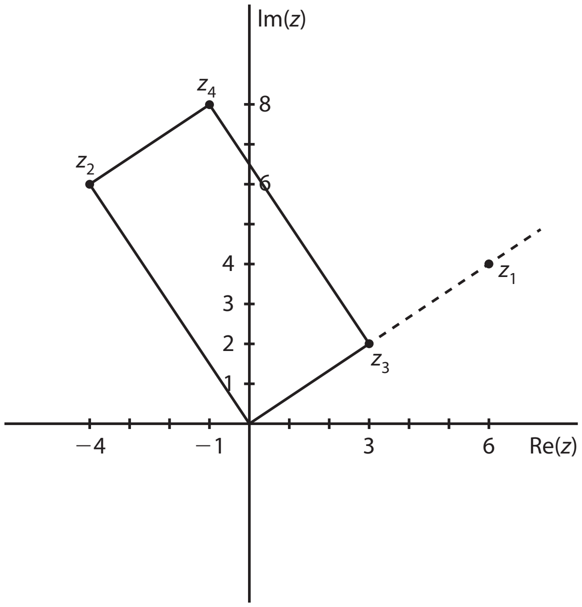
$$= \frac{-\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

(b) $z_2 = iz_1$

$z_3 = kz_1, k \in R$

$z_4 = z_2 + z_3$

(i)



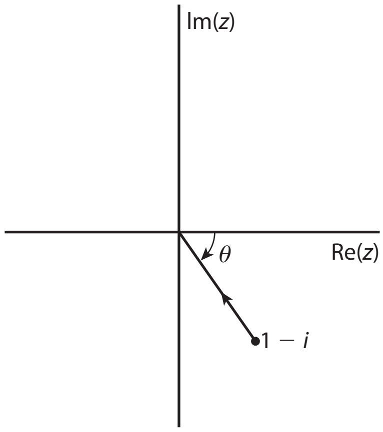
(ii) $k = \frac{1}{2}$

(iii) $z_3 = kz_1, k \in R$

meant that z_3 and z_1 were collinear with the origin.

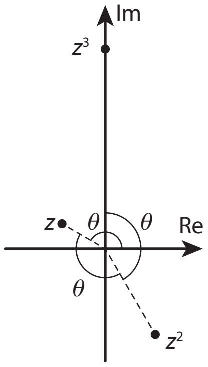
11. (a) (i) $1 - i$

$$r = |z| = \sqrt{1^2 + (-1)^2}$$
$$= \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1)$$
$$= -45°\left(-\frac{\pi}{4}\right)$$
$$\therefore 1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$



(ii) $$(1 - i)^9 = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^9$$
$$= 2^{\frac{9}{2}}\left(\cos\left(\frac{-9\pi}{4}\right) + i\sin\left(\frac{-9\pi}{4}\right)\right)$$
$$= 2^{\frac{9}{2}}\left(\frac{1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right)$$
$$= 2^4 - 2^4 i$$
$$= 16 - 16i$$

(b) (i) The points are labelled opposite.
(The reason why is given in part (iii) below.)



(ii) Let $z = r(\cos\theta + i\sin\theta)$.

Then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$.

From the diagram,

$$3\theta = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}$$
$$\theta = \frac{5\pi}{6} = 150°.$$

(iii) As $|z| = r > 1$,

$$|z^2| = r^2 > r$$

and $|z^3| = r^3 > r^2$,

z, z^2 and z^3 spiral away from the origin. Thus z is the point nearest the origin, z^2 is the next nearest, and z^3 is the point furthest away from the origin.

(c) (i) $z = a + ai \quad a > 1, \quad a \in R$.

$$z^2 = (a + ai)(a + ai)$$
$$= \cancel{a^2} + a^2i + a^2i - \cancel{a^2} \quad (i^2 = -1)$$
$$z^2 = 2a^2i$$

$$z^4 = (2a^2i)(2a^2i)$$
$$= -4a^4 \quad (i^2 = -1)$$

$$z^6 = z^4 . z^2$$
$$= (-4a^4)(2a^2i)$$
$$= -8a^6i$$

(ii) The points spiral away from the origin and are restricted to the Real and Imaginary axes.

12. (i) $z = \cos\theta + i\sin\theta$

$$z^k = (\cos\theta + i\sin\theta)^k$$
$$= \cos k\theta + i\sin k\theta$$

(ii) $$\frac{1}{z^k} = \frac{1}{\cos k\theta + i\sin k\theta}$$
$$= \frac{1}{\cos k\theta + i\sin k\theta} \cdot \frac{\cos k\theta - i\sin k\theta}{\cos k\theta - i\sin k\theta}$$
$$= \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta - i\sin^2 k\theta}$$
$$= \cos k\theta - i\sin k\theta \qquad (\cos^2\theta + \sin^2\theta = 1)$$

(iii) $z^k = \cos k\theta + i\sin k\theta \ldots A$

$z^{-k} = \cos k\theta - i\sin k\theta \ldots B$

$z^k + z^{-k} = 2\cos k\theta$ …. adding A and B

$$\Rightarrow \quad \cos k\theta = \frac{z^k + z^{-k}}{2}$$

$z^k - z^{-k} = 2i\sin k\theta$ …. subtracting B and A

$$\Rightarrow \sin k\theta = \frac{z^k - z^{-k}}{2i}$$

(iv) $$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\Rightarrow \quad \sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$
$$\sin^2\theta\cos^2\theta = \left(\frac{\sin 2\theta}{2}\right)^2$$

Let $k = 2 \Rightarrow \sin 2\theta = \dfrac{z^2 - z^{-2}}{2i}$

$$\Rightarrow \sin 2\theta = \frac{z^2 - \frac{1}{z^2}}{2i}$$

$$\sin^2\theta\cos^2\theta = \left(\frac{\sin 2\theta}{2}\right)^2 = \left(\frac{z^2 - \frac{1}{z^2}}{4i}\right)^2$$
$$= \frac{-1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$$

(v) $$\sin^2\theta\cos^2\theta = \frac{-1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$$
$$= \frac{-1}{16}\left(z^4 - 2\left(\frac{z^2}{z^2}\right) + \frac{1}{z^4}\right)$$
$$= \frac{-1}{16}\left(z^4 + \frac{1}{z^4} - 2\right)$$

But $\qquad \cos k\theta = \dfrac{z^k + z^{-k}}{2}$

Let $k = 4$, $\quad \cos 4\theta = \dfrac{z^4 + z^{-4}}{2}$

$$\Rightarrow \quad 2\cos 4\theta = z^4 + z^{-4}$$

$$\therefore \sin^2\theta\cos^2\theta = -\frac{1}{16}(2\cos 4\theta - 2)$$
$$= -\frac{1}{8}\cos 4\theta + \frac{1}{8}$$

$$\Rightarrow \quad a = \frac{1}{8}, \quad b = -\frac{1}{8}$$

Chapter 2

Exercise 2.1

1. (i) Scale factor $= 2$

(ii) $x = \frac{12}{2} = 6$ cm

$y = 2(9) = 18$ cm

(iii) Each small square has length of side $= 3$ cm

$\Rightarrow$ Area each small square $= (3)(3) = 9\text{ cm}^2$

Area object $= 5(9) = 45\text{ cm}^2$

Area image $= 20(9) = 180\text{ cm}^2 = 4(45)\text{ cm}^2$

$\Rightarrow$ Scale factor: $k = 2$

Area (image) $= 2^2 = 4$ times area (object) $= k^2$

2. Scale factor $= 2$

(i) $|BC| = 4 \Rightarrow |B'C'| = 2(4) = 8$

(ii) $|AC| = 6 \Rightarrow |A'C'| = 2(6) = 2$

(iii) $|A'B'| = 10 \Rightarrow |AB| = \frac{1}{2}(10) = 5$

Area $\triangle A'B'C' = 30$ sq. units

$\Rightarrow$ Area $\triangle ABC = \frac{30}{4} = 7\frac{1}{2}$ sq. units

3. (i) Scale factor $= 2$

(ii) Centre of enlargement is found by joining 2 sets of corresponding points and continuing the lines until they meet.

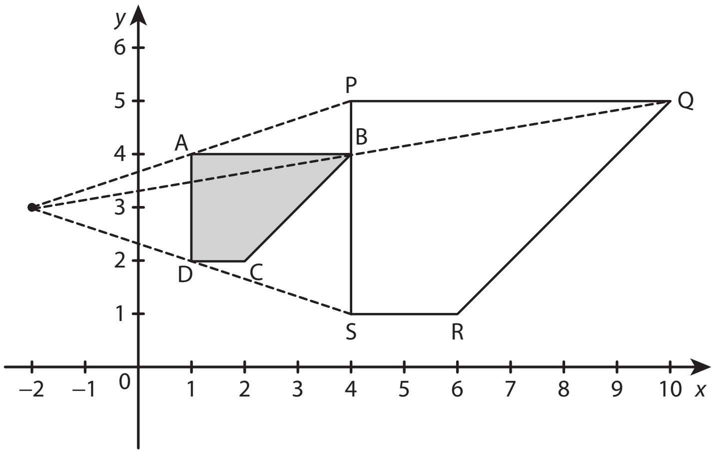
(iii) $(-2, 3)$

(iv) Area ABCD $= 30$ sq. units

$\Rightarrow$ Area PQRS $= (2^2)(30)$

$= (4)(30)$

$= 120$ sq. units



4. Scale factor $= 1\frac{1}{2}$

(i) $|AC| = 8 \Rightarrow |AC'| = (8)\left(1\frac{1}{2}\right) = 12$

(ii) $|B'C'| = 9 \Rightarrow |BC| = 9 \div 1\frac{1}{2} = 6$

(iii) $|AB| = x \Rightarrow |AB'| = x + 3$

$\Rightarrow \frac{|AB'|}{|AB|} = \frac{x+3}{x} = 1\frac{1}{2} \Rightarrow \frac{x+3}{x} = \frac{3}{2}$

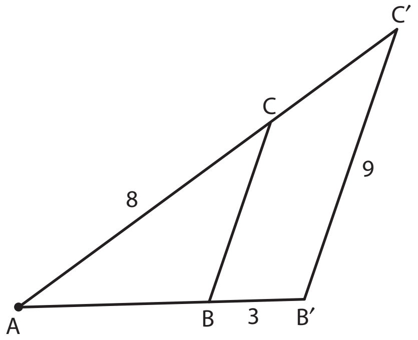
$\Rightarrow 3x = 2x + 6$

$\Rightarrow x = 6 \Rightarrow |AB| = 6$

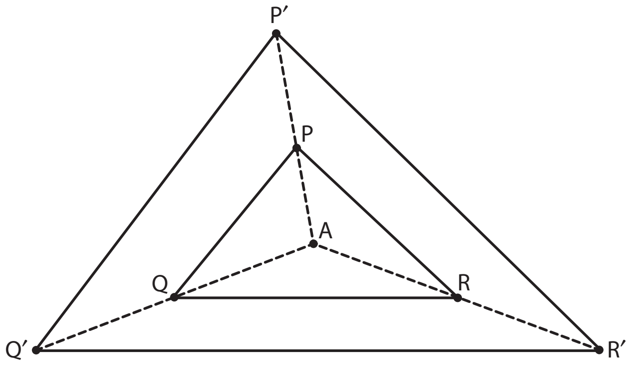
Scale factor $k = 1\frac{1}{2} = \frac{3}{2} \Rightarrow k^2 = \frac{9}{4}$

Area $\triangle ABC = 20$ sq. units

$\Rightarrow$ Area $\triangle AB'C' = (20)\left(\frac{9}{4}\right) = 45$ sq. units



5.



6. (i) Scale factor $= \frac{12.8}{3.2} = 4$

(ii) $|XZ| = 4.1$ cm $\Rightarrow |X'Z'| = (4.1)(4) = 16.4$ cm

(iii) $|X'Z'| = 12$ cm $\Rightarrow |XY| = \frac{12}{4} = 3$ cm

(iv) $|OZ'| = 4|OZ|$

$\Rightarrow |ZZ'| = 3|OZ|$

$\Rightarrow |OZ| : |ZZ'| = 1:3$

(v) $k = 4 \Rightarrow k^2 = 16$

Area $\triangle X'Y'Z' = 64$ cm^2

$\Rightarrow$ Area $\triangle XYZ = \frac{64}{16} = 4$ cm^2

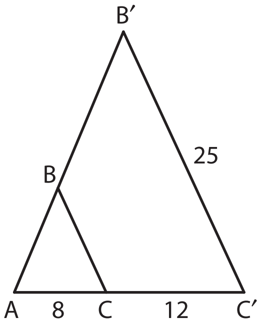
7. (i) Scale factor $= \frac{|AC'|}{|AC|} = \frac{20}{8} = 2\frac{1}{2}$

(ii) $|B'C'| = 25 \Rightarrow |BC| = 25 \div 2\frac{1}{2} = 10$

(iii) $|AB'| = 2\frac{1}{2}|AB| \Rightarrow |AB| : |AB'| = 1 : 2\frac{1}{2} = 2:5$

(iv) $k = 2\frac{1}{2} \quad \Rightarrow k^2 = \left(2\frac{1}{2}\right)^2 = 6\frac{1}{4}$

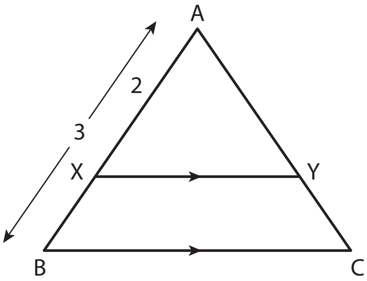
Area $\triangle ABC = 16$ sq. units $\Rightarrow$ Area $\triangle AB'C' = (16)\left(6\frac{1}{4}\right) = 100$ sq. units



8. Scale factor $= \frac{|AB|}{|AX|} = \frac{3}{2} = k \Rightarrow k^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Area $\triangle AXY = 4$ cm^2

$\Rightarrow$ Area $\triangle ABC = (4)\left(\frac{9}{4}\right) = 9$ cm^2



9. Scale factor $= k = 2.5 \Rightarrow k^2 = (2.5)^2 = 6.25$

Area design $= 176$ cm^2

$\Rightarrow$ Area completed mosaic $= (176)(6.25) = 1100$ cm^2

10. $k^2 = 4 \Rightarrow$ scale factor $k = \sqrt{4} = 2$

$k^2 = 2 \Rightarrow$ scale factor $k = \sqrt{2}$

11. Scale factor $k = \frac{12}{8} = 1.5 \quad \Rightarrow k^2 = (1.5)^2 = 2.25$

Area larger shape $= 27$ cm^2

$\Rightarrow$ Area smaller shape A $= \frac{27}{2.25} = 12$ cm^2

12. Scale factor $k = \frac{2}{3}$

(i) Original height $= 156$ mm

$\Rightarrow$ Reduced height $= (156)\left(\frac{2}{3}\right) = 104$ mm

(ii) Reduced label height $= 28$ mm

$\Rightarrow$ Original label height $= 28 \div \frac{2}{3} = 42$ mm

13. Scale factor $k = \frac{300 \text{ m}}{15 \text{ cm}} = \frac{3000 \text{ m}}{15 \text{ cm}} = 2000$

$\Rightarrow k^2 = (2000)^2 = 4\,000\,000$

Model Area $= 25.5$ cm^2

$\Rightarrow$ Tower Area $= (25.5)(4\,000\,000) = 102\,000\,000$ cm^2

$= 10\,200$ m^2

14. Scale factor $k = 25 \Rightarrow k^2 = (25)^2 = 625$

(i) Plan pond area $= 24$ cm^2

$\Rightarrow$ Real pond area $= (24)(625) = 15\,000$ cm^2

$= 1.5$ m^2

(ii) Real lawn area $= 17$ m^2 $= 170\,000$ cm^2

$\Rightarrow$ Plan lawn area $= \frac{170\,000}{625} = 272$ cm^2

15. Scale factor $= k = 2$

(i) Map scale is 1:1000

Enlarged map scale 1:500

(ii) Street length $= 6$ cm

Real-life street length $= (6)(1000)$

$= 6000$ cm $= 60$ m

(iii) Scale factor $= k = \frac{1}{2}$

Map scale is 1:1000

Sean's enlarged map scale is 1:2000

(iv) Distance railway stations $= 1$ km $= 100\,000$ cm

Distance on sean's enlarged map $= \frac{100\,000}{2000} = 50$ cm

Exercise 2.2

1. (i) Construct the parallelogram.

(ii) Yes, the diagonals bisect each other.

(iii) Given : Parallelogram ABCD with diagonals [AC] and [DB] meeting at X.

To prove : |DX| = |XB|

Proof : In △s DXC, AXB;

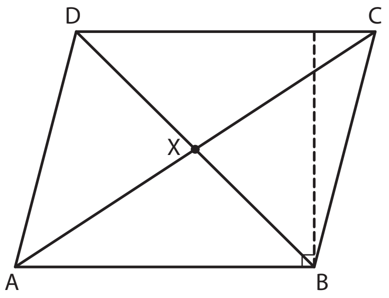
|∠XDC| = |∠XBA| (alternate angles)

|DC| = |AB| (opposite sides of parallelogram)

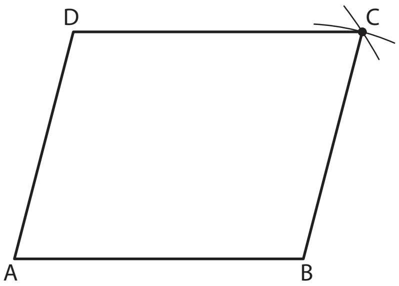
|∠XCD| = |∠XAB| (alternate angles)

$\Rightarrow$ △s are congruent by A.S.A.

$\Rightarrow$ |DX| = |XB|



2. $|\angle ABC| = 105°$



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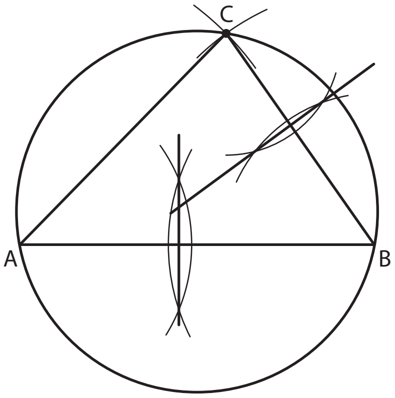

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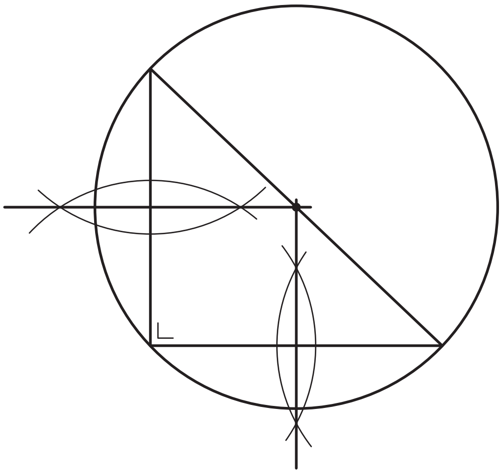
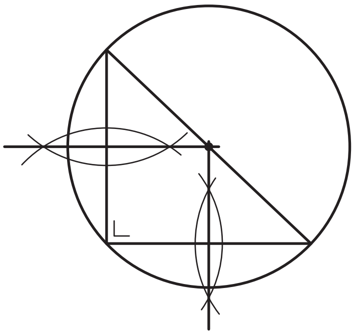
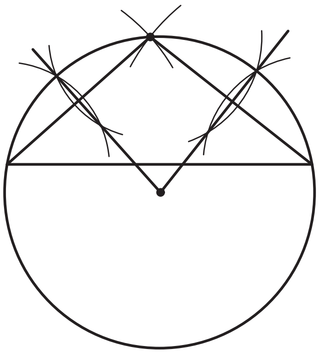


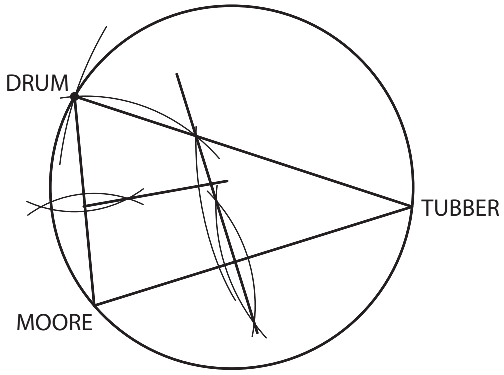


Yes : you get the same result each time.
⇒ Circumcentre is the midpoint of the hypotenuse.

5. In an obtuse-angled triangle, the circumcentre is outside the triangle.

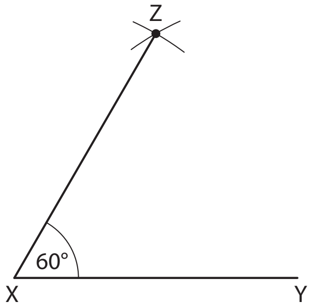


6.



School should be built at the circumcentre of the triangle.

7.



Draw a line segment [XY].

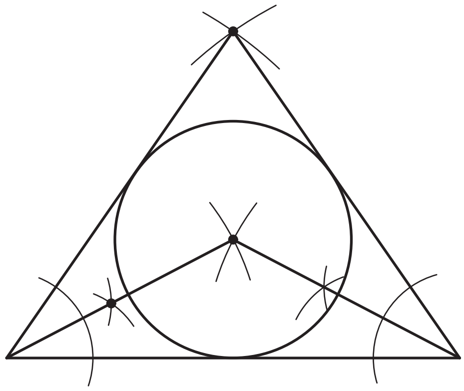
Set the compass to a radius of [XY] with X as centre and radius [XY].

Draw an arc.

Repeat at Y.

The arcs meet at Z. Join XZ ⇒ $|\angle ZXY| = 60°$

8.



9. (i) Area $\triangle BOC = \frac{1}{2}ar$

(ii) Area $\triangle ABC$ = Area $\triangle BOC$ + Area $\triangle AOC$ + Area $\triangle BOA$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a + b + c)$$

10. $|\angle G| + |\angle E| = |\angle H| + |\angle F| = 90°$

Since $|\angle G| = |\angle H| \Rightarrow |\angle E| = |\angle F|$

In $\triangle$s XKZ, XKY

$|\angle E| = |\angle F|$

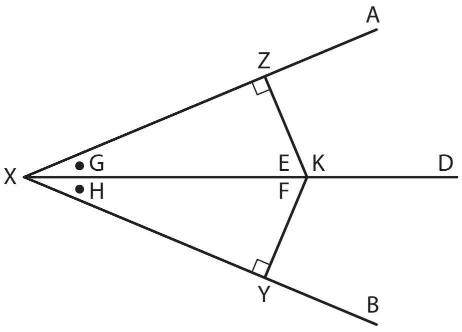
$|XK| = |XK|$

$|\angle G| = |\angle H|$

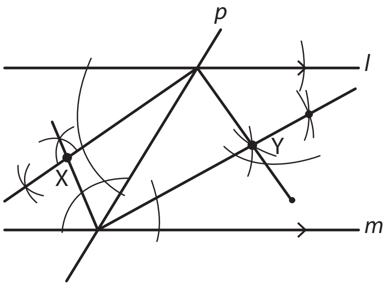
⇒ $\triangle$s are congruent by A.S.A.

⇒ $|KZ| = |KY|$

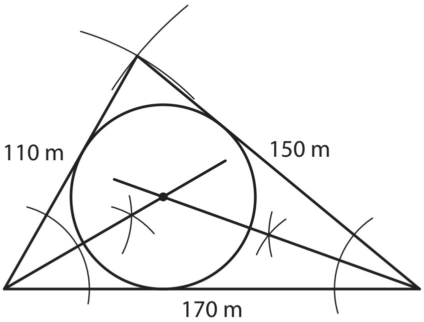
Conclusion: Any point on the bisector of an angle is equidistant from the arms of the angle.



11. Construct the bisectors of all 4 angles to locate X and Y.



12. (i) Using 20 m = 1 cm, draw a scaled diagram.

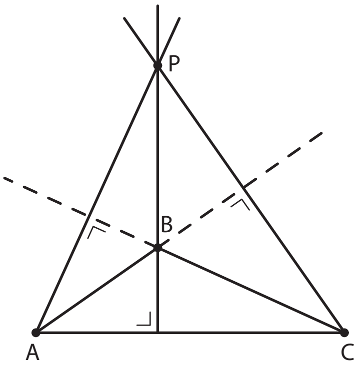


(ii) Best position to pitch a tent is the incentre.

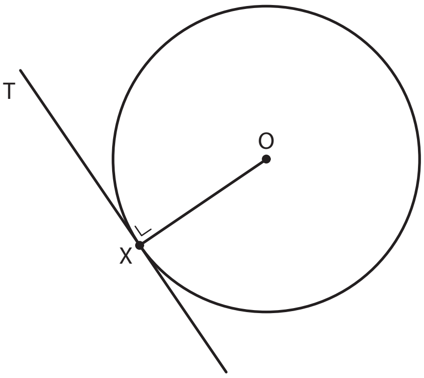
13. (i) Radius $= \frac{1}{2}(10) = 5$ cm

(ii) B is the point of intersection of all 3 altitudes (i.e. the orthocentre) of △ABC.

14. Orthocentre P is outside the triangle ABC.
Result will hold for all obtuse-angled triangles.

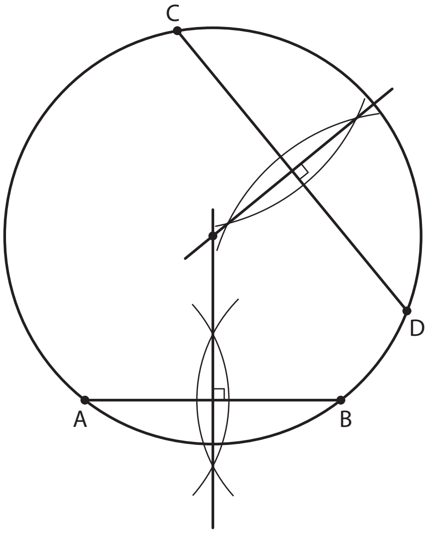


15. Tangent XT is perpendicular to radius OX.



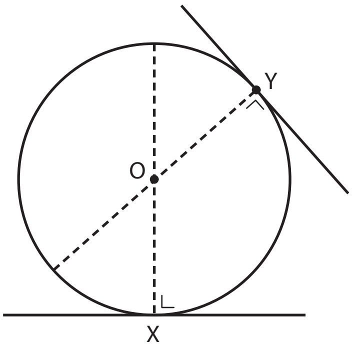
16. Perpendicular bisector of [AB] is equidistant from the end-points A and B.

Point of intersection of the perpendicular bisectors of the two chords is the centre of the circle.



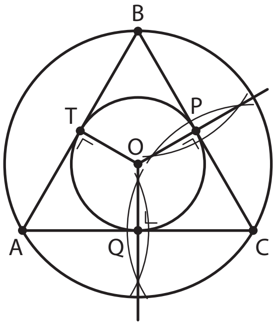
17. Draw tangents at both points X and Y.

The perpendicular lines to both tangents are diameters which meet at point O, the centre of the circle.



18. (i) Construct the perpendicular bisectors of [AC] and [BC] to meet at the point O, the circumcentre. Hence, $|OA| = |OB| = |OC|$.

(ii) In an equilateral; triangle, the bisectors of the angles at A, B, C also meet at the point O. Hence, $|OP| = |OQ| = |OT|$ = radius of incircle.



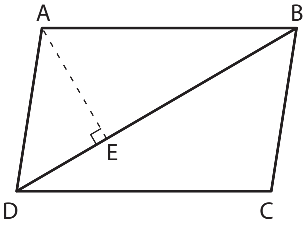
Revision Exercise 2 (Core)

1. Area parallelogram $= 2$ Area $\Delta ADB = 40$

$$= 2 \cdot \frac{1}{2}|DB| \cdot |AE| = 40$$

$$= 15 \,.\, |AE| = 40$$

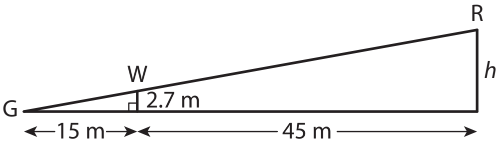
$$\Rightarrow |AE| = \frac{40}{15} = 2\frac{2}{3} \text{ cm}$$



2. $\text{Tan } \angle G = \frac{2.7}{15} = \frac{h}{60}$

$$\Rightarrow 15h = (60)(2.7) = 162$$

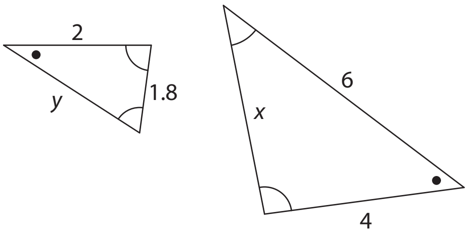
$$\Rightarrow \quad h = \frac{162}{15} = 10.8 \text{ m}$$



3. (i) Equal angles are marked on the diagram

$\Rightarrow$ the two triangles are equiangular (i.e. all the same size but not necessarily all 60°).

$\Rightarrow$ the two triangles are similar. (Not congruent as the corresponding sides differ in length).



(ii) $\frac{x}{1.8} = \frac{4}{2} = \frac{6}{y}$

$\frac{x}{1.8} = \frac{4}{2}$ and $\frac{4}{2} = \frac{6}{y}$

$\Rightarrow 2x = 7.2$ $\qquad \Rightarrow 4y = 12$

$\Rightarrow \quad x = \frac{7.2}{2} = 3.6 \quad \Rightarrow \quad y = \frac{12}{4} = 3$

4. (i) As $|DF| = \frac{1}{2}|DB|$, the scale factor for EFG is $\frac{1}{2}$.

As $|DI| = \frac{3}{2}|DB|$, the scale factor for HIJ is $\frac{3}{2}$.

(ii) The area of the triangle ABC is $\frac{1}{2}(3)(4) = 6$.

Hence the area of the triangle EFG is $\left(\frac{1}{2}\right)^2 \times 6 = \frac{3}{2}$.

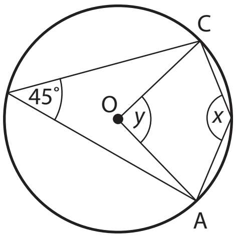
Hence the area of the triangle HIJ is $\left(\frac{3}{2}\right)^2 \times 6 = \frac{27}{2}$.

5. (i) Cyclic quadrilateral $\Rightarrow \angle x + 45° = 180°$

$\Rightarrow \angle x = 180° - 45° = 135°$

$\angle y = 2(45°) = 90°$

(ii) $\angle y = 90° \Rightarrow$ [AC] is the diameter of the circumcircle.
Hence, a circle drawn on [AC] as diameter
must pass through O.

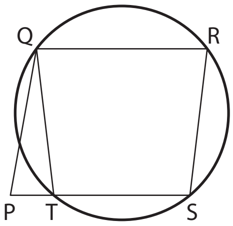


6. (i) False as QT is not parallel to RS.

(ii) False as PQRS is not a rectangle.

(iii) True as opposite angles of a cyclic quadrilateral add to 180°.

(iv) False as PQRS is not a rectangle.



7. (i) Point of intersection of the perpendicular bisectors of [AB] and [BC].

(ii) $2\angle a + 2\angle b + 70° = 180°$

$\Rightarrow 2\angle a + 2\angle b = 180° - 70° = 110°$

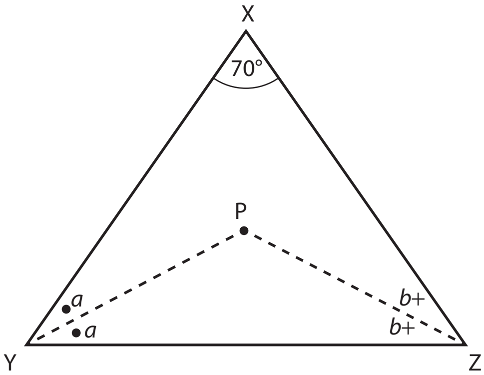
$\Rightarrow \angle a + \angle b = \frac{110°}{2} = 55°$

$\Rightarrow |\angle YPZ| + \angle a + \angle b = 180°$

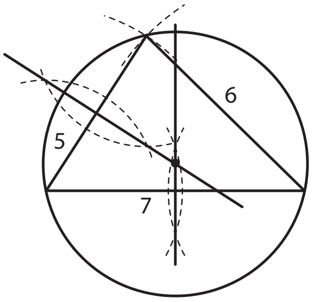
$\Rightarrow |\angle YPZ| + 55° = 180°$

$\Rightarrow |\angle YPZ| = 180° - 55° = 125°$

(iii) Yes because P is the point of intersection of the Bisectors of the angles ∠XYZ and ∠XZY.



8. The construction of the circumcircle is shown below.

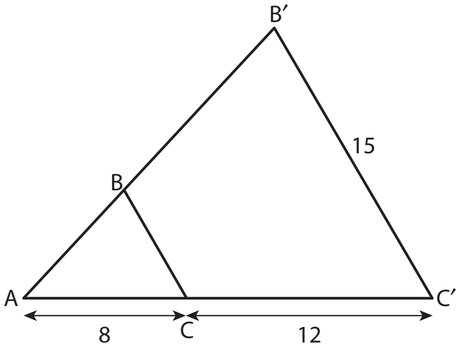


9. (i) Scale factor $= \frac{20}{8} = 2.5$

(ii) $\frac{|BC|}{15} = \frac{8}{20} \Rightarrow 20|BC| = 120$

$\Rightarrow \quad |BC| = \frac{120}{20} = 6$

(iii) $|AB| : |AB'| = 2:5$



10. Let x be the width of the picture.

Then $\frac{4x}{3}$ is the height of the picture.

Area $= 192$

$$(x)\left(\frac{4x}{3}\right) = 192$$

$$x^2 = 144$$

$$x = 12$$

Thus the width of the picture is 12 cm and its height is 16 cm.

Revision Exercises 2 (Advanced)

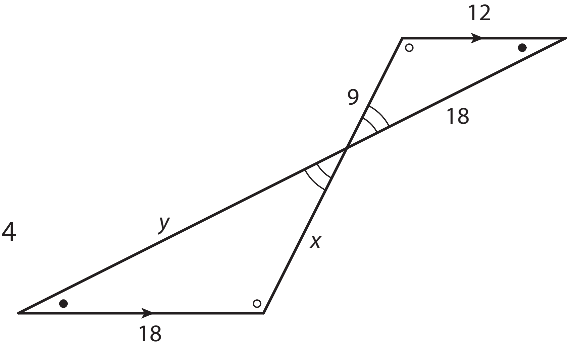
1. Similar Triangles with equal angles marked.

$\Rightarrow \frac{x}{9} = \frac{18}{12} = \frac{y}{18}$

$\Rightarrow \frac{x}{9} = \frac{18}{12}$ and $\frac{18}{12} = \frac{y}{18}$

$\Rightarrow 12x = (9)(18) = 162$ $\quad \Rightarrow 12y = (18)(18) = 324$

$\Rightarrow x = \frac{162}{12} = 13.5$ $\quad \Rightarrow y = \frac{324}{12} = 27$



2. (i) Parallel lines $\Rightarrow \left|\frac{AB}{BC}\right| = \left|\frac{DE}{CD}\right| = \frac{7}{7} = 1 \quad \Rightarrow |DE| = |CD| = 8$ cm

(ii) $\triangle$s BCD and ACE are similar $\Rightarrow \frac{|BC|}{|AC|} = \frac{|BD|}{|AE|}$

$\Rightarrow \frac{7}{14} = \frac{|BD|}{12} \Rightarrow 14|BD| = 84$

$\Rightarrow |BD| = \frac{84}{14} = 6$ cm

(iii) $\triangle$ACE is an enlargement of $\triangle$BCD as the triangles are similar and they share a common vertex.

The centre of enlargement is C, as this point remains fixed.

The scale factor is $\frac{|CA|}{|CB|} = \frac{14}{7} = 2$.

3. Area $\triangle BIC = \frac{1}{2}(6)(3) = 9 \text{ cm}^2$

Area $\triangle ABC$ = Area $\triangle BIC$ + Area $\triangle AIC$ + Area $\triangle AIB$

$$= \frac{1}{2}(6)(3) + \frac{1}{2}(4)(3) + \frac{1}{2}(5)(3)$$

$$= 9 + 6 + 7.5 = 22.5 \text{ cm}^2$$

4. (i) Scale factor $= \frac{4.2}{3.5} = 1.2$

(ii) $a = (1.2)(4.5) = 5.4$ cm

$b = \frac{7.2}{1.2} = 6$ cm

(iii) Let x be the length of a side of the image square.

The area of this square is x^2, and its perimeter is $4x$. If these are equal, then

$x^2 = 4x$

$x = 4$

The given square has a side of length $\sqrt{25} = 5$.

Thus the scale factor is $\frac{4}{5}$.

5. (i) $|AB| = 2|BD|$ and $|AD| = 3|BD| = 36$

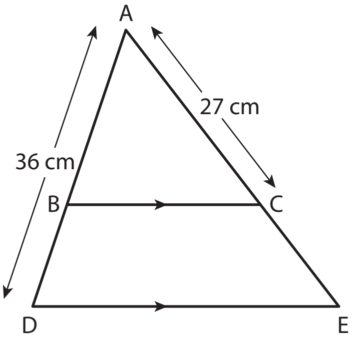
$\Rightarrow |BD| = \frac{36}{3} = 12$

$\Rightarrow |AB| = 2(12) = 24$ cm

(ii) $|AC| = 2|CE| = 27$

$\Rightarrow |CE| = \frac{27}{2} = 13.5$

$\Rightarrow |AE| = 3|CE| = 3(13.5) = 40.5$ cm



6. $|AB| = 6 \Rightarrow |BF| = \frac{1}{2}(6) = 3$ cm

Find radius $|OB|$ using pythagoras' theorem

$\Rightarrow r^2 = (3)^2 + (4)^2 = 9 + 16 = 25$

$\Rightarrow r = \sqrt{25} = 5$

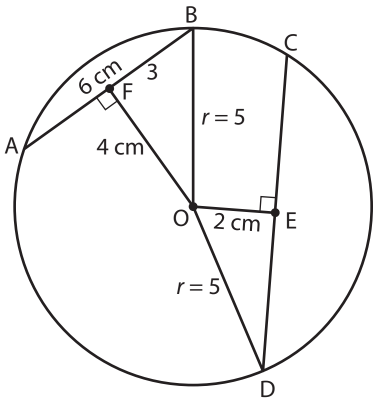
$|OD| = 5 \Rightarrow |DE|^2 + (2)^2 = 5^2$

$\Rightarrow |DE|^2 + 4 = 25$

$\Rightarrow |DE|^2 = 25 - 4 = 21$

$\Rightarrow |DE| = \sqrt{21}$

$\Rightarrow |DC| = 2|DE| = 2\sqrt{21}$



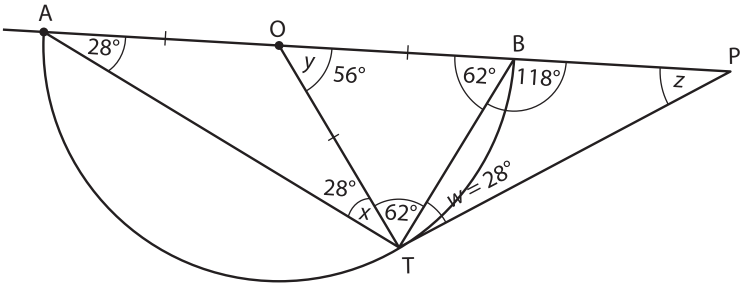
7. $|OA| = |OT| = \Rightarrow \angle x = 28°$

$\Rightarrow \angle y = 28° + \angle x = 28° + 28° = 56°$

$\angle w = 28°$

$\Rightarrow \angle z + 28° + 118° = 180°$

$\Rightarrow \angle z = 180° - 146° = 34°$



8. (i) Scale factor $= k$ $\Rightarrow k^2 = \dfrac{500}{20} = 25$

$\Rightarrow k = \sqrt{25} = 5$

Radius of small ball $= r$ cm

$\Rightarrow$ Radius of large ball $= 5r$ cm

(ii) $x^2 + (2x)^2 = (25)^2$

$\Rightarrow x^2 + 4x^2 = 5x^2 = 625$

$\Rightarrow x^2 = \dfrac{625}{5} = 125$ $\Rightarrow x = \sqrt{125} = 5\sqrt{5}$

$\Rightarrow 2x = 10\sqrt{5}$

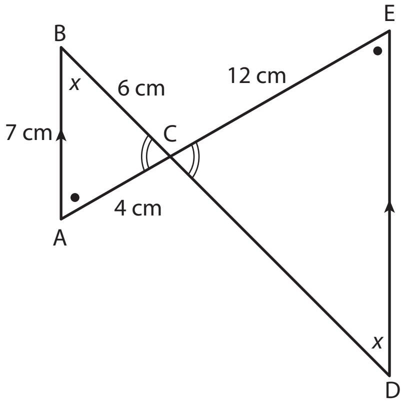
Area rectangle $= (2x)(x) = (10\sqrt{5})(5\sqrt{5}) = 250\text{ cm}^2$

9. (i) Triangles ABC, CDE are equiangular with equal angles marked,

$\Rightarrow$ Triangles are similar.

(ii) $\dfrac{|CD|}{6} = \dfrac{12}{4}$ $\Rightarrow 4|CD| = 72$

$\Rightarrow |CD| = \dfrac{72}{4} = 18$ cm



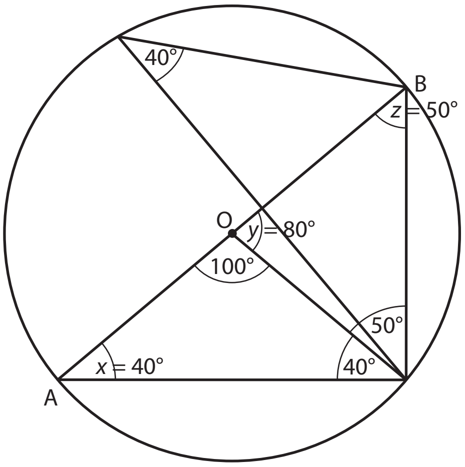
10. $\angle x = 40°$ (standing on the same arc)

$\angle y = 40° + 40° = 80°$

$\angle z = 40° + 40° = 80°$

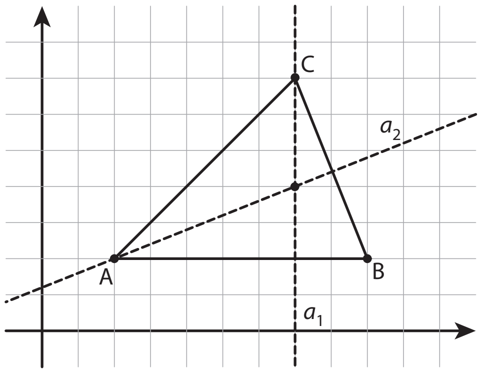
$\angle z + 130° = 180°$

$\Rightarrow \angle z = 180° - 130° = 50°$



Chapter 2: Revision Exercises 2 (Extended Response)

1. (i) The points are plotted below.



(ii) a_1 and a_2 are two of the altitudes of the triangle.
The equation of a_1 is $x = 7$.

The slope of BC is $\frac{7-2}{7-9} = \frac{5}{-2} = -\frac{5}{2}$.

Thus the slope of a_2 is $\frac{2}{5}$.

The equation of a_2 is

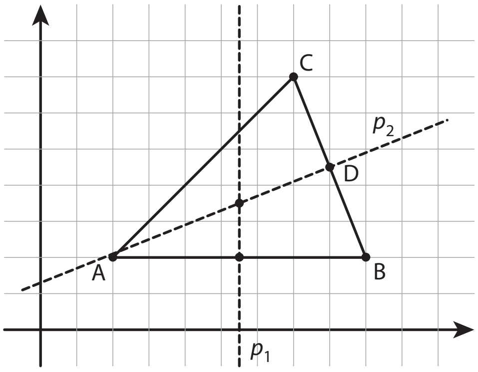
$$y - 2 = \frac{2}{5}(x - 2)$$
$$5y - 10 = 2x - 4$$
$$2x - 5y = -6$$

Solving between a_1 and a_2 to find the orthocentre:

$$2(7) - 5y = -6$$
$$-5y = -20$$
$$y = 4$$

Thus the orthocentre of the triangle is (7, 4).

(iii)



p_1 and p_2 are two of the perpendicular bisectors of the sides of the triangle.
The equation of p_1 is $x = 5.5$.
D, the midpoint of [BC], is

$$D = \left(\frac{9+7}{2}, \frac{2+7}{2}\right) = (8, 4.5).$$

From before, the slope of a line perpendicular to BC is $\frac{2}{5}$.

The equation of p_2 is

$$y - 4.5 = \frac{2}{5}(x - 8)$$
$$5y - 22.5 = 2x - 16$$
$$2x - 5y = -6.5$$

Solving between p_1 and p_2:

$$2(5.5) - 5y = -6.5$$
$$17.5 = 5y$$
$$y = 3.5$$

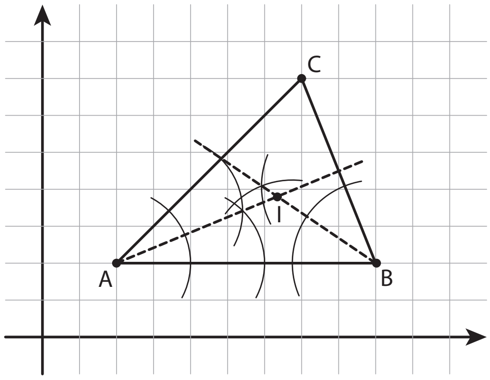
Thus the circumcentre of the triangle is (5.5, 3.5).
The radius of the circumcircle is then

$$\sqrt{(5.5 - 2)^2 + (3.5 - 2)^2} = \sqrt{14.5}$$

The equation of the circumcircle is then

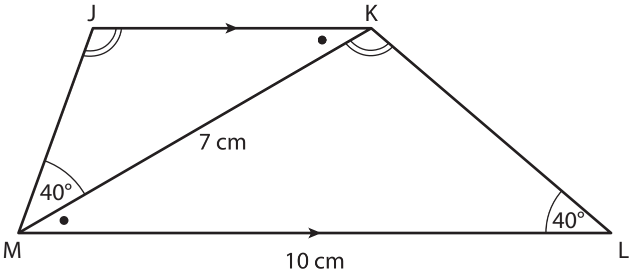
$$(x - 5.5)^2 + (y - 3.5)^2 = 14.5$$

(iv) In the diagram below, I is the incentre of the triangle.



2. (i) Triangles JKM, KML are equiangular with equal angles marked.

$\Rightarrow$ Triangles are similar.



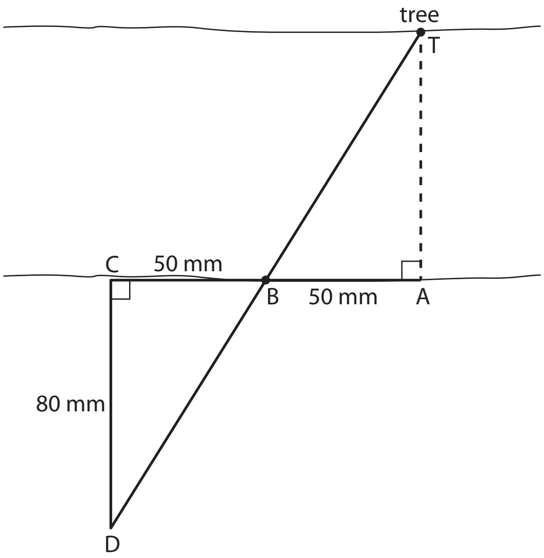
(ii) $\dfrac{|JK|}{|KM|} = \dfrac{|KM|}{|ML|}$

$\Rightarrow \dfrac{|JK|}{7} = \dfrac{7}{10} \Rightarrow 10|JK| = (7)(7) = 49$

$\Rightarrow |JK| = \dfrac{49}{10} = 4.9$ cm

(iii) Triangles BAT and DCB are similar and congruent.

$\Rightarrow$ Width of River $= |AT| = |DC| = 80$ m



3. (i) Surface Areas $= (2)^2 : (3)^2 = 4:9$

(ii) Edge of smaller cube $= 12\left(\dfrac{2}{3}\right) = 8$ cm

(iii) Total surface area of larger cube $= 54\left(\dfrac{9}{4}\right) = 121.5$ cm^2

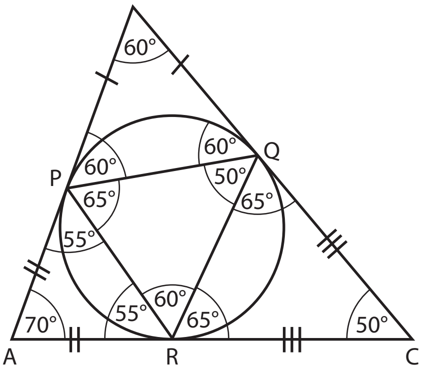
4. The lengths of two tangents from a point to a circle are equal.

Hence, $|PB| = |BQ|, |AP| = |AR|, |CQ| = |CR|$

(i) $|\angle PRQ| = 180° - (55° + 65°) = 60°$

(ii) $|\angle BPQ| = 60°$

(iii) $|\angle PQR| = 180° - (60° + 65°) = 55°$



5. (i) (a) $|\angle PRQ| = 105°$

$|\angle PSQ| + |\angle PRQ| = 180°$

$\Rightarrow |\angle PSQ| + 105° = 180°$

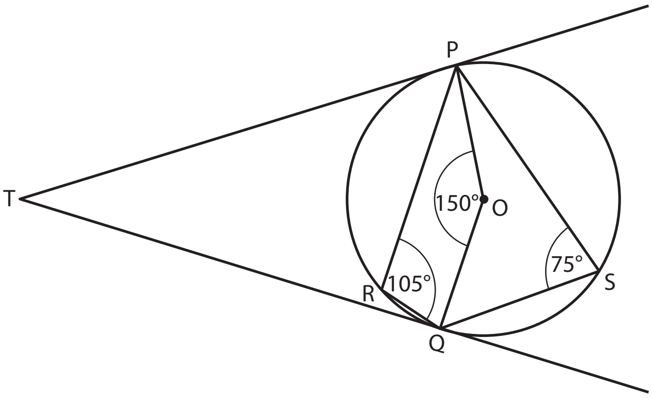
$\Rightarrow |\angle PSQ| = 180° - 105° = 75°$

$\Rightarrow |\angle POQ| = 2(75°) = 150°$

(b) POQT is a cyclic quadrilateral because $|\angle TPO| + |\angle TQO| = 180°$

(c) $|\angle PTQ| + 150° = 180°$

$\Rightarrow |\angle PTQ| = 180° - 150° = 30°$

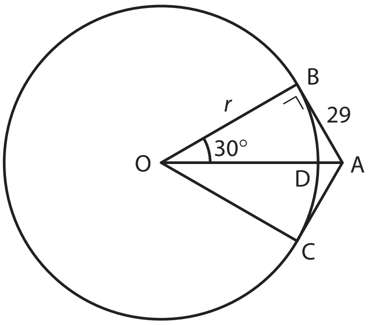


(ii) Let O be the centre of the circle, A be the point outside the circle, B and C the points of tangency and D be the nearest point to A on the circle.

As $|\angle BAC| = 120°$,

$|\angle BAO| = 60°$ and

$|\angle AOB| = 30°$.



Let r be the radius of the circle. Then

$$\frac{r}{29} = \tan 60°$$

$$r = 29 \tan 60°$$

$$r = 50.23 \text{ m}$$

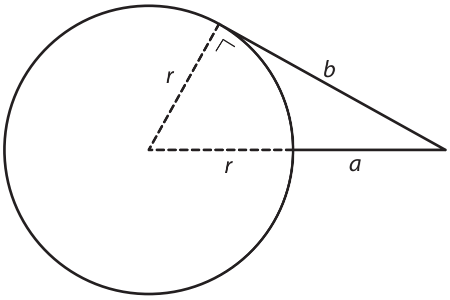
Also

$$|OA| = \sqrt{29^2 + 50.23^2} = 58$$

and so

$$|AD| = 58 - 50.23 = 7.77 \text{ m}$$

Another method is as follows:



Calculate a, the distance to the nearest point on the circle, and b, the length of the tangent to the circle. Then, if r is the radius of the circle:

$$(r + a)^2 = r^2 + b^2$$

$$r^2 + 2ar + a^2 = r^2 + b^2$$

$$2ar = b^2 - a^2$$

$$r = \frac{b^2 - a^2}{2a}.$$

6. (a) (i) From the diagram:

$|AM| = |MB| = 6.$

Then

$|OM| = \sqrt{8^2 - 6^2} = 2\sqrt{7}$ cm

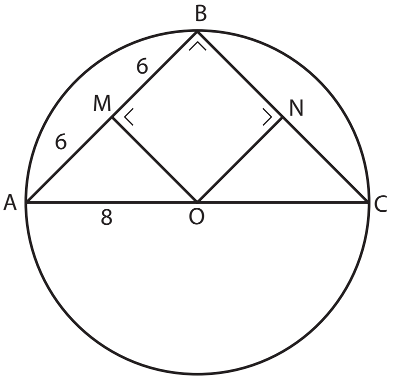
(ii) $|BC| = 2|OM| = 4\sqrt{7}$ cm

(iii) $|ON| = |MB| = 6$ cm

(iv) Area $\triangle ABC = \frac{1}{2}|AB|\ .\ |BC$

$= \frac{1}{2}(12)(4\sqrt{7})$

$= 24\sqrt{7}$ cm^2



(b) (i) $|OE|^2 + |BE|^2 = |OB|^2$

$\Rightarrow |OE|^2 + (4)^2 = (6)^2$

$\Rightarrow |OE|^2 = 36 - 16 = 20$

$\Rightarrow |OE| = \sqrt{20} = 2\sqrt{5}$

$|AE|^2 + |EO|^2 = |AO|^2$

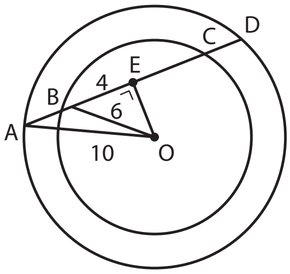
$\Rightarrow |AE|^2 + (2\sqrt{5})^2 = (10)^2$

$\Rightarrow |AE|^2 + 20 = 100$

$\Rightarrow |AE|^2 = 100 - 20 = 80$

$\Rightarrow |AE| = \sqrt{80} = 4\sqrt{5}$

Hence, $|AB| = |AE| - |BE| = 4\sqrt{5} - 4 = 4(\sqrt{5} - 1)$ cm



7. (i) $|CF| = |CD| = y$ cm

$|CB| = |CD| + |DB| = y + x = 24$

(ii) $x + z = 10$ and $z + y = 26$

$\underline{x + y = 24}$ (subtracting) $\underline{z - y = -14}$ (adding)

$\Rightarrow z - y = -14$ $\quad 2z = 12$

$\Rightarrow z = 6$

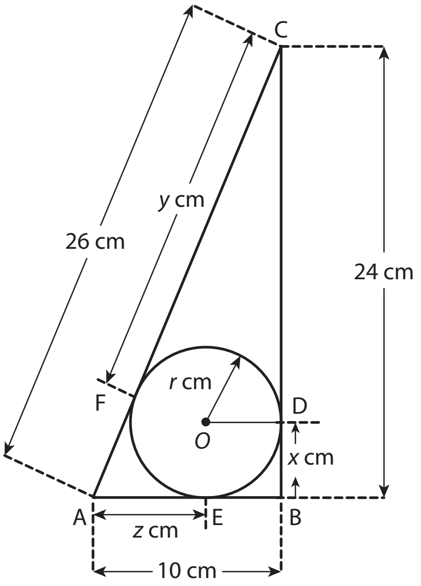
$\Rightarrow 6 - y = -14$

$\Rightarrow -y = -14 - 6 = -20$

$\Rightarrow y = 20$

$x + 6 = 10 \Rightarrow x = 10 - 6 = 4$

(iii) Radius $= |OD| = |EB| = 10 - 6 = 4$ cm



8. (i) (a) $|\angle COE| = |\angle OCB| = 52°$ (alternate angles)

$|\angle AOE| = |\angle OBC| = 52°$ (corresponding angles)

$\Rightarrow |\angle AOE| = |\angle COE|$

Hence, OD bisects $\angle AOC$

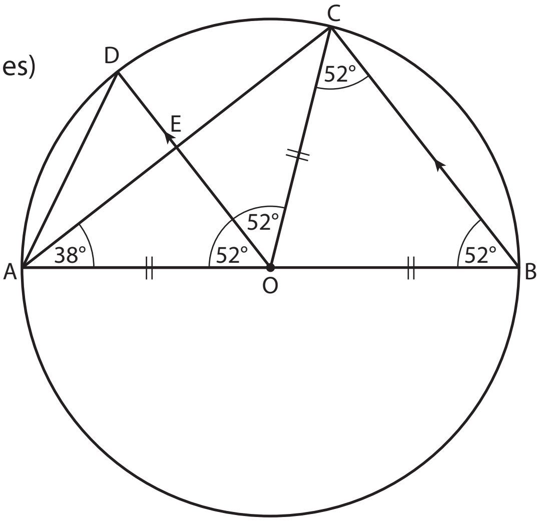
(b) $|\angle COE| = 52°$

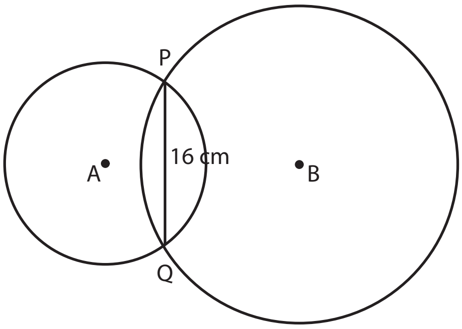
$\Rightarrow |\angle CAD| = \frac{1}{2}|\angle COE| = \frac{1}{2}(52°) = 26°$

(c) In $\triangle AEO$, $|\angle AEO| + 38° + 52° = 180°$

$\Rightarrow |\angle AEO| = 180° - 90° = 90°$

Hence, $OD \perp AC$.





(ii) In $\triangle APQ$, $|AQ|^2 + (8)^2 = (10)^2$

$\Rightarrow |AQ|^2 + 64 = 100$

$\Rightarrow |AQ|^2 = 100 - 64 = 36$

$\Rightarrow |AQ| = \sqrt{36} = 6$ cm

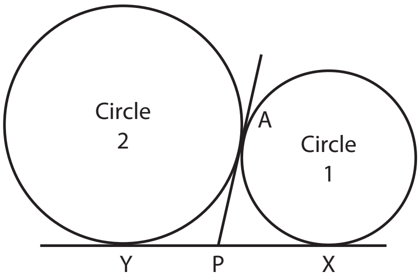
In $\triangle BPQ \Rightarrow |BQ|^2 + (8)^2 = (17)^2$

$\Rightarrow |BQ|^2 + 64 = 289$

$\Rightarrow |BQ|^2 = 289 - 64 = 225$

$\Rightarrow |BQ| = \sqrt{225} = 15$ cm

Hence, $|AB| = 6 + 15 = 21$ cm



(iii) (a) In $\triangle APY$, $|PA| = |PY|$ (equal tangents)

In $\triangle APX$, $|PA| = |PX|$ (equal tangents)

$\Rightarrow |PY| = |PX|$

$\Rightarrow$ Tangent at A bisects the line [XY]

(b) $|PA| = |PY| \Rightarrow \angle c = \angle c$

$|PA| = |PX| \Rightarrow \angle d = \angle d$

Hence, $\triangle YAX \Rightarrow \angle c + \angle c + \angle d + \angle d = 2\angle c + 2\angle d = 180°$

$\Rightarrow \angle c + \angle d = \frac{180°}{2} = 90°$

ie $|\angle XAY| = 90°$

Chapter 3

Exercise 3.1

1. (i) $\int x\,dx = \frac{x^2}{2} + c$

(ii) $\int x^2\,dx = \frac{x^3}{3} + c$

(iii) $\int (3x^2 + 4x)\,dx = \frac{3x^3}{3} + \frac{4x^2}{2} + c = x^3 + 2x^2 + c$

(iv) $\int -2x^2\,dx = \frac{-2x^3}{3} + c$

(v) $\int 3\,dx = 3x + c$

(vi) $\int (-x^2 + 3)\,dx = -\frac{x^3}{3} + 3x + c$

(vii) $\int (4x^3 + 6x)\,dx = \frac{4x^4}{4} + \frac{6x^2}{2} + c = x^4 + 3x^2 + c$

(viii) $\int (2x^2 - 3x - 1)\,dx = \frac{2x^3}{3} - \frac{3x^2}{2} - x + c$

(ix) $\int 12y^2\,dy = \frac{12y^3}{3} + c = 4y^3 + c$

2. (i) $\int x^{-2}\,dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$

(ii) $\int 2x^{-3}\,dx = \frac{2x^{-2}}{-2} + c = -\frac{1}{x^2} + c$

(iii) $\int \frac{3}{x^2}\,dx = \int 3x^{-2}\,dx = \frac{3x^{-1}}{-1} + c = -\frac{3}{x} + c$

(iv) $\int -\frac{2}{x^3}\,dx = \int -2x^{-3}\,dx = \frac{-2x^{-2}}{-2} + c = \frac{1}{x^2} + c$

(v) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

(vi) $\int 3x^{\frac{1}{2}}\,dx = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = 2\sqrt{x^3} + c$

(vii) $\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{\frac{1}{2}}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c$

(viii) $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4}\sqrt[3]{x^4} + c$

(ix) $\int 4\pi r^2\,dr = \frac{4\pi r^3}{3} + c = \frac{4\pi r^3}{3} + c$

3. (i) $\int \left(2x^3 + \frac{3}{x^2}\right)dx = \int (2x^3 + 3x^{-2})\,dx = 2\frac{x^4}{4} + 3\frac{x^{-1}}{-1} + c$

$= \frac{x^4}{2} - \frac{3}{x} + c$

(ii) $\int \left(\frac{4}{x^2} - 2 + x^3\right)dx = \int (4x^{-2} - 2 + x^3)\,dx$

$= 4\frac{x^{-1}}{-1} - 2x + \frac{x^4}{4} + c$

$= -\frac{4}{x} - 2x + \frac{x^4}{4} + c$

(iii) $\int(4\sqrt{x} - 3)\,dx = \int\left(4x^{\frac{1}{2}} - 3\right)dx$

$$= 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3x + c = \frac{8}{3}\sqrt{x^3} - 3x + c$$

(iv) $\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{2}{3}\sqrt{x^3} + 2\sqrt{x} + c$$

(v) $\int\left(2\sqrt{x} - \frac{2}{x^2}\right)dx = \int\left(2x^{\frac{1}{2}} - 2x^{-2}\right)dx$

$$= 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{x^{-1}}{-1} + c = \frac{4}{3}\sqrt{x^3} + \frac{2}{x} + c$$

(vi) $\int\left(\frac{1}{x^2} - \frac{x}{\sqrt{x}}\right)dx = \int\left(x^{-2} - x^{\frac{1}{2}}\right)dx$

$$= \frac{x^{-1}}{-1} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{1}{x} - \frac{2}{3}\sqrt{x^3} + c$$

4. (i) $\frac{dy}{dx} = x^2 + 3x$

$\Rightarrow y = \int(x^2 + 3x)\,dx = \frac{x^3}{3} + \frac{3x^2}{2} + c$

(ii) $\frac{dy}{dx} = 6x^3 - 4x^2 + x - 5$

$\Rightarrow y = \int(6x^3 - 4x^2 + x - 5)\,dx$

$$= \frac{6x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} - 5x + c$$

$$= \frac{3x^4}{2} - \frac{4x^3}{3} + \frac{x^2}{2} - 5x + c$$

5. (i) $\int(x - 3)^2\,dx = \int(x^2 - 6x + 9)\,dx$

$$= \frac{x^3}{3} - \frac{6x^2}{2} + 9x + c$$

$$= \frac{x^3}{3} - 3x^2 + 9x + c$$

(ii) $\int\left(x - \frac{1}{x}\right)^2 dx = \int\left(x^2 - 2 + \frac{1}{x^2}\right)dx$

$$= \int(x^2 - 2 + x^{-2})\,dx$$

$$= \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1} + c$$

$$= \frac{x^3}{3} - 2x - \frac{1}{x} + c$$

(iii) $\int\sqrt{x}(x - 3)\,dx = \int\left(x^{\frac{3}{2}} - 3x^{\frac{1}{2}}\right)dx$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{5}\sqrt{x^5} - 2\sqrt{x^3} + c$$

6. (i) $\int \frac{x^4 - 3x^3 + 4x}{x}\,dx = \int (x^3 - 3x^2 + 4)\,dx$

$$= \frac{x^4}{4} - \frac{3x^3}{3} + 4x + c$$

$$= \frac{x^4}{4} - x^3 + 4x + c$$

(ii) $\int \frac{3x^3 - x^2 + 6}{x^2}\,dx$

$$= \int (3x - 1 + 6x^{-2})\,dx$$

$$= \frac{3x^2}{2} - x + 6\frac{x^{-1}}{-1} + c$$

$$= \frac{3x^2}{2} - x - \frac{6}{x} + c$$

(iii) $\int \frac{x^2 - 2x + 6}{\sqrt{x}}\,dx$

$$= \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right) dx$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{6x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{5}x^{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{3} + 12\sqrt{x} + c$$

7. $f'(x) = 2x$

$\Rightarrow f(x) = \int 2x\,dx = x^2 + c$

Point $(-1, 4) \Rightarrow f(-1) = (-1)^2 + c = 4$

$\Rightarrow 1 + c = 4 \quad \Rightarrow c = 3$

$\Rightarrow f(x) = x^2 + 3$

8. $f'(x) = 2x - 5$

$\Rightarrow f(x) = \int (2x - 5)\,dx$

$= x^2 - 5x + c$

Point $(1, 7) \Rightarrow f(1) = (1)^2 - 5(1) + c = 7$

$\Rightarrow 1 - 5 + c = 7$

$\Rightarrow c = 7 + 4 = 11$

$\Rightarrow f(x) = x^2 - 5x + 11$

9. $y = \int (6x + 5)\,dx = \frac{6x^2}{2} + 5x + c = 3x^2 + 5x + c$

When $x = 2 \Rightarrow y = 3(2)^2 + 5(2) + c = 19$

$\Rightarrow 12 + 10 + c = 19$

$\Rightarrow c = 19 - 22 = -3$

10. $y = \int (6x^2 - 8x + 5)\,dx = \frac{6x^3}{3} - \frac{8x^2}{2} + 5x + c$

$= 2x^3 - 4x^2 + 5x + c$

When $x = 2 \Rightarrow y = 2(2)^3 - 4(2)^2 + 5(2) + c = 7$

$\Rightarrow 16 - 16 + 10 + c = 7$

$\Rightarrow c = 7 - 10 = -3$

11. (i) $\frac{dy}{dx} = x^2 + 2x$

$\Rightarrow y = \int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 + c$

When $x = 0 \Rightarrow y = 2(0)^3 + (0)^2 + c = 2$

$\Rightarrow 0 + 0 + c = 2 \Rightarrow c = 2$

$\Rightarrow y = \frac{x^3}{3} + x^2 + 2$

(ii) $\frac{dy}{dx} = 3 - x^2$

$\Rightarrow y = \int (3 - x^2)\,dx = 3x - \frac{x^3}{3} + c$

When $x = 3 \Rightarrow y = 3(3) - \frac{(3)^3}{3} + c = 2$

$\Rightarrow 9 - 9 + c = 2 \Rightarrow c = 2$

$\Rightarrow y = 3x - \frac{x^3}{3} + 2$

12. (i) $\frac{dV}{dt} = t^2 - t$

$\Rightarrow V = \int (t^2 - t)\,dt = \frac{t^3}{3} - \frac{t^2}{2} + c$

When $t = 3 \Rightarrow V = \frac{(3)^3}{3} - \frac{(3)^2}{2} + c = 9$

$\Rightarrow 9 - \frac{9}{2} + c = 9 \Rightarrow c = 4\frac{1}{2}$

$\Rightarrow V = \frac{t^3}{3} - \frac{t^2}{2} + 4\frac{1}{2}$

(ii) $t = 10 \Rightarrow V = \frac{(10)^3}{3} - \frac{(10)^2}{2} + 4\frac{1}{2} = 287\frac{5}{6}$

13. (i) $f'(x) = 4x + k$

Turning point at $(-2, -1) \Rightarrow f'(-2) = 4(-2) + k = 0$

$\Rightarrow -8 + k = 0 \Rightarrow k = 8$

(ii) $f'(x) = 4x + 8$

$\Rightarrow f(x) = \int (4x + 8)\,dx = \frac{4x^2}{2} + 8x + c$

$\Rightarrow f(x) = 2x^2 + 8x + c$

Point $(-2, -1) \Rightarrow f(-2) = 2(-2)^2 + 8(-2) + c = -1$

$\Rightarrow 8 - 16 + c = -1 \Rightarrow c = 7$

$\Rightarrow f(x) = 2x^2 + 8x + 7$

On y-axis, $x = 0 \Rightarrow f(0) = 2(0)^2 + 8(0) + 7 = 7$

$\Rightarrow$ point on y-axis $= (0, 7)$

14. Tangent at (3, 6) passes through the origin (0, 0)

$\Rightarrow$ slope m $= \frac{6-0}{3-0} = \frac{6}{3} = 2$

(i) When $x = 3 \Rightarrow$ slope $= \frac{dy}{dx} = 2(3) + k = 2$

$\Rightarrow 6 + k = 2 \Rightarrow k = -4$

(ii) $\frac{dy}{dx} = 2x - 4$

$\Rightarrow y = \int (2x - 4)\,dx = x^2 - 4x + c$

Point $(3, 6) \Rightarrow 6 = (3)^2 - 4(3) + c$

$\Rightarrow 6 = 9 - 12 + c$

$\Rightarrow 6 = -3 + c \Rightarrow c \Rightarrow 9$

Hence $y = x^2 - 4x + 9$

Exercise 3.2

1. (i) $\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$

(ii) $\int 3e^x\,dx = 3e^x + c$

(iii) $\int 2e^{4x}\,dx = \frac{2\,.\,e^{4x}}{4} + c = \frac{e^{4x}}{2} + c$

(iv) $\int e^{-3x}\,dx = \frac{-e^{-3x}}{3} + c$

2. (i) $\int (e^{3x} + 4)\,dx = \frac{e^{3x}}{3} + 4x + c$

(ii) $\int 4e^{\frac{1}{2}x}\,dx = \frac{4\,.\,e^{\frac{1}{2}x}}{\frac{1}{2}} + c = 8e^{\frac{1}{2}x} + c$

(iii) $$\int \left(e^{4x} + \frac{1}{e^{4x}}\right) dx = \int e^{4x} + e^{-4x}\,dx$$
$$= \frac{e^{4x}}{4} + \frac{e^{-4x}}{-4} + c$$
$$= \frac{e^{4x}}{4} - \frac{e^{-4x}}{4} + c$$

3. $y = e^{x^2} \Rightarrow \frac{dy}{dx} = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2x\,.\,e^{x^2}$

Since $\frac{dy}{dx} = 2x\,e^{x^2} \Rightarrow \int 2x\,e^{x^2}\,dx = e^{x^2} + c$

4. (i) $\int \cos 3x\,dx = \frac{\sin 3x}{3} + c$

(ii) $\int \sin 4x\,dx = -\frac{\cos 4x}{4} + c$

(iii) $\int -\sin 5x\,dx = -\left(-\frac{\cos 5x}{5}\right) + c = \frac{\cos 5x}{5} + c$

(iv) $\int \cos kx\,dx = \frac{\sin kx}{k} + c$

5. (i) $\int 3\cos 6x\,dx = 3 \cdot \frac{\sin 6x}{6} + c = \frac{\sin 6x}{2} + c$

(ii) $$\int (\cos 2x - \sin 5x)\,dx = \frac{\sin 2x}{2} - \left(\frac{-\cos 5x}{5}\right) + c$$
$$= \frac{\sin 2x}{2} + \frac{\cos 5x}{5} + c$$

(iii) $$\int 3\cos(-9x)\,dx = 3 \cdot \frac{\sin(-9x)}{-9} + c$$
$$= -\frac{\sin(-9x)}{3} + c$$

6. $\int 3(e^x - 4\sin 3x + 2)\,dx$

$= 3\left[e^x - 4\left(\frac{-\cos 3x}{3}\right) + 2x\right] + c$

$= 3e^x + 4\cos 3x + 6x + c$

7. (i) $\int (4e^{2x} + 4\sin 3x)\,dx$

$= 4\frac{e^{2x}}{2} + 4\left(\frac{-\cos 3x}{3}\right) + c$

$= 2e^{2x} - \frac{4\cos 3x}{3} + c$

(ii) $\int (3\cos x - 2\cos 4x)\,dx$

$= 3\sin x - \frac{2\sin 4x}{4} + c$

$= 3\sin x - \frac{\sin 4x}{2} + c$

8. $y = \cos 4x^2 \Rightarrow \dfrac{dy}{dx} = -\sin 4x^2 \cdot \dfrac{d}{dx}(4x^2)$

$$= -\sin 4x^2 \,.\, 8x = -8x \sin 4x^2$$

Since $\dfrac{dy}{dx} = -8x \sin 4x^2 \Rightarrow \int -8x \sin 4x^2 \, dx = \cos 4x^2 + c$

9. (i) $\displaystyle\int \frac{e^{2x} + 4}{e^x} dx = \int \frac{e^{2x}}{e^x} + \frac{4}{e^x} dx = \int (e^x + 4e^{-x}) \, dx$

$$= e^x + \frac{4e^{-x}}{-1} + c$$

$$= e^x - \frac{4}{e^x} + c$$

(ii) $\displaystyle\int \frac{e^{x+2} + 3}{e^x} dx = \int \frac{e^{x+2}}{e^x} + \frac{3}{e^x} dx$

$$= \int (e^2 + 3e^{-x}) \, dx$$

$$= xe^2 + 3\frac{e^{-x}}{-1} + c$$

$$= xe^2 - \frac{3}{e^x} + c$$

(iii) $\displaystyle\int \frac{1 + 3e^x}{e^{2x}} dx = \int \frac{1}{e^{2x}} + \frac{3e^x}{e^{2x}} dx$

$$= \int (e^{-2x} + 3e^{-x}) \, dx$$

$$= \frac{e^{-2x}}{-2} + 3\frac{e^{-x}}{-1} + c$$

$$= -\frac{1}{2}e^{-2x} - 3e^{-x} + c$$

10. (i) $\int (e^x - e^{-x})^2 \, dx = \int [\,(e^x)^2 - 2e^x \,.\, e^{-x} + (e^{-x})^2\,] \, dx$

$$= \int (e^{2x} - 2 + e^{-2x}) \, dx$$

$$= \frac{1}{2}e^{2x} - 2x + \frac{e^{-2x}}{-2} + c$$

$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + c$$

(ii) $\int (3 + e^x)(2 + e^{-x}) \, dx$

$$= \int (6 + 3e^{-x} + 2e^x + e^x \,.\, e^{-x}) \, dx$$

$$= \int (6 + 3e^{-x} + 2e^x + 1) \, dx$$

$$= \int (7 + 3e^{-x} + 2e^x) \, dx$$

$$= 7x + \frac{3e^{-x}}{-1} + 2e^x + c$$

$$= 7x - \frac{3}{e^x} + 2e^x + c$$

11. $y = 7^x \Rightarrow \ln y = \ln 7^x$

$$\Rightarrow \ln y = x \ln 7x$$

$$\Rightarrow \quad x = \frac{\ln y}{\ln 7} = \frac{1}{\ln 7} \cdot \ln y$$

(i) $\dfrac{dx}{dy} = \dfrac{1}{\ln 7} \cdot \dfrac{1}{y}$

(ii) $\dfrac{dy}{dx} = \ln 7 \,.\, y = \ln 7 \,.\, 7^x$ OR $7^x \ln 7$

(iii) Since $\frac{dy}{dx} = 7^x \cdot \ln 7$

$$\int 7^x \, dx = \frac{7^x}{\ln 7} + c$$

12. $\frac{dy}{dx} = ae^{-x} + 2$

When $x = 0 \Rightarrow \frac{dy}{dx} = ae^{-(0)} + 2 = 5$

$$\Rightarrow a \,.\, 1 + 2 = 5 \quad \Rightarrow a = 3$$

Hence $\frac{dy}{dx} = 3e^{-x} + 2$

$$\Rightarrow \quad y = \int (3e^{-x} + 2)\, dx$$

$$= 3\frac{e^{-x}}{-1} + 2x + c = \frac{-3}{e^x} + 2x + c$$

$y = -3$ when $x = 0 \Rightarrow -3 = \frac{-3}{e^0} + 2(0) + c$

$$\Rightarrow -3 = \frac{-3}{1} + 0 + c \quad \Rightarrow c = 0$$

$$\Rightarrow \quad y = \frac{-3}{e^x} + 2x$$

13. Tangent at $(1, e^2)$ passes through the origin $(0, 0)$

$\Rightarrow$ gradient $\quad m = \frac{e^2 - 0}{1 - 0} = e^2$

(i) When $x = 1 \quad \Rightarrow$ gradient $= \frac{dy}{dx} = e^{k(1)} = e^2 \Rightarrow k = 2$

(ii) $\frac{dy}{dx} = e^{2x} \Rightarrow y = \int e^{2x}\, dx = \frac{e^{2x}}{2} + c$

Point $(1, e^2) \Rightarrow e^2 = \frac{e^{2(1)}}{2} + c \quad \Rightarrow c = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$

$$\Rightarrow \quad y = \frac{1}{2}e^{2x} + \frac{1}{2}e^2$$

14. (i) $f(x) = 2x\, e^x \Rightarrow$ product rule: $u = 2x$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 2x \,.\, e^x + e^x \,.\, 2 = 2x\, e^x + 2e^x$$

(ii) $\int (2x\, e^x + 2e^x)\, dx = 2x\, e^x + c$

$$\Rightarrow \quad \int 2x\, e^x dx + \int 2e^x dx = 2x\, e^x + c$$

$$\Rightarrow \quad \int 2x\, e^x dx + 2e^x = 2x\, e^x + c$$

$$\Rightarrow \quad \int 2x\, e^x dx = 2x\, e^x - 2e^x + c$$

15. $f(x) = x \sin x \Rightarrow$ product rule: $u = x$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x \,.\, \cos x + \sin x \,.\, 1 = \sin x + x \cos x$$

$$\int(\sin x + x\cos x)\,\mathrm{d}x = x\sin x + c$$

$$\Rightarrow \int \sin x\,\mathrm{d}x + \int x\cos x\,\mathrm{d}x = x\sin x + c$$

$$\Rightarrow \quad -\cos x + \int x\cos x\,\mathrm{d}x = x\sin x + c$$

$$\Rightarrow \qquad \int x\cos x\,\mathrm{d}x = x\sin x + \cos x + c$$

16. (i) $f(x) = 4xe^{2x} \Rightarrow$ product rule: $u = 4x$ and $v = e^{2x}$

$$\Rightarrow \frac{du}{dx} = 4 \quad \Rightarrow \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = f'(x) = u\frac{dv}{dx} + v\frac{du}{dx} = 4x\cdot 2e^{2x} + e^{2x}\cdot 4$$

$$= 8x\,e^{2x} + 4e^{2x}$$

(ii) $\int(8x\,e^{2x} + 4e^{2x})dx = 4x\,e^{2x} + c$

$$\Rightarrow \quad \int 8x\,.\,e^{2x}\,dx + \int 4\,e^{2x}\,dx = 4x\,e^{2x} + c$$

$$\Rightarrow \quad \int 8x\,e^{2x}\,dx + 4\frac{e^{2x}}{2} = 4x\,e^{2x} + c$$

$$\Rightarrow \quad \int 8x\,e^{2x}\,dx = 4x\,e^{2x} - 2e^{2x} + c$$

17. $y = 2x\,.\,e^{3x} + \cos x$

$$\frac{dy}{dx} = 2x\,.\,e^{3x}\,.\,3 + e^{3x}\,.\,2 - \sin x$$

$$= 6x\,e^{3x} + 2e^{3x} - \sin x$$

Hence $\int(6x\,e^{3x} + 2e^{3x} - \sin x)\,dx = 2x\,e^{3x} + \cos x + c$

$$\Rightarrow \quad \int 6x\,e^{3x}\,dx + \int 2e^{3x}\,dx - \int \sin x\,dx = 2x\,e^{3x} + \cos x + c$$

$$\Rightarrow \quad \int 6x\,e^{3x}\,dx + \frac{2\,.\,e^{3x}}{3} + \cos x = 2x\,e^{3x} + \cos x + c$$

$$\Rightarrow \quad \int 6x\,e^{3x}\,dx = 2x\,e^{3x} - \frac{2}{3}e^{3x} + c$$

Exercise 3.3

1. (i) $v = \frac{ds}{dt} = 5t + 4$

$$\Rightarrow s = \int(5t + 4)\,dt$$

$$= \frac{5t^2}{2} + 4t + c$$

$s = 0$ when $t = 0 \Rightarrow \frac{5}{2}(0)^2 + 4(0) + c = 0 \quad \Rightarrow c = 0$

$$\Rightarrow \quad s = \frac{5t^2}{2} + 4t$$

(ii) $t = 4 \Rightarrow s = \frac{5(4)^2}{2} + 4(4) = 56$ metres

2. $v = \frac{ds}{dt} = t^2 - 4t + 3$

(i) $a = \frac{d^2s}{dt^2} = 2t - 4$

$t = 5 \quad \Rightarrow \quad a = 2(5) - 4 = 6$ m/sec^2

(ii) $s = \int(t^2 - 4t + 3)\,dt$

$$= \frac{t^3}{3} - \frac{4t^2}{2} + 3t + c = \frac{t^3}{3} - 2t^2 + 3t + c$$

$s = 4$ when $t = 3 \quad \Rightarrow \frac{(3)^3}{3} - 2(3)^2 + 3(3) + c = 4$

$\Rightarrow 9 - 18 + 9 + c = 4 \quad \Rightarrow c = 4$

$\Rightarrow s = \frac{t^3}{3} - 2t^2 + 3t + 4$

(iii) $t = 1 \quad \Rightarrow \quad s = \frac{(1)^3}{3} - 2(1)^2 + 3(1) + 4 = 5\frac{1}{3}\text{m}$

3. $a = 6t - 12$

(i) $v = \int(6t - 12)\,dt = \frac{6t^2}{2} - 12t + c$

$= 3t^2 - 12t + c$

$v = 9$ when $t = 0 \quad \Rightarrow \quad 3(0)^2 - 12(0) + c = 9$

$\Rightarrow c = 9$

$\Rightarrow v = 3t^2 - 12t + 9$

(ii) $s = \int(3t^2 - 12t + 9)dt$

$= \frac{3t^3}{3} - 12\frac{t^2}{2} + 9t + c$

$= t^3 - 6t^2 + 9t + c$

$s = 6$ when $t = 0 \quad \Rightarrow \quad (0)^3 - 6(0)^2 + 9(0) + c = 6$

$\Rightarrow \quad c = 6$

$\Rightarrow \quad s = t^3 - 6t^2 + 9t + 6$

(iii) Body at rest $\quad \Rightarrow \quad v = 0$

$\Rightarrow \quad 3t^2 - 12t + 9 = 0$

$\Rightarrow \quad t^2 - 4t + 3 = 0$

$\Rightarrow \quad (t - 1)(t - 3) = 0$

$\Rightarrow \quad t = 1$ OR $t = 3$

4. $a = (2t - 3)$ cm/sec^2

(i) $v = \int(2t - 3)\,dt = t^2 - 3t + c$

$v = 3$ when $t = 0 \quad \Rightarrow \quad (0)^2 - 3(0) + c = 3$

$\Rightarrow c = 3$

$\Rightarrow v = t^2 - 3t + 3$

$s = \int(t^2 - 3t + 3)\,dt$

$= \frac{t^3}{3} - 3\frac{t^2}{2} + 3t + c$

$s = 2$ when $t = 0 \quad \Rightarrow \quad \frac{(0)^3}{3} - \frac{3(0)^2}{2} + 3(0) + c = 2 \quad \Rightarrow c = 2$

$\Rightarrow s = \frac{t^3}{3} - \frac{3t^2}{2} + 3t + 2$

(ii) $t = 2 \quad \Rightarrow \quad v = (2)^2 - 3(2) + 3 = 1$ m/sec

$t = 2 \quad \Rightarrow \quad s = \frac{(2)^3}{3} - 3\,\frac{(2)^2}{3} + 3(2) + 2 = 4\frac{2}{3}\text{ m}$

5. $a = -10$ m/sec^2

(i) $v = \int -10dt = -10t + c$

$v = 25$ when $t = 0 \quad \Rightarrow \quad -10(0) + c = 25 \quad \Rightarrow c = 25$

$\Rightarrow v = (-10t + 25)$ m/sec

(ii) $s = \int(-10t + 25)\,dt$

$= -10\frac{t^2}{2} + 25t + c = -5t^2 + 25t + c$

$s = 0$ when $t = 0 \quad \Rightarrow \quad -5(0)^2 + 25(0) + c = 0 \quad \Rightarrow c = 0$

$\Rightarrow s = (-5t^2 + 25t)$ m

(iii) Maximum height occurs when $v = 0$

$\Rightarrow -10t + 25 = 0$

$\Rightarrow -10t = -25 \Rightarrow t = \frac{5}{2}$ sec

(iv) $t = \frac{5}{2} \Rightarrow s = -5\left(\frac{5}{2}\right)^2 + 25\left(\frac{5}{2}\right) = \frac{125}{4}$ m

(v) Return to the point of projection $\Rightarrow s = 0$

$\Rightarrow -5t^2 + 25t = 0$

$\Rightarrow t^2 - 5t = 0$

$\Rightarrow t(t-5) = 0$

$t = 0$ OR $t = 5$

$\Rightarrow$ Answer: $t = 5$ sec

6. $\frac{dN}{dt} = 4e^t + 10$

(i) $N = \int(4e^t + 10)\,dt = 4e^t + 10t + c$

(ii) $t = 0 \quad \Rightarrow \quad N = 4e^0 + 10(0) + c = 10$

$4 + c = 10 \quad \Rightarrow \quad c = 6$

$\Rightarrow \quad N = 4e^t + 10t + 6$

$t = 5 \quad \Rightarrow \quad N = 4e^5 + 10(5) + 6 = 4e^5 + 56$

$= 649.65 = 650$

7. $v = 0.6t - 0.004t^2$

(i) $s = \int(0.6t - 0.004t^2)\,dt$

$= 0.6 \cdot \frac{t^2}{2} - 0.004 \cdot \frac{t^3}{3} + c$

$s = 0$ when $t = 0 \quad \Rightarrow \quad 0.3(0)^2 - 0.004\frac{(0)^3}{3} + c = 0 \quad \Rightarrow c = 0$

$\Rightarrow s = 0.3t^2 - \frac{0.004t^3}{3}$

(ii) $t = 2\frac{1}{2}$ mins $= 150$ secs

$\Rightarrow \quad s = 0.3(150)^2 - \frac{0.004(150)^3}{3} = 2250$ m

8. $\frac{dh}{dt} = 2t - 3$

(i) $h = \int(2t-3)dt = t^2 - 3t + c$

$h = 4$ when $t = 0 \quad \Rightarrow \quad (0)^2 - 3(0) + c = 4 \quad \Rightarrow \quad c = 4$

$\Rightarrow \quad h = t^2 - 3t + 4$

(ii) Height of container = 36 cm

$\Rightarrow \quad 36 = t^2 - 3t + 4$

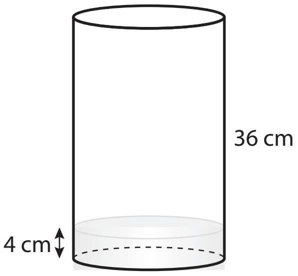
$\Rightarrow \quad t^2 - 3t - 32 = 0$

$\Rightarrow \quad t = \frac{3 \pm \sqrt{9 + 128}}{2}$

$\Rightarrow \quad t = 7.4$ OR $t = -4.4$

$\Rightarrow \quad t = 7.4$

Hence, $t = 7.4$ secs



Exercise 3.4

1. $$\int_1^2 6x\,dx = \left[\frac{6x^2}{2}\right]_1^2 = [3x^2]_1^2$$
$$= [3(2)^2] - [3(1)^2] = 12 - 3 = 9$$

2. $$\int_1^3 (3x^2 - 2x)\,dx = \left[\frac{3x^3}{3} - \frac{2x^2}{2}\right]_1^3 = [x^3 - x^2]_1^3$$
$$= [(3)^3 - (3)^2] - [(1)^3 - (1)^2]$$
$$= [27 - 9] - [1 - 1] = 18$$

3. $$\int_1^4 (3x^2 - 4)\,dx = \left[\frac{3x^3}{3} - 4x\right]_1^4 = [x^3 - 4x]_1^4$$
$$= [(4)^3 - 4(4)] - [(1)^3 - 4(1)]$$
$$= (64 - 16) - (1 - 4) = 51$$

4. $$\int_1^2 (x^3 + 2x)\,dx = \left[\frac{x^4}{4} + \frac{2x^2}{2}\right]_1^2 = \left[\frac{x^4}{4} + x^2\right]_1^2$$
$$= \left[\frac{(2)^4}{4} + (2)^2\right] - \left[\frac{(1)^4}{4} + (1)^2\right]$$
$$= (4 + 4) - \left(\frac{1}{4} + 1\right) = 6\frac{3}{4}$$

5. $$\int_1^3 (x^2 - x + 1)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_1^3$$
$$= \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} + 3\right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} + 1\right]$$
$$= \left[9 - \frac{9}{2} + 3\right] - \left[\frac{1}{3} - \frac{1}{2} + 1\right]$$
$$= 7\frac{1}{2} - \frac{5}{6} = 6\frac{2}{3}$$

6. $$\int_{-1}^2 (2x - 5)\,dx = \left[\frac{2x^2}{2} - 5x\right]_{-1}^2 = [x^2 - 5x]_{-1}^2$$
$$= [(2)^2 - 5(2)] - [(-1)^2 - 5(-1)]$$
$$= (4 - 10) - (1 + 5) = -6 - 6 = -12$$

7. $$\int_0^1 x^2(3 - x)\,dx = \int_0^1 (3x^2 - x^3)dx = \left[\frac{3x^3}{3} - \frac{x^4}{4}\right]_0^1$$
$$= \left[x^3 - \frac{x^4}{4}\right]_0^1 = \left[(1)^3 - \frac{(1)^4}{4}\right] - \left[(0)^3 - \frac{(0)^4}{4}\right]$$
$$= 1 - \frac{1}{4} - 0 = \frac{3}{4}$$

8. $$\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{\frac{1}{2}}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9 = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^9$$
$$= \left[\frac{2}{3}(9)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}(27) - \frac{2}{3} = 18 - \frac{2}{3} = 17\frac{1}{3}$$

9. $$\int_2^4 \frac{1}{x^2}\,dx = \int_2^4 x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_2^4 = \left[-\frac{1}{x}\right]_2^4$$
$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

10. $\displaystyle \int_4^9 \frac{dx}{\sqrt{x}} = \int_4^9 x^{-\frac{1}{2}}\,dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9 = [2\sqrt{x}]_4^9$

$\displaystyle = [2\sqrt{9}] - [2\sqrt{4}] = 6 - 4 = 2$

11. $\displaystyle \int_0^2 \frac{x^3 - 2x^2 + 4x}{x}\,dx = \int_0^2 (x^2 - 2x + 4)\,dx$

$\displaystyle = \left[\frac{x^3}{3} - x^2 + 4x\right]_0^2 = \left[\frac{(2)^3}{3} - (2)^2 + 4(2)\right] - \left[\frac{(0)^3}{3} - (0)^2 + 4(0)\right]$

$\displaystyle = \left[\frac{8}{3} - 4 + 8\right] - [0 - 0 + 0] = 6\frac{2}{3}$

12. $\displaystyle \int_1^4 (\sqrt{x} - 2)^2\,dx = \int_1^4 (x - 4\sqrt{x} + 4)dx = \int_1^4 (x - 4x^{\frac{1}{2}} + 4)\,dx$

$\displaystyle = \left[\frac{x^2}{2} - 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4x\right]_1^4 = \left[\frac{x^2}{2} - \frac{8}{3}x^{\frac{3}{2}} + 4x\right]_1^4$

$\displaystyle = \left[\frac{(4)^2}{2} - \frac{8}{3}(4)^{\frac{3}{2}} + 4(4)\right] - \left[\frac{(1)^2}{2} - \frac{8}{3}(1)^{\frac{3}{2}} + 4(1)\right]$

$\displaystyle = \left(8 - \frac{64}{3} + 16\right) - \left(\frac{1}{2} - \frac{8}{3} + 4\right)$

$\displaystyle = 2\frac{2}{3} - 1\frac{5}{6} = \frac{5}{6}$

13. $\displaystyle \int_{-2}^{-1} \frac{2}{x^3}\,dx = \int_{-2}^{-1} 2x^{-3}\,dx = \left[\frac{2x^{-2}}{-2}\right]_{-2}^{-1} = \left[-\frac{1}{x^2}\right]_{-2}^{-1}$

$\displaystyle = \left[-\frac{1}{-1^2}\right] - \left[-\frac{1}{-2^2}\right]$

$\displaystyle = -1 + \frac{1}{4} = -\frac{3}{4}$

14. $\displaystyle \int_1^{16} \left(\frac{\sqrt{x} - 4}{\sqrt{x}}\right)dx = \int_1^{16}\left(\frac{\sqrt{x}}{\sqrt{x}} - \frac{4}{\sqrt{x}}\right)dx = \int_1^{16} (1 - 4x^{-\frac{1}{2}})\,dx$

$\displaystyle = \left[x - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^{16} = [x - 8\sqrt{x}]_1^{16}$

$\displaystyle = (16 - 8\sqrt{16}) - (1 - 8\sqrt{1}) = -16 + 7 = -9$

15. $\displaystyle \int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int_1^4 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)dx$

$\displaystyle = \left[\frac{2x^{\frac{3}{2}}}{3} + 2\sqrt{x}\right]_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}} + 2\sqrt{4}\right] - \left[\frac{2(1)^{\frac{3}{2}}}{3} + 2\sqrt{1}\right]$

$\displaystyle = \left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right) = 9\frac{1}{3} - 2\frac{2}{3} = 6\frac{2}{3}$

16. $\displaystyle \int_1^2 (x - 1)(x - 2)\,dx = \int_1^2 (x^2 - 3x + 2)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$

$\displaystyle = \left[\frac{(2)^3}{3} - \frac{3(2)^2}{2} + 2(2)\right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} + 2(1)\right]$

$\displaystyle = \left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right) = \frac{2}{3} - \frac{5}{6} = -\frac{1}{6}$

17. $$\frac{x^2-16}{2x+8}=\frac{(x-4)(x+4)}{2(x+4)}=\frac{x-4}{2}$$

$$\int_0^1 \frac{x^2-16}{2x+8}dx=\frac{1}{2}\int_0^1 (x-4)dx=\frac{1}{2}\left[\frac{x^2}{2}-4x\right]_0^1$$

$$=\frac{1}{2}\left[\frac{(1)^2}{2}-4(1)\right]-\frac{1}{2}\left[\frac{(0)^2}{2}-4(0)\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}-4\right]-\frac{1}{2}[0-0]=-1\frac{3}{4}$$

18. $$\int_0^k (2x-4)\,dx=-3$$

$$\Rightarrow [x^2-4x]_0^k=-3$$

$$\Rightarrow [k^2-4k]-[(0)^2-4(0)]=-3$$

$$\Rightarrow k^2-4k+3=0$$

$$\Rightarrow (k-1)(k-3)=0 \quad \Rightarrow k=1 \quad \text{OR} \quad k=3$$

19. $$\int_0^k (x^2-3x)\,dx=0$$

$$\Rightarrow \left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^k=0$$

$$\Rightarrow \left[\frac{k^3}{3}-\frac{3k^2}{2}\right]-\left[\frac{(0)^3}{3}-\frac{3(0)^2}{2}\right]=0$$

$$\Rightarrow \frac{k^3}{3}-\frac{3k^2}{2}=0$$

$$\Rightarrow 2k^3-9k^2=0$$

$$\Rightarrow k^2(2k-9)=0$$

$$\Rightarrow k^2=0 \quad \text{OR} \quad 2k-9=0$$

$$\Rightarrow k=0 \quad \text{OR} \quad k=\frac{9}{2}$$

Since $k>0 \quad \Rightarrow \quad k=\frac{9}{2}$

20. $x^3-8=(x-2)(x^2+2x+4)$

$$\int_0^2 \frac{x^3-8}{x-2}dx=\int_0^2 \frac{(x-2)(x^2+2x+4)}{x-2}dx$$

$$=\int_0^2 (x^2+2x+4)dx$$

$$=\left[\frac{x^3}{3}+x^2+4x\right]_0^2$$

$$=\left[\frac{(2)^3}{3}+(2)^2+4(2)\right]-\left[\frac{(0)^3}{3}+(0)^2+4(0)\right]$$

$$=\frac{8}{3}+4+8-0$$

$$=14\frac{2}{3}$$

21. $$\int_0^1 nx^2\,dx=1$$

$$\Rightarrow \left[\frac{nx^3}{3}\right]_0^1=1$$

$$\Rightarrow \left[\frac{n(1)^3}{3}\right]-\left[\frac{n(0)^3}{3}\right]=1$$

$$\Rightarrow \frac{n}{3}-0=1 \quad \Rightarrow \quad n=3$$

22. (i) $$\int_0^{\frac{\pi}{4}} \cos 2x dx = \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \left[\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right] - \left[\frac{\sin 2(0)}{2}\right]$$

$$= \left[\frac{\sin \frac{\pi}{2}}{2}\right] - \left[\frac{\sin 0}{2}\right]$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

(ii) $$\int_0^{\frac{\pi}{6}} \sin 3x dx = \left[\frac{-\cos 3x}{3}\right]_0^{\frac{\pi}{6}} = \left[\frac{-\cos 3\left(\frac{\pi}{6}\right)}{3}\right] - \left[\frac{-\cos 3(0)}{3}\right]$$

$$= -\frac{\cos \frac{\pi}{2}}{3} + \frac{\cos 0}{3} = 0 + \frac{1}{3} = \frac{1}{3}$$

(iii) $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 5 \sin x dx = [-5 \cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \left[-5 \cos\left(\frac{\pi}{2}\right)\right] - \left[-5 \cos\left(\frac{\pi}{3}\right)\right]$$

$$= -5(0) + 5\left(\frac{1}{2}\right) = \frac{5}{2}$$

(iv) $$\int_0^{\frac{\pi}{2}} [2 \cos x + 1]dx = [2 \sin x + x]_0^{\frac{\pi}{2}}$$

$$= \left[2 \sin \frac{\pi}{2} + \frac{\pi}{2}\right] - [2 \sin 0 + 0]$$

$$= 2 + \frac{\pi}{2} - 0 = 2 + \frac{\pi}{2}$$

23. (i) $$\int_0^2 e^{4x}dx = \left[\frac{e^{4x}}{4}\right]_0^2 = \frac{e^{4(2)}}{4} - \frac{e^{4(0)}}{4} = \frac{1}{4}[e^8 - 1]$$

(ii) $$\int_{-1}^1 e^{x+3}dx = [e^{x+3}]_{-1}^1 = e^{1+3} - e^{-1+3} = e^4 - e^2$$

(iii) $$\int_0^1 e^{\frac{x}{2}}dx = \left[\frac{e^{\frac{x}{2}}}{\frac{1}{2}}\right]_0^1 = \left[2e^{\frac{x}{2}}\right]_0^1 = 2e^{\frac{1}{2}} - 2e^{\frac{0}{2}}$$

$$= 2e^{\frac{1}{2}} - 2$$

(iv) $$\int_0^1 (e^{-2x} + 1)dx = \left[\frac{e^{-2x}}{-2} + x\right]_0^1 = \left[-\frac{1}{2}e^{-2x} + x\right]_0^1$$

$$= \left[-\frac{1}{2}e^{-2(1)} + 1\right] - \left[-\frac{1}{2}e^{-2(0)} + 0\right]$$

$$= -\frac{1}{2}e^{-2} + 1 + \frac{1}{2} = \frac{3}{2} - \frac{1}{2e^2}$$

$$= \frac{1}{2}\left[3 - \frac{1}{e^2}\right]$$

24. (i) $$\int_0^1 (2e^{\frac{x}{3}} + 2)dx = \left[\frac{2e^{\frac{x}{3}}}{\frac{1}{3}} + 2x\right]_0^1$$

$$= \left[6e^{\frac{x}{3}} + 2x\right]_0^1$$

$$= \left[6e^{\frac{1}{3}} + 2(1)\right] - \left[6e^{\frac{0}{3}} + 2(0)\right]$$

$$= 6e^{\frac{1}{3}} + 2 - 6 - 0 = 6e^{\frac{1}{3}} - 4$$

(ii) $$\int_{-2}^{2} \frac{e^x + e^{-x}}{2} dx = \frac{1}{2}\int_{-2}^{2}(e^x + e^{-x})dx$$

$$= \frac{1}{2}\left[e^x + \frac{e^{-x}}{-1}\right]_{-2}^{2}$$

$$= \frac{1}{2}[e^x - e^{-x}]_{-2}^{2} = \frac{1}{2}[e^2 - e^{-2}] - \frac{1}{2}[e^{-2} - e^{-(-2)}]$$

$$= \frac{1}{2}e^2 - \frac{1}{2e^2} - \frac{1}{2e^2} + \frac{1}{2}e^2$$

$$= e^2 - \frac{1}{e^2}$$

(iii) $$\int_{1}^{3} 5^x dx = \left[\frac{5^x}{\ln 5}\right]_{1}^{3} = \frac{5^3}{\ln 5} - \frac{5^1}{\ln 5}$$

$$= \frac{125}{\ln 5} - \frac{5}{\ln 5} = \frac{120}{\ln 5}$$

(iv) $$\int_{0}^{e} 7^x dx = \left[\frac{7^x}{\ln 7}\right]_{0}^{e} = \frac{7^e}{\ln 7} - \frac{7^0}{\ln 7} = \frac{7^e}{\ln 7} - \frac{1}{\ln 7}$$

25. $f(x) = \frac{\cos x}{\sin x} \Rightarrow$ Quotient rule: $u = \cos x$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = -\sin x \quad \Rightarrow \frac{dv}{dx} = \cos x$$

$$f'(x) = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

Hence $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx = -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\left[\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}\right] + \left[\frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}\right]$$

$$= -\left(\frac{0}{1}\right) + \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 0 + 1 = 1$$

26. $f(x) = x \sin 3x \Rightarrow$ product rule: $u = x$ and $v = \sin 3x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \cos 3x\,.\,3$$

$$= 3\cos 3x$$

$$f'(x) = \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x\,.\,(3\cos 3x) + \sin 3x\,.\,1$$

$$= 3x\cos 3x + \sin 3x$$

Hence $\int(3x\cos 3x + \sin 3x)\,dx = x\sin 3x$

$$\Rightarrow \int_{0}^{\frac{\pi}{6}} 3x\cos 3x + \int_{0}^{\frac{\pi}{6}} \sin 3x\,dx = [x\sin 3x]_{0}^{\frac{\pi}{6}}$$

$$\Rightarrow \int_{0}^{\frac{\pi}{6}} 3x\cos 3x + \left[\frac{-\cos 3x}{3}\right]_{0}^{\frac{\pi}{6}} = \frac{\pi}{6}\sin 3\left(\frac{\pi}{6}\right) - 0\,.\,\sin 3(0)$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x\cos 3x + \left[\frac{-\cos 3\left(\frac{\pi}{6}\right)}{3}\right] - \left[\frac{-\cos 3(0)}{3}\right] = \frac{\pi}{6}\sin\frac{\pi}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x\cos 3x + \left[\frac{-\cos\frac{\pi}{2}}{3}\right] + \frac{\cos 0}{3} = \frac{\pi}{6}\,.\,1$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x\cos 3x - \frac{0}{3} + \frac{1}{3} = \frac{\pi}{6}$$

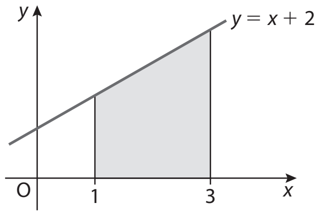
$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x\cos 3x = \frac{\pi}{6} - \frac{1}{3}$$

Exercise 3.5

1. Area $= \int_1^3 (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_1^3$

$$= \left[\frac{(3)^2}{2} + 2(3)\right] - \left[\frac{(1)^2}{2} + 2(1)\right]$$

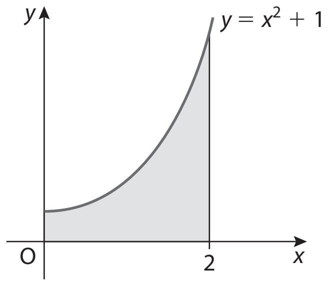
$$= \frac{9}{2} + 6 - \frac{1}{2} - 2 = 10\tfrac{1}{2} - 2\tfrac{1}{2} = 8 \text{ sq.units}$$



2. Area $= \int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^2$

$$= \left[\frac{(2)^3}{3} + 2\right] - \left[\frac{(0)^3}{3} + 0\right]$$

$$= \frac{8}{3} + 2 - 0 = 4\tfrac{2}{3} \text{ sq.units}$$

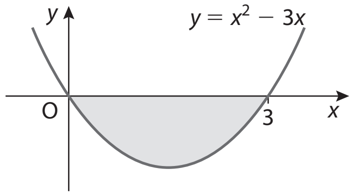


3. Area $= \int_0^3 (x^2-3x)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$

$$= \left[\frac{(3)^3}{3} - \frac{3(3)^2}{2}\right] - \left[\frac{(0)^3}{3} - \frac{3(0)^2}{2}\right]$$

$$= 9 - \frac{27}{2} - 0 = -4\tfrac{1}{2}$$

Hence Area $= 4\tfrac{1}{2}$ sq.units

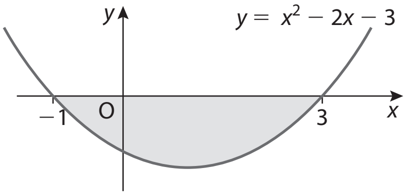


4. Area $= \int_{-1}^3 (x^2-2x-3)\,dx = \left[\frac{x^3}{3} - x^2 - 3x\right]_{-1}^3$

$$= \left[\frac{(3)^3}{3} - (3)^2 - 3(3)\right] - \left[\frac{(-1)^3}{3} - (-1)^2 - 3(-1)\right]$$

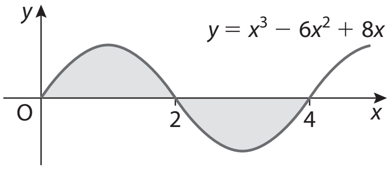
$$= (9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3\right)$$

$$= -9 - 1\tfrac{2}{3} = -10\tfrac{2}{3}$$



Hence Area $= 10\tfrac{2}{3}$ sq.units

5. Area $= \int_0^2 (x^3 - 6x^2 + 8x)\,dx + \int_2^4 (x^3 - 6x^2 + 8x)\,dx$



$$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2}\right]_0^2 + \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2}\right]_2^4$$

$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 + \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4$$

$$= \left[\frac{(2)^4}{4} - 2(2)^3 + 4(2)^2\right] - \left[\frac{0}{4} - 2(0) + 4(0)\right] + \left[\frac{(4)^4}{4} - 2(4)^3 + 4(4)^2\right] - \left[\frac{(2)^4}{4} - 2(2)^3 + 4(2)^2\right]$$

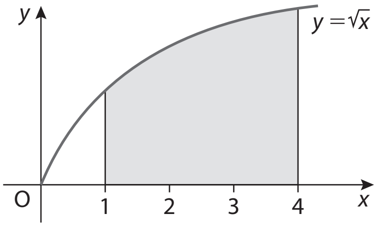
$$= [4 - 16 + 16] - [0] + [64 - 128 + 64] - [4 - 16 + 16]$$

$$= 4 + [-4] \Rightarrow \text{Area} = 4 + 4 = 8 \text{ sq.units}$$

6. Area $= \int_1^4 \sqrt{x}\,dx = \int_1^4 x^{\frac{1}{2}}dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^4$

$$= \left[\frac{2(4)^{\frac{3}{2}}}{3}\right] - \left[\frac{2(1)^{\frac{3}{2}}}{3}\right] = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

$$= 4\frac{2}{3} \text{ sq.units}$$

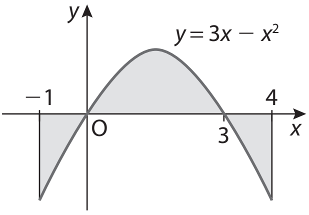


7. Area $= \int_{-1}^{0}(3x - x^2)dx + \int_0^3 (3x - x^2)\,dx + \int_3^4 (3x - x^2)\,dx$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 + \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_3^4$$

$$= \left[\frac{3(0)^2}{3} - \frac{(0)^3}{3}\right] - \left[\frac{3(-1)^2}{2} - \frac{(-1)^3}{3}\right] + \left[\frac{3(3)^2}{2} - \frac{(3)^3}{3}\right] - \left[\frac{3(0)^2}{2} - \frac{(0)^3}{3}\right]$$

$$+ \left[\frac{3(4)^2}{2} - \frac{(4)^3}{3}\right] - \left[\frac{3(3)^2}{2} - \frac{(3)^3}{3}\right]$$



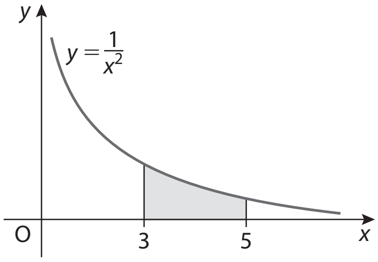
$$= 0 - \frac{11}{6} + 4\frac{1}{2} - 0 + 2\frac{2}{3} - 4\frac{1}{2}$$

$$= -\frac{11}{6} + 4\frac{1}{2} - \frac{11}{6}$$

$$\Rightarrow \text{Area} = \frac{11}{6} + 4\frac{1}{2} + \frac{11}{6} = 8\frac{1}{6} \text{ sq.units}$$

8. Area $= \int_3^5 \frac{1}{x^2}dx = \int_3^5 x^{-2}dx = \left[\frac{x^{-1}}{-1}\right]_3^5$

$$= \left[-\frac{1}{x}\right]_3^5 = \left[-\frac{1}{5}\right] - \left[-\frac{1}{3}\right] = \frac{2}{15} \text{ sq.units}$$



9. $y = x^2 - 3x - 4$

Points A and B are on x-axis $\Rightarrow y = 0$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = 4$ OR $x = -1$

$\Rightarrow A = (4, 0)$ and $B = (-1, 0)$

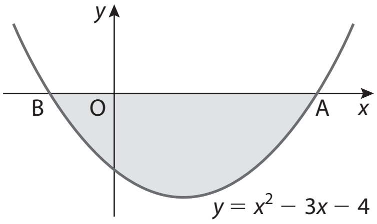
Area $= \int_{-1}^{4}(x^2 - 3x - 4)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_{-1}^{4}$

$$= \left[\frac{(4)^3}{3} - \frac{3(4)^2}{2} - 4(4)\right] - \left[\frac{(-1)^3}{3} - \frac{3(-1)^2}{2} - 4(-1)\right]$$

$$= \left[\frac{64}{3} - 24 - 16\right] - \left[-\frac{1}{3} - \frac{3}{2} + 4\right]$$

$$= -18\frac{2}{3} - 2\frac{1}{6} = -20\frac{5}{6}$$

$\Rightarrow$ Area $= 20\frac{5}{6}$ sq.units

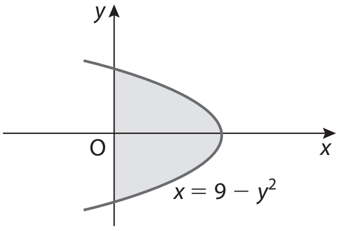


10. $x = 9 - y^2$

On y-axis, $x = 0 \Rightarrow 9 - y^2 = 0$

$\Rightarrow (3 + y)(3 - y) = 0 \Rightarrow y = -3$ OR $y = 3$

Points are (0, 3) and (0, −3)



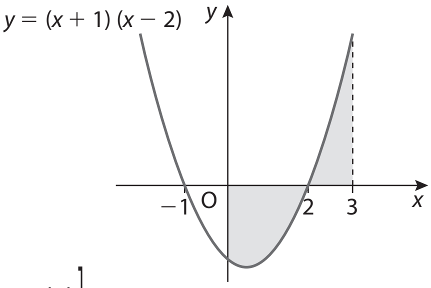
$$\text{Area} = \int_a^b x\,dy = \int_{-3}^{3}(9 - y^2)\,dy$$

$$= \left[9y - \frac{y^3}{3}\right]_{-3}^{3}$$

$$= \left[9(3) - \frac{(3)^3}{3}\right] - \left[9(-3) - \frac{(-3)^3}{3}\right]$$

$$= (27 - 9) - (-27 + 9) = 18 + 18 = 36 \text{ sq.units}$$

11. $y = (x + 1)(x - 2) = x^2 - x - 2$



$$\text{Area} = \int_0^2 (x^2 - x - 2)\,dx + \int_2^3 (x^2 - x - 2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_0^2 + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_2^3$$

$$= \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right] - \left[\frac{(0)^3}{3} - \frac{(0)^2}{2} - 2(0)\right] + \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} - 2(3)\right] - \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right]$$

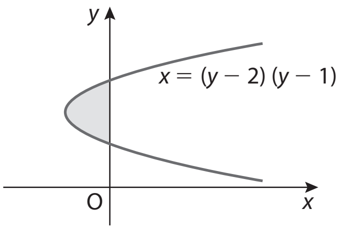
$$= \left(\frac{8}{3} - 2 - 4\right) - 0 + \left(9 - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - 2 - 4\right)$$

$$= -3\frac{1}{3} + 1\frac{5}{6}$$

Hence Area $= 3\frac{1}{3} + 1\frac{5}{6} = \frac{31}{6}$ sq.units

12. $x = (y - 2)(y - 1) \Rightarrow$ on y-axis, $x = 0 \Rightarrow (y - 2)(y - 1) = 0$

$\Rightarrow y = 2 \quad \text{or} \quad y = 1$



$x = (y - 2)(y - 1) = y^2 - 3y + 2$

$$\text{Area} = \int_1^2 (y^2 - 3y + 2)\,dy = \left[\frac{y^3}{3} - \frac{3y^2}{2} + 2y\right]_1^2$$

$$= \left[\frac{(2)^3}{3} - \frac{3(2)^2}{2} + 2(2)\right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} + 2(1)\right]$$

$$= \left[\frac{8}{3} - 6 + 4\right] - \left[\frac{1}{3} - \frac{3}{2} + 2\right]$$

$$= \frac{2}{3} - \frac{5}{6} = -\frac{1}{6}$$

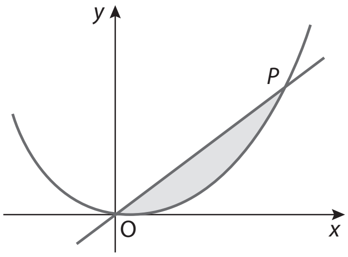
Hence Area $= \frac{1}{6}$ sq.units

13. (i) $y = 2x \cap y = x^2$

$\Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x - 2) = 0 \Rightarrow x = 0, x = 2$

At P: $x = 2 \Rightarrow y = 2(2) = 4 \Rightarrow P = (2, 4)$

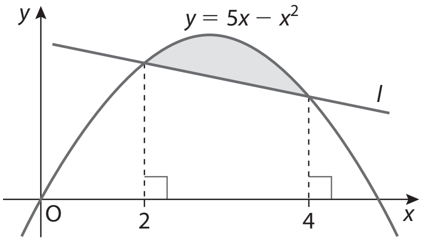


(ii) $$\text{Area} = \int_0^2 2x\,dx - \int_0^2 x^2\,dx$$

$$= [x^2]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= [[(2)^2] - [(0)^2]] - \left[\left(\frac{(2)^3}{3}\right) - \left(\frac{(0)^3}{3}\right)\right]$$

$$= (4 - 0) - \left(\frac{8}{3} - 0\right) = 4 - 2\frac{2}{3} = 1\frac{1}{3} \text{ sq.units}$$



14. $y = -x + 8 \cap y = 5x - x^2$

$\Rightarrow -x + 8 = 5x - x^2$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow (x - 2)(x - 4) = 0 \Rightarrow x = 2 \quad \text{OR} \quad x = 4$

$$\text{Area} = \int_2^4 (5x - x^2)\,dx - \int_2^4 (-x + 8)\,dx$$

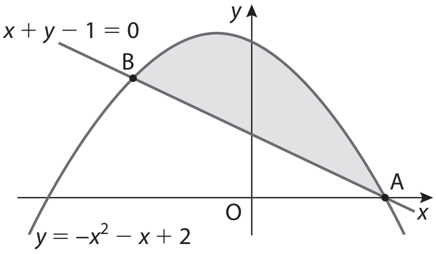
$$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_2^4 - \left[\frac{-x^2}{2} + 8x\right]_2^4$$

$$= \left[\left[\frac{5(4)^2}{2} - \frac{(4)^3}{3}\right] - \left[\frac{5(2)^2}{2} - \frac{(2)^3}{3}\right]\right] - \left[\left[\frac{-(4)^2}{2} + 8(4)\right] - \left[\frac{-(2)^2}{2} + 8(2)\right]\right]$$

$$= \left[40 - 21\frac{1}{3}\right] - \left[10 - \frac{8}{3}\right] - [(-8 + 32) - (-2 + 16)]$$

$$= \left[18\frac{2}{3} - 7\frac{1}{3}\right] - [24 - 14]$$

$$= 11\frac{1}{3} - 10 = 1\frac{1}{3}\ \text{sq.units}$$



15. (i) $x + y - 1 = 0 \Rightarrow y = -x + 1$

Line $y = -x + 1 \quad \cap \quad$ Curve $y = -x^2 - x + 2$

$\Rightarrow -x + 1 = -x^2 - x + 2$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x + 1)(x - 1) = 0 \Rightarrow x = -1 \quad \text{OR} \quad x = 1$

At B, $x = -1 \Rightarrow y = -(-1) + 1 = 2 \Rightarrow \text{B} = (-1, 2)$

At A, $x = 1 \Rightarrow y = -1 + 1 = 0 \qquad \Rightarrow \text{A} = (1, 0)$

(ii) Shaded Area = Area under the curve − Area under the line

$$= \int_{-1}^{1} (-x^2 - x + 2)\,dx - \int_{-1}^{1} (-x + 1)\,dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-1}^{1} - \left[-\frac{x^2}{2} + x\right]_{-1}^{1}$$

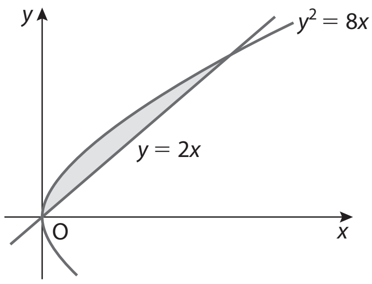
$$= \left[-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 2(1)\right] - \left[-\frac{(-1)^3}{3} - \frac{(-1)^2}{2} + 2(-1)\right]$$

$$- \left[\left[\frac{-(1)^2}{2} + (1)\right] - \left[\frac{-(-1)^2}{2} + (-1)\right]\right]$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{3} - \frac{1}{2} - 2\right)\right] - \left[\left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{2} - 1\right)\right]$$

$$= \left(1\frac{1}{6} + 2\frac{1}{6}\right) - \left(\frac{1}{2} + 1\frac{1}{2}\right)$$

$$= 3\frac{1}{3} - 2 = 1\frac{1}{3}\ \text{sq.units}$$



16. $y = 2x \cap y^2 = 8x \quad \text{OR} \quad y = \sqrt{8}x^{\frac{1}{2}}$

$\Rightarrow \quad y^2 = 4x^2 \cap y^2 = 8x$

$\Rightarrow \quad 4x^2 = 8x \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x - 2) = 0$

$\Rightarrow x = 0 \quad \text{OR} \quad x = 2$

When $x = 0 \Rightarrow y = 2(0) = 0 \Rightarrow$ Point $(0, 0)$

When $x = 2 \Rightarrow y = 2(2) = 4 \Rightarrow$ Point $(2, 4)$

Shaded area = Area under the curve − area under the line

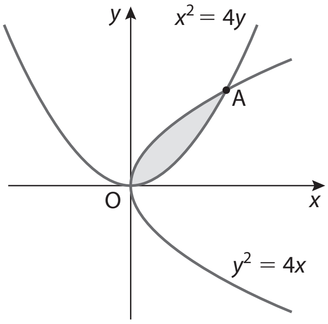
$$= \int_0^2 \sqrt{8}x^{\frac{1}{2}}dx - \int_0^2 2x\,dx$$

$$= \left[\sqrt{8}\,.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 - [x^2]_0^2$$

$$= \left[\frac{2\sqrt{8}}{3}(2)^{\frac{3}{2}} - 0\right] - [[2^2] - 0]$$

$$= \frac{2\sqrt{8}}{3}\cdot 2\sqrt{2} - 4 = \frac{16}{3} - 4 = 1\frac{1}{3} \text{ sq.units}$$

17. (i) $y^2 = 4x \cap x^2 = 4y \Rightarrow y = \frac{x^2}{4}$



$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow \frac{x^4}{16} = 4x$$

$$\Rightarrow x^4 = 64x$$

$$\Rightarrow x^4 - 64x = 0$$

$$\Rightarrow x(x - 4)(x^2 + 4x + 16) = 0$$

$$\Rightarrow x = 0, x = 4$$

When $x = 4 \Rightarrow y = \frac{(4)^2}{4} = 4 \Rightarrow A = (4, 4)$

(ii) $y^2 = 4x \Rightarrow y = \sqrt{4x} = 2\sqrt{x} = 2x^{\frac{1}{2}}$

Shaded Area = Area under curve $y = 2x^{\frac{1}{2}}$ − area under curve $y = \frac{x^2}{4}$

Shaded area $= \int_0^4 2x^{\frac{1}{2}}dx - \int_0^4 \frac{x^2}{4}dx$

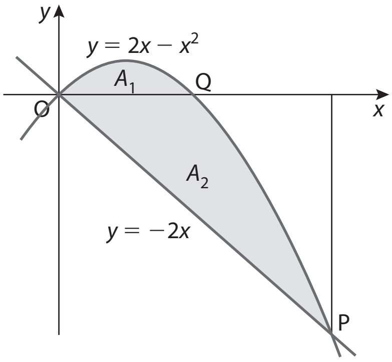
$$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 - \left[\frac{1}{4}\cdot\frac{x^3}{3}\right]_0^4$$

$$= \left[\frac{4}{3}(x)^{\frac{3}{2}}\right]_0^4 - \left[\frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}(4)^{\frac{3}{2}} - 0\right] - \left[\frac{(4)^3}{12} - 0\right]$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} = 5\frac{1}{3} \text{ sq.units}$$

18. (i) $y = 2x - x^2 \Rightarrow$ on x-axis, $y = 0$



$$\Rightarrow 2x - x^2 = 0$$

$$\Rightarrow x(2 - x) = 0 \Rightarrow x = 0 \quad \text{OR} \quad x = 2$$

$$\Rightarrow Q = (2, 0)$$

Line: $y = -2x \cap$ Curve: $y = 2x - x^2$

$$\Rightarrow -2x = 2x - x^2$$

$$\Rightarrow x^2 - 4x = 0$$

$$\Rightarrow x(x - 4) = 0$$

$$\Rightarrow x = 0 \quad \text{OR} \quad x = 4$$

When $x = 4 \Rightarrow y = -2(4) = -8 \Rightarrow P = (4, -8)$

(ii) Area $A_1 = \int_0^2 (2x - x^2)\,dx$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= \left[(2)^2 - \frac{(2)^3}{3}\right] - \left[(0)^2 - \frac{(0)^3}{3}\right]$$

$$= 4 - \frac{8}{3} - 0 = 1\frac{1}{3}$$

$$\text{Area } A_2 = \int_0^4 (-2x)\,dx - \int_2^4 (2x - x^2)\,dx$$

$$= [-x^2]_0^4 - \left[x^2 - \frac{x^3}{3}\right]_2^4$$

$$= [-(4)^2 - (-0)^2] - \left[\left[(4)^2 - \frac{(4)^3}{3}\right] - \left[(2)^2 - \frac{(2)^3}{3}\right]\right]$$

$$= -16 - \left[\left(16 - \frac{64}{3}\right) - \left(4 - \frac{8}{3}\right)\right]$$

$$= -16 - \left[\frac{-16}{3} - \frac{4}{3}\right]$$

$$= -16 + \frac{20}{3} = -\frac{28}{3}$$

Hence Area $A_2 = \frac{28}{3} = 9\frac{1}{3}$

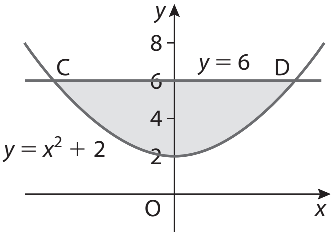
$\Rightarrow$ Area shaded region $= 1\frac{1}{3} + 9\frac{1}{3} = 10\frac{2}{3}$ sq.units

19. (i) $y = x^2 + 2 \cap y = 6$

$\Rightarrow x^2 + 2 = 6 \Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0 \Rightarrow x = -2$ OR $x = 2$

Hence C = $(-2, 6)$ and D = $(2, 6)$



(ii) Shaded area = area under line: $y = 6$ − area under curve: $y = x^2 + 2$

$$= \int_{-2}^{2} 6\,dx - \int_{-2}^{2} (x^2 + 2)\,dx$$

$$= [6x]^{2}_{-2} - \left[\frac{x^3}{3} + 2x\right]_{-2}^{2}$$

$$= [6(2) - 6(-2)] - \left[\left[\frac{(2)^3}{3} + 2(2)\right] - \left[\frac{(-2)^3}{3} + 2(-2)\right]\right]$$

$$= 12 + 12 - \left[\left(\frac{8}{3} + 4\right) - \left(\frac{-8}{3} - 4\right)\right]$$

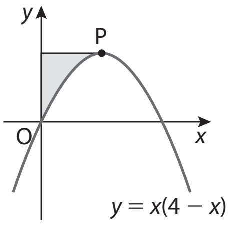
$$= 24 - 13\frac{1}{3} = 10\frac{2}{3} \text{ sq.units}$$

20. (i) $y = x(4 - x) = 4x - x^2$

$\frac{dy}{dx} = 4 - 2x = 0$ for maximum

$\Rightarrow 2x = 4 \Rightarrow x = 2$

$\Rightarrow y = 4(2) - (2)^2 = 4 \Rightarrow P = (2, 4)$



(ii) Shaded area = area between line: $x = 2$ and y-axis

− area between curve: $y = 4x^2 - x^2$ [OP] and x-axis

$$\Rightarrow \text{Shaded area} = \int_0^4 2\,dy - \int_0^2 (4x - x^2)\,dx$$

$$= [2y]_0^4 - \left[2x^2 - \frac{x^3}{3}\right]_0^2$$

$$= [2(4) - 2(0)] - \left[\left[2(2)^2 - \frac{(2)^3}{3}\right] - \left[2(0)^2 - \frac{(0)^3}{3}\right]\right]$$

$$= (8 - 0) - \left[\left(8 - \frac{8}{3}\right) - (0 - 0)\right]$$

$$= 8 - 8 + \frac{8}{3} = 2\frac{2}{3} \text{ sq.units}$$

21. $\int_0^{2k}(x^2+4x+4)\,dx = 4\int_0^{k}(x^2+4x+4)\,dx$

$= \left[\frac{x^3}{3}+2x^2+4x\right]_0^{2k} = 4\left[\frac{x^3}{3}+2x^2+4x\right]_0^{k}$

$= \left[\frac{(2k)^3}{3}+2(2k)^2+4(2k)\right]-\left[\frac{(0)^3}{3}+2(0)^2+4(0)\right] = 4\left[\frac{k^3}{3}+2k^2+4k\right]-4(0)$

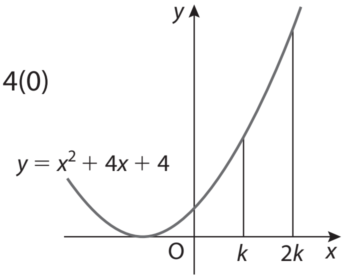
$= \frac{8k^3}{3}+8k^2+8k-0 = \frac{4k^3}{3}+8k^2+16k-0$

$\Rightarrow 8k^3+24k = 4k^3+48k$

$\Rightarrow 4k^3-24k = 0$

$\Rightarrow k(k^2-6) = 0 \Rightarrow k = 0$ OR $k^2 = 6$

$\Rightarrow k = \sqrt{6}$



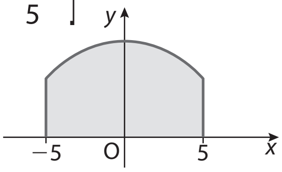
22. (i) Area $= \int_{-5}^{5}(6-0.08x^2-0.0006x^4)\,dx$

$= \left[6x-0.08\frac{x^3}{3}-0.0006\frac{x^5}{5}\right]_{-5}^{5}$

$= \left[6(5)-0.08\frac{(5)^3}{3}-0.0006\frac{(5)^5}{5}\right]-\left[6(-5)-0.08\frac{(-5)^3}{3}-0.0006\frac{(-5)^5}{5}\right]$

$= \left(30-\frac{10}{3}-\frac{3}{8}\right)-\left(-30+\frac{10}{3}+\frac{3}{8}\right) = 52.58 = 52.6$ sq.units

(ii) Volume of the tunnel $= 52.6\times 14 = 736.4 = 736.4 = 736\text{ m}^3$



23. (i) $y = 2x^2-3x+2 \cap y = x^2+x+7$

$\Rightarrow 2x^2-3x+2 = x^2+x+7$

$\Rightarrow x^2-4x-5 = 0$

$\Rightarrow (x+1)(x-5) = 0 \quad \Rightarrow x = -1$ OR $x = 5$

$x = -1 \Rightarrow g(-1) = (-1)^2+(-1)+7 = 1-1+7 = 7 \Rightarrow$ point $(-1, 7)$

$x = 5 \Rightarrow g(5) = (5)^2+(5)+7 = 25+5+7 = 37 \Rightarrow$ point $(5, 37)$

(ii) Shaded area = area under $g(x)$ − area under $f(x)$

$= \int_{-1}^{5}(x^2+x+7)\,dx - \int_{-1}^{5}(2x^2-3x+2)\,dx$

$= \int_{-1}^{5}(x^2+x+7)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}+7x\right]_{-1}^{5}$

$= \left[\frac{(5)^3}{3}+\frac{(5)^2}{2}+7(5)\right]-\left[\frac{(-1)^3}{3}+\frac{(-1)^2}{2}+7(-1)\right]$

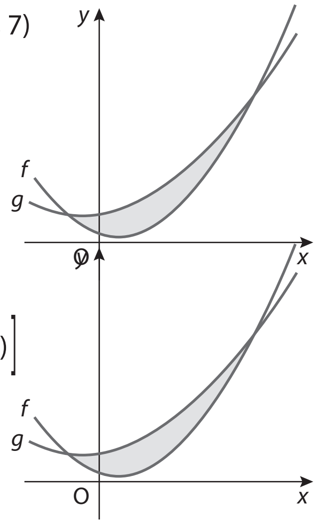
$= \left(\frac{125}{3}+\frac{25}{2}+35\right)-\left(-\frac{1}{3}+\frac{1}{2}-7\right) = 96$

$\int_{-1}^{5}(2x^2-3x+2)\,dx = \left[\frac{2x^3}{3}-\frac{3x^2}{2}+2x\right]_{-1}^{5}$

$= \left[\frac{2(5)^3}{3}-\frac{3(5)^2}{2}+2(5)\right]-\left[\frac{2(-1)^3}{3}-\frac{3(-1)^2}{2}+2(-1)\right]$

$= \left(\frac{250}{3}-\frac{75}{2}+10\right)-\left(-\frac{2}{3}-\frac{3}{2}-2\right) = 60$

$\Rightarrow$ Shaded Area $= 96-60 = 36$ sq.units

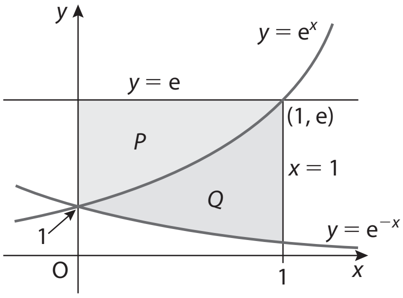


24. (i) Area $P = \int_0^1 \mathrm{e}\,dx - \int_0^1 \mathrm{e}^x\,dx$

$= [\mathrm{e}x]_0^1 - [\mathrm{e}^x]_0^1$

$= [\mathrm{e}(1)-\mathrm{e}(0)]-[\mathrm{e}^1-\mathrm{e}^0]$

$= (\mathrm{e}-0)-(\mathrm{e}-1) = \mathrm{e}-\mathrm{e}+1 = 1$ sq.units



(ii) Area $Q = \int_0^1 e^x\,dx - \int_0^1 e^{-x}\,dx$

$$= [e^x]_0^1 - \left[\frac{e^{-x}}{-1}\right]_0^1$$

$$= [e^x]_0^1 - \left[\frac{-1}{e^x}\right]_0^1$$

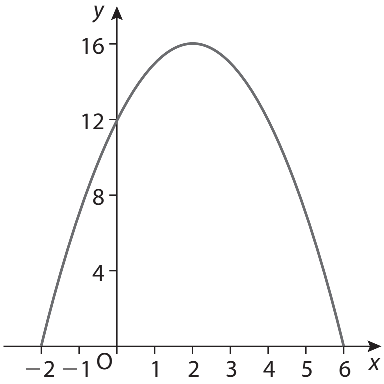
$$= [e^1 - e^0] - \left[-\frac{1}{e^1} + \frac{1}{e^0}\right]$$

$$= e - 1 + \frac{1}{e} - 1 = (e + \frac{1}{e} - 2) \text{ sq.units}$$

Exercise 3.6

1. (i) Average value $= \frac{f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6)}{7}$

$$= \frac{12 + 15 + 16 + 15 + 12 + 7 + 0}{7} = \frac{77}{7} = 11$$



(ii) Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{6-0}\int_0^6 (-x^2 + 4x + 12)\,dx$$

$$= \frac{1}{6}\left[\frac{-x^3}{3} + 2x^2 + 12x\right]_0^6$$

$$= \frac{1}{6}\left[\frac{-(6)^3}{3} + 2(6)^2 + 12(6)\right] - \frac{1}{6}\left[\frac{-(0)^3}{3} + 2(0)^2 + 12(0)\right]$$

$$= \frac{1}{6}[-72 + 72 + 72] - 0 = 12$$

(iii) Method (ii) gives the exact estimate

2. (i) $f(x) = 2x - 4$

$f(2) = 2(2) - 4 = 0$

$f(3) = 2(3) - 4 = 2$

$f(4) = 2(4) - 4 = 4$

$f(5) = 2(5) - 4 = 6$

$\Rightarrow$ average value $= \frac{0 + 2 + 4 + 6}{4} = \frac{12}{4} = 3$

OR

Average value $= \frac{1}{5-2}\int_2^5 (2x - 4)\,dx$

$$= \frac{1}{3}[x^2 - 4x]_2^5$$

$$= \frac{1}{3}[(5)^2 - 4(5)] - \frac{1}{3}[(2)^2 - 4(2)]$$

$$= \frac{1}{3}[5] - \frac{1}{3}[-4] = \frac{5}{3} + \frac{4}{3} = 3$$

(ii) Average value $= \frac{1}{2-0}\int_0^2 (x^2 - x)\,dx$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{(2)^3}{3} - \frac{(2)^2}{2}\right] - \frac{1}{2}\left[\frac{(0)^3}{3} - \frac{(0)^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{8}{3} - 2\right] - 0 = \frac{1}{3}$$

(iii) Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 (2x - x^2)\,dx$

$$= \frac{1}{2}\left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left[(2)^2 - \frac{(2)^3}{3}\right] - \frac{1}{2}\left[(0)^2 - \frac{(0)^3}{3}\right] = \frac{2}{3}$$

3. Average value $= \dfrac{1}{4-0}\displaystyle\int_0^4 x^3\,dx$

$$= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^4$$

$$= \frac{1}{4}\left[\frac{(4)^4}{4}\right] - \frac{1}{4}\left[\frac{(0)^4}{4}\right]$$

$$= \frac{1}{4}(64) - 0 = 16$$

4. $f(x) = x^2 + 4$

Average value $= \dfrac{1}{3-(-2)}\displaystyle\int_{-2}^3 (x^2 + 4)\,dx$

$$= \frac{1}{5}\left[\frac{x^3}{3} + 4x\right]_{-2}^3$$

$$= \frac{1}{5}\left[\frac{(3)^3}{3} + 4(3)\right] - \frac{1}{5}\left[\frac{(-2)^3}{3} + 4(-2)\right]$$

$$= \frac{1}{5}(21) - \frac{1}{5}\left(\frac{-32}{3}\right) = 6\frac{1}{3}$$

5. (i) $f(x) = \sin x$

Average value $= \dfrac{1}{\frac{\pi}{2} - 0}\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$

$$= \frac{2}{\pi}[-\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\cos\frac{\pi}{2}\right] - [-\cos 0]$$

$$= \frac{2}{\pi}(-0) + (1) = \frac{2}{\pi}$$

(ii) $f(x) = \cos x$

Average value $= \dfrac{1}{2\pi - 0}\displaystyle\int_0^{2\pi} \cos x\,dx$

$$= \frac{1}{2\pi}[\sin x]_0^{2\pi}$$

$$= \frac{1}{2\pi}(\sin 2\pi) - \frac{1}{2\pi}(\sin 0) = 0 - 0 = 0$$

(iii) $f(x) = e^x$

Average value $= \dfrac{1}{3-0}\displaystyle\int_0^3 e^x\,dx$

$$= \frac{1}{3}[e^x]_0^3$$

$$= \frac{1}{3}(e^3) - \frac{1}{3}(e^0) = \frac{1}{3}(e^3 - 1)$$

(iv) $f(x) = e^{4x}$

$$\text{Average value} = \frac{1}{2-0}\int_0^2 e^{4x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{4}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{e^{4(2)}}{4}\right) - \frac{1}{2}\left[\frac{e^{4(0)}}{4}\right]$$

$$= \frac{e^8}{8} - \frac{1}{8}$$

6. $f(x) = x + 1$

$$\text{Average value} = \frac{1}{k-2}\int_2^k (x+1)\,dx = 8$$

$$= \frac{1}{k-2}\left[\frac{x^2}{2} + x\right]_2^k = 8$$

$$= \frac{1}{k-2}\left[\frac{k^2}{2} + k\right] - \frac{1}{k-2}\left[\frac{(2)^2}{2} + 2\right] = 8$$

$$\Rightarrow \frac{k^2}{2} + k - 4 = 8\,(k-2) = 8k - 16$$

$$\Rightarrow k^2 + 2k - 8 = 16k - 32$$

$$\Rightarrow k^2 - 14k + 24 = 0$$

$$\Rightarrow (k-2)(k-12) = 0$$

$$\Rightarrow k = 2 \quad \text{OR} \quad k = 12 \;\Rightarrow\; k = 12$$

7. $f(x) = x^3$

$$\text{Average value} = \frac{1}{k-0}\int_0^k x^3\,dx = 16$$

$$\Rightarrow \frac{1}{k}\left[\frac{x^4}{4}\right]_0^k = 16$$

$$\Rightarrow \frac{1}{k}\left[\frac{k^4}{4}\right] - \frac{1}{k}\left[\frac{(0)^4}{4}\right] = 16$$

$$\Rightarrow \frac{k^3}{4} = 16$$

$$\Rightarrow k^3 = 64 \Rightarrow k = \sqrt[3]{64} = 4$$

8. (i) $f(x) = \dfrac{1}{x^2} = x^{-2}$

$$\text{Average value} = \frac{1}{5-1}\int_1^5 x^{-2}\,dx$$

$$= \frac{1}{4}\left[\frac{x^{-1}}{-1}\right]_1^5$$

$$= \frac{1}{4}\left[-\frac{1}{x}\right]_1^5$$

$$= \frac{1}{4}\left[-\frac{1}{5}\right] - \frac{1}{4}\left[-\frac{1}{1}\right]$$

$$= -\frac{1}{20} + \frac{1}{4} = \frac{1}{5}$$

(ii) $f(x) = 5\left(\cos\frac{x}{2}\right)$

$$\text{Average value} = \frac{1}{2\pi - 0}\int_0^{2\pi} 5\left(\cos\frac{x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{5\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{2\pi} = \frac{1}{2\pi}\left[10\sin\frac{x}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[10\sin\frac{2\pi}{2}\right] - \frac{1}{2\pi}\left[10\sin\frac{0}{2}\right]$$

$$= \frac{1}{2\pi}[10(0)] - \frac{1}{2\pi}[10(0)]$$

$$= \frac{1}{2\pi}(0) - \frac{1}{2\pi}(0) = 0$$

9. $V = \pi\frac{h^3}{12}$

$$\text{Average volume} = \frac{1}{8-2}\int_2^8 \pi\frac{h^3}{12}\,dh$$

$$= \frac{1}{6}\left[\frac{\pi}{12}\frac{h^4}{4}\right]_2^8$$

$$= \frac{\pi}{288}[h^4]_2^8$$

$$= \frac{\pi}{288}(8)^4 - \frac{\pi}{288}(2)^4$$

$$= \frac{\pi}{288}[4096 - 16] = \frac{85\pi}{6}\text{ cm}^3$$

10. $v = 9.8t$

$$\text{Average velocity} = \frac{1}{3-0}\int_0^3 (9.8t)\,dt$$

$$= \frac{1}{3}\left[9.8\frac{t^2}{2}\right]_0^3$$

$$= \frac{1}{3}\left[9.8\frac{(3)^2}{2}\right] - \frac{1}{3}\left[9.8\frac{(0)^2}{2}\right]$$

$$= \frac{1}{3}\left(\frac{441}{10}\right) - 0 = \frac{147}{10}\text{ m/sec}$$

11. (i) $v = 3t^2 - 4$

$$\text{Average velocity} = \frac{1}{3-1}\int_1^3 (3t^2 - 4)\,dt$$

$$= \frac{1}{2}\left[\frac{3t^3}{3} - 4t\right]_1^3$$

$$= \frac{1}{2}[t^3 - 4t]_1^3$$

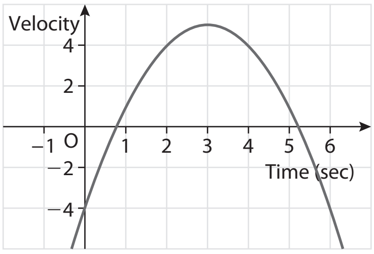
$$= \frac{1}{2}[(3)^3 - 4(3)] - \frac{1}{2}[(1)^3 - 4(1)]$$

$$= \frac{1}{2}(15) - \frac{1}{2}(-3) = 9\text{ m/sec}$$

(ii) $v = 3t^2 - 4 \Rightarrow a = \dfrac{dv}{dt} = 6t$

$$\text{Average acceleration} = \frac{1}{3-1}\int_1^3 6t\,dt$$
$$= \frac{1}{2}\left[\frac{6t^2}{2}\right]_1^3$$
$$= \frac{1}{2}[3t^2]_1^3$$
$$= \frac{1}{2}[3(3)^2] - \frac{1}{2}[3(1)^2]$$
$$= \frac{27}{2} - \frac{3}{2} = 12\text{ m/sec}^2$$

12. (i) $v = 5 - (t-3)^2 = 5 - (t^2 - 6t + 9) = -t^2 + 6t - 4$



$$\text{Average velocity} = \frac{1}{6-0}\int_0^6 (-t^2 + 6t - 4)\,dt$$
$$= \frac{1}{6}\left[\frac{-t^3}{3} + \frac{6t^2}{2} - 4t\right]_0^6$$
$$= \frac{1}{6}\left[\frac{-t^3}{3} + 3t^2 - 4t\right]_0^6$$
$$= \frac{1}{6}\left[\frac{-(6)^3}{3} + 3(6)^2 - 4(6)\right] - \frac{1}{6}\left[\frac{-(0)^3}{3} + 3(0)^2 - 4(0)\right]$$
$$= \frac{1}{6}[-72 + 108 - 24] - \frac{1}{6}(0) = 2\text{ m/sec}$$

(ii) $\text{Velocity} = 2 \Rightarrow -t^2 + 6t - 4 = 2$

$$\Rightarrow t^2 - 6t + 6 = 0$$
$$\Rightarrow t = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm \sqrt{12}}{2}$$
$$= \frac{6 \pm 2\sqrt{3}}{2} = (3 \pm \sqrt{3})\text{ sec}$$

13. $T = 30x$

$$\text{Average tension} = \frac{1}{0.2 - 0.1}\int_{0.1}^{0.2} 30x\,dx$$
$$= \frac{1}{0.1}\left[30\frac{x^2}{2}\right]_{0.1}^{0.2}$$
$$= 10[15x^2]_{0.1}^{0.2}$$
$$= 10[15(0.2)^2] - 10[15(0.1)^2]$$
$$= 6 - 1.5 = 4.5\text{ newtons}$$

14. (i) $y = x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{x}} \Rightarrow \dfrac{1}{y} = \sqrt{x}$

$$\text{Average value} = \frac{1}{4-1}\int_1^4 x^{\frac{1}{2}}\,dx$$
$$= \frac{1}{3}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{9}\left[x^{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{9}(4)^{\frac{3}{2}} - \frac{2}{9}(1)^{\frac{3}{2}}$$
$$= \frac{16}{9} - \frac{2}{9} = \frac{14}{9}$$

(ii) $$\text{Area} = \int_1^4 x^{-\frac{1}{2}}\,\mathrm{d}x = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= [2\sqrt{x}]_1^4 = 2(\sqrt{4}) - 2(\sqrt{1}) = 2 \text{ sq. units}$$

15. $$pv^{\frac{3}{4}} = 30 \Rightarrow p = \frac{30}{p^{\frac{3}{4}}} = 30p^{-\frac{3}{4}}$$

$$\text{Average pressure} = \frac{1}{16-1}\int_1^{16} 30p^{-\frac{3}{4}}\,\mathrm{d}p$$

$$= \frac{30}{15}\left[\frac{p^{\frac{1}{4}}}{\frac{1}{4}}\right]_1^{16} = 2.4\left[p^{\frac{1}{4}}\right]_1^{16}$$

$$= 8(16)^{\frac{1}{4}} - 8(1)^{\frac{1}{4}}$$

$$= 16 - 8 = 8$$

16. $v = 40 - 10t$

$$\text{Average velocity} = \frac{1}{3-1}\int_1^3 (40 - 10t)\,\mathrm{d}t$$

$$= \frac{1}{2}[40t - 5t^2]_1^3$$

$$= \frac{1}{2}[40(3) - 5(3)^2] - \frac{1}{2}[40(1) - 5(1)^2]$$

$$= \frac{1}{2}[120 - 45] - \frac{1}{2}[40 - 5]$$

$$= \frac{75}{2} - \frac{35}{2} = \frac{40}{2} = 20 \text{ m/sec}$$

Revision Exercise 3 (Core)

1. (i) $\int(2x + 5)\,\mathrm{d}x = x^2 + 5x + c$

(ii) $\int(3x^2 - 2x + 4)\,\mathrm{d}x = x^3 - x^2 + 4x + c$

(iii) $$\int\left(x^2 + \frac{1}{x^2}\right)\mathrm{d}x = \int(x^2 + x^{-2})\,\mathrm{d}x$$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + c = \frac{x^3}{3} - \frac{1}{x} + c$$

2. (i) $\int\sin 3x\,\mathrm{d}x = -\dfrac{\cos 3x}{3} + c$

(ii) $\int\cos 5x\,\mathrm{d}x = \dfrac{\sin 5x}{5} + c$

(iii) $\int(2\sin x + 3\cos 2x)\,\mathrm{d}x$

$$= -2\cos x + 3\frac{\sin 2x}{2} + c$$

$$= -2\cos x + \frac{3}{2}\sin 2x + c$$

3. (i) $\int e^{5x}\,\mathrm{d}x = \dfrac{e^{5x}}{5} + c$

(ii) $\int(e^{2x} + e^{-x})\,\mathrm{d}x = \dfrac{e^{2x}}{2} + \dfrac{e^{-x}}{-1} + c = \dfrac{e^{2x}}{2} - e^{-x} + c$

(iii) $\int(4 + e^{3x})\,\mathrm{d}x = 4x + \dfrac{e^{3x}}{3} + c$

4. $\dfrac{dy}{dx} = x^2 - 3x + 2$

$$y = \int (x^2 - 3x + 2)\,dx$$

$$= \frac{x^3}{3} - 3\frac{x^2}{2} + 2x + c = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$$

5. (i) $\displaystyle\int \frac{x^3 - 2}{x^2}\,dx = \int \left(\frac{x^3}{x^2} - \frac{2}{x^2}\right) dx$

$$= \int (x - 2x^{-2})\,dx$$

$$= \frac{x^2}{2} - \frac{2x^{-1}}{-1} + c = \frac{x^2}{2} + \frac{2}{x} + c$$

(ii) $\displaystyle\int (\sqrt{x} - 3)\,dx = \int (x^{\frac{1}{2}} - 3)\,dx$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3x + c = \frac{2}{3}x^{\frac{3}{2}} - 3x + c$$

(iii) $\displaystyle\int (\sqrt{x} + 3)^2\,dx = \int (x + 6\sqrt{x} + 9)\,dx$

$$= \int (x + 6x^{\frac{1}{2}} + 9)\,dx$$

$$= \frac{x^2}{2} + 6\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 9x + c$$

$$= \frac{x^2}{2} + 4x^{\frac{3}{2}} + 9x + c$$

6. (i) $\displaystyle\int_0^3 (2x^2 - 4x + 1)\,dx$

$$= \left[2\frac{x^3}{3} - 2x^2 + x\right]_0^3$$

$$= \left[\frac{2}{3}(3)^3 - 2(3)^2 + 3\right] - \left[\frac{2}{3}(0)^3 - 2(0)^2 + 0\right]$$

$$= 18 - 18 + 3 - 0 = 3$$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x\,dx = \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$$= \frac{\sin 2\left(\frac{\pi}{4}\right)}{2} - \frac{\sin 2(0)}{2}$$

$$= \frac{\sin \frac{\pi}{2}}{2} - \sin 0 = \frac{1}{2} - 0 = \frac{1}{2}$$

(iii) $\displaystyle\int_0^{\frac{\pi}{3}} (\cos 3\theta + \sin 3\theta)\,d\theta$

$$= \frac{\sin 3\theta}{3} - \frac{\cos 3\theta}{3}\Bigg]_0^{\frac{\pi}{3}}$$

$$= \left[\frac{\sin 3\left(\frac{\pi}{3}\right)}{3} - \frac{\cos 3\left(\frac{\pi}{3}\right)}{3}\right] - \left[\frac{\sin 3(0)}{3} - \frac{\cos 3(0)}{3}\right]$$

$$= \left[\frac{\sin \pi}{3} - \frac{\cos \pi}{3}\right] - \left[\frac{\sin 0}{3} - \frac{\cos 0}{3}\right]$$

$$= \left[\frac{0}{3} - \frac{(-1)}{3}\right] - \left[\frac{0}{3} - \frac{1}{3}\right] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

7. $\int_0^1 \left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$

$$= \left[\frac{2}{3}(1)^{\frac{3}{2}} + \frac{2}{5}(1)^{\frac{5}{2}}\right] - \left[\frac{2}{3}(0)^{\frac{3}{2}} + \frac{2}{5}(0)^{\frac{5}{2}}\right]$$

$$= \frac{2}{3} + \frac{2}{5} - 0 - 0 = \frac{16}{15}$$

8. (i) $\int_0^3 (e^{2x} + 1)\, dx = \left[\frac{e^{2x}}{2} + x\right]_0^3 = \left[\frac{e^{2(3)}}{2} + 3\right] - \left[\frac{e^{2(0)}}{2} + 0\right]$

$$= \frac{e^6}{2} + 3 - \frac{1}{2} - 0 = \frac{1}{2}e^6 + 2\frac{1}{2}$$

(ii) $\int_0^2 2e^{-2x}\, dx = \left[\frac{2e^{-2x}}{-2}\right]_0^2 = \left[\frac{-1}{e^{2x}}\right]_0^2$

$$= \left[-\frac{1}{e^{2(2)}}\right] - \left[\frac{-1}{e^{2(0)}}\right] = \frac{-1}{e^4} + 1$$

(iii) $\int_1^2 \left(e^{2x} + \frac{4}{x^2}\right) dx = \int_1^2 (e^{2x} + 4x^{-2})\, dx$

$$= \left[\frac{e^{2x}}{2} + \frac{4x^{-1}}{-1}\right]_1^2 = \left[\frac{e^{2x}}{2} - \frac{4}{x}\right]_1^2$$

$$= \left[\frac{e^{2(2)}}{2} - \frac{4}{2}\right] - \left[\frac{e^{2(1)}}{2} - \frac{4}{1}\right]$$

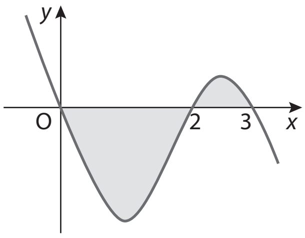
$$= \frac{1}{2}e^4 - 2 - \frac{1}{2}e^2 + 4$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^2 + 2$$

9. $y = x(2 - x)(x - 3) = -x^3 + 5x^2 - 6x$

On x-axis $\Rightarrow y = 0 \Rightarrow x(2 - x)(x - 3) = 0$

$\Rightarrow x = 0, x = 2$ or $x = 3$



Shaded Area $= \int_0^2 (-x^3 + 5x^2 - 6x)\, dx + \int_2^3 (-x^3 + 5x^2 - 6x)\, dx$

$$= \left[-\frac{x^4}{4} + 5\frac{x^3}{3} - 3x^2\right]_0^2 + \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 3x^2\right]_2^3$$

$$= \left[\left[-\frac{(2)^4}{4} + \frac{5(2)^3}{3} - 3(2)^2\right] - \left[-\frac{(0)^4}{4} + \frac{5(0)^3}{3} - 3(0)^2\right]\right]$$

$$+ \left[\left[-\frac{(3)^4}{4} + \frac{5(3)^3}{3} - 3(3)^2\right] - \left[-\frac{(2)^4}{4} + \frac{5(2)^3}{3} - 3(2)^2\right]\right]$$

$$= \left[\left(-4 + \frac{40}{3} - 12\right) - (0 + 0 + 0)\right] + \left[\left(-\frac{81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right)\right]$$

$$= \left[-2\frac{2}{3} - 0\right] + \left[-2\frac{1}{4} + 2\frac{2}{3}\right]$$

Area $= 2\frac{2}{3} + \frac{5}{12} = 3\frac{1}{12}$ sq.units

10. $\frac{\mathrm{dy}}{\mathrm{dx}} = 15x^2 - 12x$

$\Rightarrow y = \int (15x^2 - 12x)\,\mathrm{dx}$

$= 15\frac{x^3}{3} - 12\frac{x^2}{2} + c = 5x^3 - 6x^2 + c$

Point $(1, 3) \Rightarrow 5(1)^3 - 6(1)^2 + c = 3$

$\Rightarrow 5 - 6 + c = 3 \Rightarrow c = 4$

Hence $f(x) = y = 5x^3 - 6x^2 + 4$

11. $f(x) = 2x^2 - x$

Average value $= \frac{1}{4-0}\int_0^4 (2x^2 - x)\,\mathrm{dx}$

$= \frac{1}{4}\left[2\frac{x^3}{3} - \frac{x^2}{2}\right]_0^4$

$= \frac{1}{4}\left[2\frac{(4)^3}{3} - \frac{(4)^2}{2}\right] - \frac{1}{4}\left[2\frac{(0)^3}{3} - \frac{(0)^2}{2}\right]$

$= \frac{1}{4}\left(\frac{128}{3} - 8\right) - \frac{1}{4}(0) = 8\frac{2}{3}$

12. $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{e}^{2x} - x$

$y = \int (\mathrm{e}^{2x} - x)\,\mathrm{dx} = \frac{\mathrm{e}^{2x}}{2} - \frac{x^2}{2} + c$

$y = 5$ when $x = 0 \Rightarrow \frac{\mathrm{e}^{2(0)}}{2} - \frac{(0)^2}{2} + c = 5$

$\Rightarrow \frac{1}{2} - 0 + c = 5 \Rightarrow c = 4\frac{1}{2}$

Hence $y = \frac{1}{2}\mathrm{e}^{2x} - \frac{x^2}{2} + 4\frac{1}{2}$

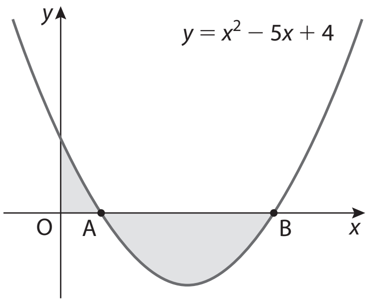
13. (i) $y = x^2 - 5x + 4$

On x axis, $y = 0 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 1)(x - 4) = 0$

$\Rightarrow x = 1 \quad \text{OR} \quad x = 4$

$\Rightarrow A = (1, 0) \quad \text{and} \quad B = (4, 0)$



(ii) Shaded Area $= \int_0^1 (x^2 - 5x + 4)\,\mathrm{dx} + \int_1^4 (x^2 - 5x + 4)\,\mathrm{dx}$

$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^1 + \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_1^4$

$= \left[\left[\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right] - \left[\frac{(0)^3}{3} - \frac{5(0)^2}{2} + 4(0)\right]\right]$

$+ \left[\left[\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 4(4)\right] - \left[\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right]\right]$

$= \left[\left(\frac{1}{3} - \frac{5}{2} + 4\right) - (0 - 0 + 0)\right] + \left[\left(\frac{64}{3} - 40 + 16\right) - \left(\frac{1}{3} - \frac{5}{2} + 4\right)\right]$

$= 1\frac{5}{6} + \left(-4\frac{1}{2}\right)$

Area $= 1\frac{5}{6} + 4\frac{1}{2} = 6\frac{1}{3}$ sq. units

14. (i) $v = 6t + 12t^2$

$$\text{Average velocity} = \frac{1}{2-0}\int_0^2 (6t + 12t^2)\,dt$$
$$= \frac{1}{2}[3t^2 + 4t^3]_0^2$$
$$= \frac{1}{2}[3(2)^2 + 4(2)^3] - \frac{1}{2}[3(0)^2 + 4(0)^3]$$
$$= \frac{1}{2}[12 + 32] - \frac{1}{2}(0) = 22 \text{ m/sec}$$

(ii) $v = 6t + 12t^2 \Rightarrow a = \dfrac{dv}{dt} = 6 + 24t$

$$\text{Average acceleration} = \frac{1}{5-1}\int_1^5 (6 + 24t)\,dt$$
$$= \frac{1}{4}[6t + 12t^2]_1^5$$
$$= \frac{1}{4}[6(5) + 12(5)^2] - \frac{1}{4}[6(1) + 12(1)^2]$$
$$= \frac{1}{4}(330) - \frac{1}{4}(18) = 78 \text{ m/sec}^2$$

15. $f(x) = x\,.\sin 2x \Rightarrow$ Product Rule: $u = x$ and $v = \sin 2x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = 2\cos 2x$$

$$\frac{dy}{dx} = f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x \cdot 2\cos 2x + \sin 2x\,.1 = 2x\cos 2x + \sin 2x$$

$$\int(2x\cos 2x + \sin 2x)\,dx = x\sin 2x + c$$
$$= \int 2x\cos 2x\,dx + \int \sin 2x = x\sin 2x + c$$
$$\Rightarrow \int 2x\cos 2x\,dx - \frac{\cos 2x}{2} = x\sin 2x + c$$
$$\Rightarrow \int 2x\cos 2x\,dx = x\sin 2x + \frac{\cos 2x}{2} + c$$

Revision Exercise 3 (Advanced)

1. (i) $\displaystyle\int_1^3 (9x^2 - 4x)\,dx = \left[\frac{9x^3}{3} - \frac{4x^2}{2}\right]_1^3 = [3x^3 - 2x^2]_1^3$

$$= [3(3)^3 - 2(3)^2] - [3(1)^3 - 2(1)^2]$$
$$= (81 - 18) - (3 - 2) = 63 - 1 = 62$$

(ii) $\displaystyle\int_0^a (9x^2 - 4x)\,dx = 0$

$$\Rightarrow [3x^3 - 2x^2]_0^a = 0$$
$$\Rightarrow (3a^3 - 2a^2) - [3(0)^3 - 2(0)^2] = 0$$
$$\Rightarrow 3a^3 - 2a^2 = 0 \Rightarrow a^2(3a - 2) = 0$$
$$\Rightarrow a = 0 \quad \text{OR} \quad a = \frac{2}{3}$$

Since $a > 0 \quad \Rightarrow \quad a = \dfrac{2}{3}$

2. $f(x) = (x+3)(2x-5) = 2x^2 + x - 15$

$$\text{Average value} = \frac{1}{5-1}\int (2x^2 + x - 15)\,dx$$

$$= \frac{1}{4}\left[\frac{2x^3}{3} + \frac{x^2}{2} - 15x\right]_1^5$$

$$= \frac{1}{4}\left[2\frac{(5)^3}{3} + \frac{(5)^2}{2} - 15(5)\right] - \frac{1}{4}\left[2\frac{(1)^3}{3} + \frac{(1)^2}{2} - 15(1)\right]$$

$$= \frac{1}{4}\left(20\frac{5}{6}\right) - \frac{1}{4}\left(-13\frac{5}{6}\right) = 8\frac{2}{3}$$

3. (i) $y = x^2 \quad \cap \quad y = -2x + 15$

$\Rightarrow x^2 = -2x + 15 \Rightarrow x^2 + 2x - 15 = 0$

$\Rightarrow (x+5)(x-3) = 0$

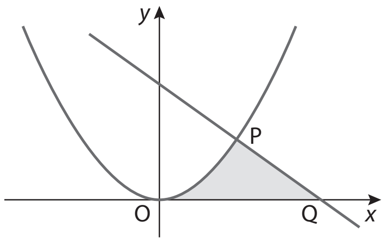
$\Rightarrow x = -5 \quad \text{OR} \quad x = 3$

When $x = 3 \Rightarrow y = (3)^2 = 9 \Rightarrow \text{P} = (3, 9)$

LINE: $2x + y = 15$

On x-axis, $y = 0 \Rightarrow 2x + 0 = 15 \Rightarrow x = 7\frac{1}{2}$

$$\Rightarrow \text{Q} = \left(7\frac{1}{2}, 0\right)$$



(ii) Shaded Area = Area under curve OP + Area under line PQ

$$= \int_0^3 x^2 dx + \int_3^{7\frac{1}{2}} (-2x + 15)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^3 + \left[-x^2 + 15x\right]_3^{7\frac{1}{2}}$$

$$= \left[\frac{(3)^3}{3} - \frac{(0)^3}{3}\right] + \left[-\left(7\frac{1}{2}\right)^2 + 15\left(7\frac{1}{2}\right)\right] - [-(3)^2 + 15(3)]$$

$$= 9 - 0 + \left[-56\frac{1}{4} + 112\frac{1}{2}\right] - [-9 + 45]$$

$$= 9 + 56\frac{1}{4} - 36 = 29.25 \text{ sq. units}$$

4. $V = \frac{1}{3}\pi(30h^2 - h^3)$

$$\text{Average value} = \frac{1}{4-0}\int_0^4 \frac{1}{3}\pi(30h^2 - h^3)\,dh$$

$$= \frac{\pi}{12}\left[30\frac{h^3}{3} - \frac{h^4}{4}\right]_0^4 = \frac{\pi}{12}\left[10h^3 - \frac{h^4}{4}\right]_0^4$$

$$= \frac{\pi}{12}\left[10(4)^3 - \frac{(4)^4}{4}\right]_0^4 = -\frac{\pi}{12}\left[10(0)^3 - \frac{(0)^4}{4}\right]$$

$$= \frac{\pi}{12}[640 - 64] = 48\pi \text{ cm}^3$$

5. $y = x^2 - 3x^2 + 5$

$$\frac{dy}{dx} = 3x^2 - 6x = 0 \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$$

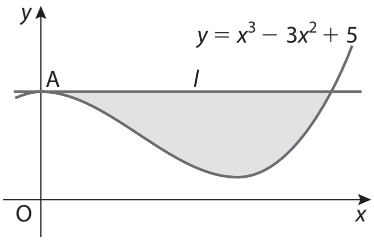
$x = 0 \Rightarrow y = (0)^3 - 3(0)^2 + 5 = 5 \Rightarrow \text{A} = (0, 5)$

Line $\ell : y = 5$ meets curve $\Rightarrow x^3 - 3x^2 + 5 = 5$

$\Rightarrow x^3 - 3x^2 = 0$

$\Rightarrow x^2(x-3) = 0 \Rightarrow x = 0, x = 3$

2nd point = (3, 5)



Shared Area = Area under line ℓ – Area under curve

$$= \int_0^3 5dx - \int_0^3 (x^3 - 3x^2 + 5)dx$$

$$= [5x]_0^3 - \left[\frac{x^4}{4} - x^3 + 5x\right]_0^3$$

$$= (15 - 0) - \left[\left[\frac{(3)^4}{4} - (3)^3 + 5(3)\right] - \left[\frac{(0)^4}{4} - (0)^3 + 5(0)\right]\right]$$

$$= 15 - 8\frac{1}{4} = 6\frac{3}{4} = \frac{27}{4} \text{ sq. units}$$

6. $\frac{dy}{dx} = ae^{-x} + 1$

$\frac{dy}{dx} = 3$ when $x = 0 \Rightarrow ae^{-0} + 1 = 3$

$\Rightarrow a\,.\,1 = 2 \Rightarrow a = 2$

$\frac{dy}{dx} = 2e^{-x} + 1$

$y = \int (2e^{-x} + 1)\,dx$

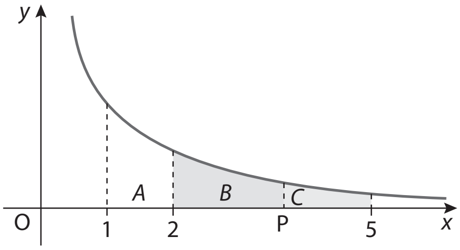
$= 2\frac{e^{-x}}{-1} + x + c = \frac{-2}{e^x} + x + c$

$y = 5$ when $x = 0 \Rightarrow \frac{-2}{e^0} + 0 + c = 5 \Rightarrow c = 7$

$\Rightarrow y = \frac{-2}{e^x} + x + 7$

When $x = 2 \Rightarrow y = \frac{-2}{e^2} + 2 + 7 = 9 - 2e^{-2}$

7. (i) $y = \frac{10}{x^2} = 10x^{-2}$



Area $A = \int_1^2 10x^{-2}dx = \left[10\frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{-10}{x}\right]_1^2$

$= \left(-\frac{10}{2}\right) - \left(-\frac{10}{1}\right) = -5 + 10 = 5$ sq. units

(ii) Area B = Area C

$\Rightarrow \int_2^p 10x^{-2}dx = \int_p^5 10x^{-2}dx$

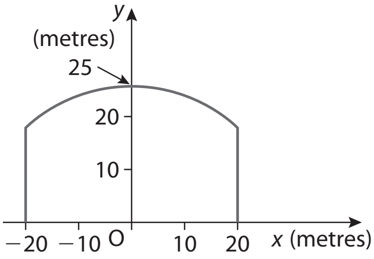
$= \left[\frac{-10}{x}\right]_2^p = \left[\frac{-10}{x}\right]_p^5$

$= \left(-\frac{10}{p}\right) - \left(-\frac{10}{2}\right) = \left(-\frac{10}{5}\right) - \left(-\frac{10}{p}\right)$

$= -\frac{10}{p} + 5 = -2 + \frac{10}{p}$

$\Rightarrow \frac{-20}{p} = -7 \Rightarrow 7p = 20 \Rightarrow p = \frac{20}{7}$

8. Area $= \int_{-20}^{20} (25 - 0.02x^2)\,dx$



$$= \left[25x - 0.02\frac{x^3}{3}\right]_{-20}^{20}$$

$$= \left[25(20) - 0.02\frac{(20)^3}{3}\right] - \left[25(-20) - 0.02\,\frac{(-20)^3}{3}\right]$$

$$= \left(500 - \frac{160}{3}\right) - \left(-500 + \frac{160}{3}\right) = 893\frac{1}{3}\text{ m}^2$$

Volume $= 893\frac{1}{3} \times 80 = 71{,}466\frac{2}{3}$ m^3

9. (i) $f(x) = x^2 . \ln 3x \Rightarrow$ product rule: $u = x^2$ and $v = \ln 3x$

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{dv}{dx} = \frac{1}{3x}\cdot 3 = \frac{1}{x}$$

$$\frac{dy}{dx} = f'(x) = u\frac{dv}{dx} + v\frac{du}{dx} = x^2\cdot\frac{1}{x} + \ln 3x\,.\,2x = 2x\ln 3x + x$$

(ii) $\int (2x\ln 3x + x)\,dx = x^2\ln 3x$

$$\Rightarrow \int 2x\ln 3x + \int x dx = x^2\ln 3x + c$$

$$\Rightarrow \int 2x\ln 3x + \frac{x^2}{2} = x^2\ln 3x + c$$

$$\Rightarrow \int 2x\ln 3x\,dx = x^2\ln 3x - \frac{x^2}{2} + c$$

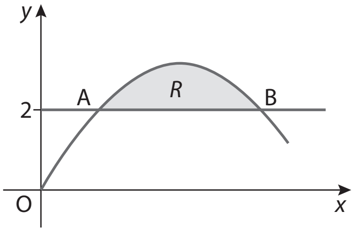
10. (i) $y = 2 \quad \cap \quad y = -2x^2 + 5x$

$\Rightarrow 2 = -2x^2 + 5x$

$\Rightarrow 2x^2 - 5x + 2 = 0$

$\Rightarrow (2x - 1)(x - 2) = 0 \Rightarrow x = \frac{1}{2}$ OR $x = 2$

Hence $A = \left(\frac{1}{2}, 2\right)$ and $B = (2, 2)$



(ii) Shaded area $R = \int_{\frac{1}{2}}^{2} (-2x^2 + 5x)\,dx - \int_{\frac{1}{2}}^{2} 2\,dx$

$$= \left[\frac{-2x^3}{3} + \frac{5x^2}{2}\right]_{\frac{1}{2}}^{2} - [2x]_{\frac{1}{2}}^{2}$$

$$= \left[\left(\frac{-2(2)^3}{3} + \frac{5(2)^2}{2}\right) - \left(\frac{-2\left(\frac{1}{2}\right)^3}{3} + \frac{5\left(\frac{1}{2}\right)^2}{2}\right)\right] - \left[2(2) - 2\left(\frac{1}{2}\right)\right]$$

$$= \left(\frac{-16}{3} + 10\right) - \left(-\frac{1}{12} + \frac{5}{8}\right) - 3 = \frac{9}{8} \text{ sq. units}$$

11. $k = 5v^2$

Average kinetic energy $= \frac{1}{7-1}\int_1^7 5v^2\,dv$

$$= \frac{1}{6}\left[\frac{5v^3}{3}\right]_1^7 = \frac{1}{6}\left[5\frac{(7)^3}{3}\right] - \frac{1}{6}\left[\frac{5(1)^3}{3}\right]$$

$$= \frac{1}{6}\left(\frac{1715}{3}\right) - \frac{1}{6}\left(\frac{5}{3}\right) = 95 \text{ joules}$$

12. (i) $a = 6t + 10$

$v = \int (6t + 10)dt = 3t^2 + 10t$

When $t = 5 \quad \Rightarrow v = 3(5)^2 + 10(5) = 125$ m/sec

(ii) $s = \int (3t^2 + 10t)dt$

$$= \frac{3t^3}{3} + \frac{10t^2}{2} + c = t^3 + 5t^2 + c$$

$s = 3$ when $t = 0 \quad \Rightarrow (0)^3 + 5(0)^2 + c = 3 \Rightarrow c = 3$

$\Rightarrow s = t^3 + 5t^2 + 3$

(iii) $t = 3 \Rightarrow s = (3)^3 + 5(3)^2 + 3 = 75$ m

(iv) Average speed $= \dfrac{1}{4-1}\displaystyle\int_1^4 (3t^2 + 10t)\,dt$

$$= \frac{1}{3}\left[\frac{3t^3}{3} + \frac{10t^2}{2}\right]_1^4 = \frac{1}{3}\left[t^3 + 5t^2\right]_1^4$$

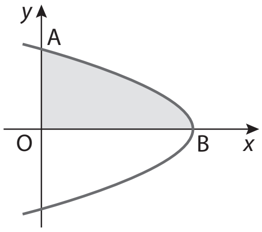
$$= \frac{1}{3}[(4)^3 + 5(4)^2] - \frac{1}{3}[(1)^3 + 5(1)^2]$$

$$= \frac{1}{3}(64 + 80) - \frac{1}{3}(1 + 5) = 46 \text{ m/sec.}$$

13. (i) $y^2 = 9(1-x) \Rightarrow y = \sqrt{9(1-x)} = 3\sqrt{1-x}$

At A, $x = 0 \Rightarrow y = 3\sqrt{1-0} = 3 \Rightarrow \text{A} = (0, 3)$

At B, $y = 0 \Rightarrow 3\sqrt{1-x} = 0 \Rightarrow x = 1 \Rightarrow \text{B} = (1, 0)$



(ii) $\displaystyle\int_0^3 \left|1 - \frac{y^2}{9}\right| dy = \left[y - \frac{1}{9}\left(\frac{y^3}{3}\right)\right]_0^3 = \left[y - \frac{y^3}{27}\right]_0^3$

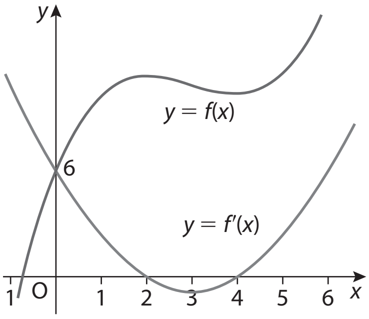
$$= \left[3 - \frac{(3)^3}{27}\right] - \left[0 - \frac{(0)^3}{27}\right] = 3 - 1 = 2 \text{ sq. units}$$

14. (i) $f'(x) = k(x-a)(x-b)$

(a) $a = 2, b = 4$

(b) $f'(x) = k(x-2)(x-4)$

Point $(0, 6) \Rightarrow f'(0) = k(0-2)(0-4) = 6 \Rightarrow 8k = 6 \Rightarrow k = \dfrac{3}{4}$



(ii) $f'(x) = \dfrac{3}{4}(x-2)(x-4) = \dfrac{3}{4}(x^2 - 6x + 8)$

$$y = f(x) = \frac{3}{4}\int (x^2 - 6x + 8)dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3} - 3x^2 + 8x + c\right] = \frac{x^3}{4} - \frac{9}{4}x^2 + 6x + \frac{3}{4}c$$

Point $(0, 6) \Rightarrow \dfrac{3}{4}\left[\dfrac{(0)^3}{3} - 3(0)^2 + 8(0) + c\right] = 6$

$$\Rightarrow 0 - 0 + 0 + c = 6 \cdot \frac{4}{3} = 8 \Rightarrow c = 8$$

Hence $y = \dfrac{1}{4}x^3 - \dfrac{9}{4}x^2 + 6x + 6$

15. $\dfrac{dy}{dx} = ax + \dfrac{b}{x^2} = ax + bx^{-2}$

Turning Point $(1, 0) \Rightarrow \dfrac{dy}{dx} = a(1) + \dfrac{b}{(1)^2} = 0 \Rightarrow a + b = 0$

$$y = \int (ax + bx^{-2})\,dx = a\frac{x^2}{2} + b\frac{x^{-1}}{-1} + c$$

$$= a\frac{x^2}{2} - \frac{b}{x} + c$$

Point $(-1, -4)$ $\quad \Rightarrow a\dfrac{(-1)^2}{2} - \dfrac{b}{(-1)} + c = -4$

$$\Rightarrow \frac{a}{2} + b + c = -4$$

Point $(1, 0)$ $\quad \Rightarrow a\dfrac{(1)^2}{2} - \dfrac{b}{1} + c = 0$

$$\Rightarrow \frac{a}{2} - b + c = 0$$

and $\quad -\dfrac{a}{2} - b - c = 4$

add $\quad \Rightarrow \quad -2b = 4 \quad \Rightarrow \quad b = -2$

and $\quad a - 2 = 0 \quad \Rightarrow \quad a = 2$

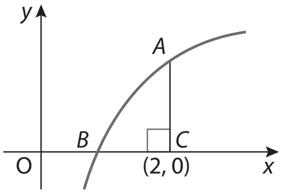
$$\frac{a}{2} - b + c = 0 \Rightarrow \frac{2}{2} - (-2) + c = 0$$

$$\Rightarrow 1 + 2 + c = 0 \Rightarrow c = -3$$

Hence $y = \frac{2x^2}{2} - \frac{-2}{x} - 3 = x^2 + \frac{2}{x} - 3$

Revision Exercise 3 (Extended – Response)

1. (a) $y = x - \frac{1}{x^2} = x - x^{-2}$



At A, $x = 2 \Rightarrow y = 2 - \frac{1}{2^2} = \frac{7}{4} \Rightarrow A = \left(2, \frac{7}{4}\right)$

$$\frac{dy}{dx} = 1 - (-2x^{-3}) = 1 + \frac{2}{x^3}$$

When $x = 2 \Rightarrow$ slope $= \frac{dy}{dx} = 1 + \frac{2}{(2)^3} = 1 + \frac{1}{4} = \frac{5}{4}$

$\Rightarrow$ Equation of Tangent: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - \frac{7}{4} = \frac{5}{4}(x - 2)$$

$$\Rightarrow 4y - 7 = 5x - 10$$

$$\Rightarrow 5x - 4y - 3 = 0$$

(b) On x-axis $\Rightarrow y = 0 \Rightarrow 5x - 4(0) - 3 = 0$

$$\Rightarrow 5x = 3$$

$$\Rightarrow x = \frac{3}{5}$$

$$\Rightarrow \text{point } T = \left(\frac{3}{5}, 0\right)$$

(c) $y = x - \frac{1}{x^2} = \frac{x^3 - 1}{x^2}$

On x-axis, $y = 0 \Rightarrow \frac{x^3 - 1}{x^2} = 0 \Rightarrow x^3 - 1 = 0$

$$\Rightarrow x^3 = 1 \Rightarrow x = \sqrt[3]{1} = 1$$

$\Rightarrow B = (1, 0)$

(d) Required Area = Area under line TA − area under curve BA

Area under line TA $\Rightarrow 5x - 4y - 3 = 0$

$$4y = 5x - 3$$

$$\Rightarrow y = \frac{1}{4}(5x - 3)$$

$$\text{Area} = \frac{1}{4}\int_{\frac{3}{5}}^{2} (5x - 3)dx$$

$$= \frac{1}{4}\left[\frac{5x^2}{2} - 3x\right]_{\frac{3}{5}}^{2}$$

$$= \frac{1}{4}\left[\frac{5(2)^2}{2} - 3(2)\right] - \frac{1}{4}\left[5\frac{\left(\frac{3}{5}\right)^2}{2} - 3\left(\frac{3}{5}\right)\right]$$

$$= \frac{1}{4}[10 - 6] - \frac{1}{4}\left[\frac{9}{10} - \frac{9}{5}\right]$$

$$= 1 - \frac{1}{4}\left(-\frac{9}{10}\right) = 1\frac{9}{40}$$

$$\text{Area under curve BA} = \int_1^2 (x - x^{-2})\,dx$$
$$= \left[\frac{x^2}{2} - \frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2$$
$$= \left[\frac{2^2}{2} + \frac{1}{2}\right] - \left[\frac{(1)^2}{2} + \frac{1}{1}\right]$$
$$= 2\tfrac{1}{2} - 1\tfrac{1}{2} = 1$$

Hence required area $= 1\frac{9}{40} - 1 = \frac{9}{40}$ sq. units

(e) Triangle ATC:

$$\begin{array}{ccc} C(2, 0) & A\left(2, \frac{7}{4}\right) & T\left(\frac{3}{5}, 0\right) \\ \downarrow & \downarrow & \\ (0, 0) & \left(0, \frac{7}{4}\right) & \left(-\frac{7}{5}, 0\right) \\ & x_1\,y_1 & x_2\,y_2 \end{array}$$

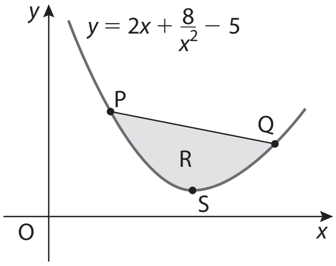
$$\text{Area Triangle ATC} = \frac{1}{2}|x_1 y_2 - x_2 y_1|$$
$$= \frac{1}{2}\left|(0)(0) - \left(-\frac{7}{5}\right)\left(\frac{7}{4}\right)\right| = \frac{49}{40}$$

Ratio $= \frac{9}{40} : \frac{49}{40} = 9 : 49$

2. (a) $y = 2x + \frac{8}{x^2} - 5 = 2x + 8x^{-2} - 5$

At P, $x = 1 \Rightarrow y = 2(1) + \frac{8}{(1)^2} - 5 = 2 + 8 - 5 = 5 \Rightarrow P = (1, 5)$

At Q, $x = 4 \Rightarrow y = 2(4) + \frac{8}{(4)^2} - 5 = 8 + \frac{1}{2} - 5 = 3\frac{1}{2} \Rightarrow Q = \left(4, 3\frac{1}{2}\right)$



Slope PQ $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3\frac{1}{2} - 5}{4 - 1} = \frac{-1\frac{1}{2}}{3} = -\frac{1}{2}$

Equation PQ: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 5 = \frac{-1}{2}(x - 1)$$
$$\Rightarrow 2y - 10 = -x + 1$$
$$\Rightarrow x + 2y - 11 = 0$$
$$2y = -x + 11$$
$$\Rightarrow y = \frac{1}{2}(-x + 11)$$

Shaded Area R = Area under line PQ − Area under curve PQ

$$\text{Area under line PQ} = \frac{1}{2}\int_1^4 (-x + 11)\,dx$$
$$= \frac{1}{2}\left[\frac{-x^2}{2} + 11x\right]_1^4$$
$$= \frac{1}{2}\left[\frac{-(4)^2}{2} + 11(4)\right] - \frac{1}{2}\left[\frac{-(1)^2}{2} + 11(1)\right]$$
$$= \frac{1}{2}[-8 + 44] - \frac{1}{2}\left[-\frac{1}{2} + 11\right]$$
$$= \frac{1}{2}[36] - \frac{1}{2}\left[10\frac{1}{2}\right]$$
$$= 18 - 5\frac{1}{4}$$
$$= 12\frac{3}{4}$$

Area under curve PQ $= \int_1^4 (2x + 8x^{-2} - 5)\,dx$

$= \left[\frac{2x^2}{2} + 8\frac{x^{-1}}{-1} - 5x\right]_1^4$

$= \left[x^2 - \frac{8}{x} - 5x\right]_1^4$

$= \left[(4)^2 - \frac{8}{4} - 5(4)\right] - \left[(1)^2 - \frac{8}{1} - 5(1)\right]$

$= [16 - 2 - 20] - [1 - 8 - 5]$

$= -6 + 12 = 6$

Hence required area $= 12\frac{3}{4} - 6 = 6\frac{3}{4}$ sq. units

(b) $y = 2x + 8x^{-2} - 5$

$\Rightarrow \frac{dy}{dx} = 2 - 16x^{-3} = 2 - \frac{16}{x^3}$

When $x > 2 \Rightarrow 2 - \frac{16}{x^3} > 0$

Since $\frac{dy}{dx} > 0$, y is increasing for $x > 2$.

(c) $\frac{dy}{dx} = 2 - \frac{16}{x^3}$

Turning point $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2 - \frac{16}{x^3} = 0$

$\Rightarrow 2x^3 - 16 = 0$

$\Rightarrow x^3 - 8 = 0$

$\Rightarrow x^3 = 8$

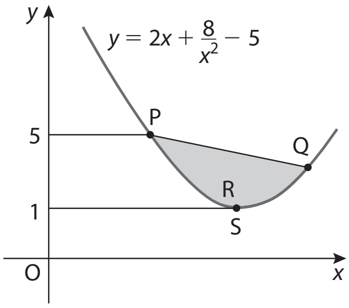
$\Rightarrow x = \sqrt[3]{8} = 2$

$x = 2 \Rightarrow y = 2(2) + \frac{8}{(2)^2} - 5$

$= 4 + \frac{8}{4} - 5 = 1$

$\Rightarrow$ Turning point $S = (2, 1)$

(d) Required area = area under line $y = 5$ − Area under line $y = 1$ between $x = 0$ and $x = 1$ + area under curve PS − area under line $y = 1$ between $x = 1$ and $x = 2$



Area $= \int_0^1 5dx - \int_0^1 1dx$

$= [5x]_0^1 - [x]_0^1$

$= [5(1) - 5(0)] - [1 - 0] = 5 - 1 = 4$

Area $= \int_1^2 (2x + 8x^{-2} - 5)dx - \int_1^2 1dx$

$= \left[x^2 - \frac{8}{x} - 5x\right]_1^2 - [x]_1^2$

$= \left[\left[(2)^2 - \frac{8}{2} - 5(2)\right] - \left[(1)^2 - \frac{8}{1} - 5(1)\right]\right] - [(2) - (1)]$

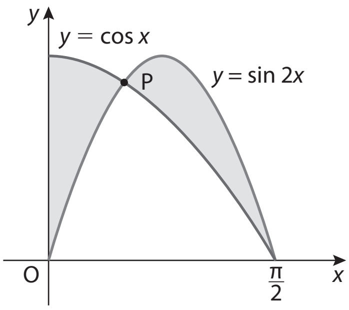
$= [(4 - 4 - 10) - (1 - 8 - 5)] - 1$

$= [-10 + 12] - 1$

$= 2 - 1 = 1$

Hence required area $= 4 + 1$

$= 5$ sq. units



3. (a) $y = \cos x \quad \cap \quad y = \sin 2x$

$\Rightarrow \sin 2x = \cos x$

$\Rightarrow 2 \sin x \cos x - \cos x = 0$

$\Rightarrow \cos x(2 \sin x - 1) = 0$

$\Rightarrow \cos x = 0$ OR $2 \sin x = 1$

$\Rightarrow x = \cos^{-1} 0$ $\quad\Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{2}$ (already on graph) $\quad\Rightarrow x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Answer $= \frac{\pi}{6}$

(b) Required area = area under curve $y = \cos x$ − area under curve $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{6}$ + area under curve $y = \sin 2x$ − area under curve $y = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$

$$\text{Area} = \int_0^{\frac{\pi}{6}} \cos x \, dx - \int_0^{\frac{\pi}{6}} \sin 2x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{6}} - \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \left[\sin \frac{\pi}{6} - \sin 0\right] - \left[\left(\frac{-\cos \frac{2\pi}{6}}{2}\right) - \left(\frac{-\cos 2(0)}{2}\right)\right]$$

$$= \left[\frac{1}{2} - 0\right] - \left[\frac{-\frac{1}{2}}{2} - \left(\frac{-1}{2}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x \, dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[\frac{-\cos 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{-\cos 2\left(\frac{\pi}{2}\right)}{2}\right) - \left(\frac{-\cos \frac{2\pi}{6}}{2}\right)\right] - \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right]$$

$$= \left[-\left(\frac{-1}{2}\right) - \left(\frac{-\frac{1}{2}}{2}\right)\right] - \left(1 - \frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(\frac{1}{2}\right) = \frac{1}{4}$$

Hence required area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ square unit

4. (a) $\frac{dV}{dt} = 120 + 26t - t^2$

Initial rate, $t = 0 \Rightarrow \frac{dV}{dt} = 120 + 26(0) - (0)^2 = 120$ *l*/min

Twice the initial rate = 2(120) = 240 *l*/min

$\Rightarrow \quad 120 + 26t - t^2 = 240$

$\Rightarrow \quad t^2 - 26t + 120 = 0$

$\Rightarrow \quad (t - 6)(t - 20) = 0$

$\Rightarrow \quad t = 6$ mins OR $t = 20$ mins

(b) $\frac{dV}{dt} = 120 + 26t - t^2$

$\Rightarrow V = \int (120 + 26t - t^2)dt$

$= 120t + 26\frac{t^2}{2} - \frac{t^3}{3} + c$

$= 120t + 13t^2 - \frac{t^3}{3} + c$

$V = 0$ when $t = 0 \Rightarrow 120(0) + 13(0)^2 - \frac{(0)^3}{3} + c = 0$

$\Rightarrow c = 0$

$\Rightarrow V = 120t + 13t^2 - \frac{t^3}{3}$

(c) $t = 30 \quad \Rightarrow V = 120(30) + 13(30)^2 - \frac{(30)^3}{3}$

$= 3600 + 11{,}700 - 9000 = 6300$ litres

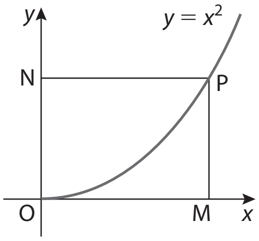
Initially, tank = 1500 litres

$\Rightarrow$ Total water = 1500 + 6300 = 7800 litres

Tank = 7000 litres

$\Rightarrow$ Water lost = 7800 − 7000 = 800 litres

5. (a) $y = x^2$



Let point M = $(k, 0) \quad \Rightarrow$ P = (k, k^2)

Area rectangle OMPN = $|OM|\,.\,|MP| = k\,.\,k^2 = k^3$

Area under curve $y = x^2 \Rightarrow \int_0^k x^2\,dx$

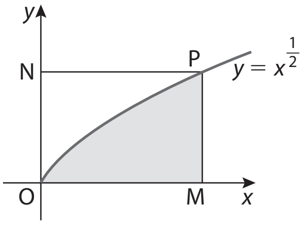
$= \left[\frac{x^3}{3}\right]_0^k$

$= \frac{k^3}{3} - \frac{(0)^3}{3} = \frac{k^3}{3}$

Hence other region ONP $= k^3 - \frac{k^3}{3} = \frac{2k^3}{3}$

$\Rightarrow$ Ratio of areas $= \frac{2k^3}{3} : \frac{k^3}{3} = 2 : 1$

(b) Let point M = $(k, 0) \quad \Rightarrow$ P = $\left(k, k^{\frac{1}{2}}\right)$



Area rectangle OMPN = $|OM|\,.\,|MP| = k\,.\,k^{\frac{1}{2}} = k^{\frac{3}{2}}$

Area under curve $y = x^{\frac{1}{2}} \Rightarrow \int_0^k x^{\frac{1}{2}}dx$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^k$

$= \left[\frac{2}{3}k^{\frac{3}{2}}\right] - \left[\frac{2}{3}(0)^{\frac{3}{2}}\right] = \frac{2}{3}k^{\frac{3}{2}}$

$\Rightarrow$ Shaded area $= \frac{2}{3}\left[k^{\frac{3}{2}}\right] = \frac{2}{3}$ area rectangle OMPN

(c) Let point M = $(k, 0) \Rightarrow$ P = (k, k^n)

Area rectangle OMPN = $|OM|.|MP|$

$= k\,.\,k^n = k^{n+1}$

Area under curve $y = x^n$:

Area $= \int_0^k x^n\,dx$

$= \left[\frac{x^{n+1}}{n+1}\right]_0^k$

$$= \left[\frac{k^{n+1}}{n+1}\right] - \left[\frac{(0)^{n+1}}{n+1}\right]$$

$$= \frac{1}{n+1}[k^{n+1}]$$

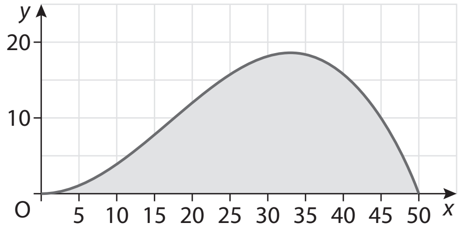
$$= \frac{1}{n+1}\text{ area rectangle OMPN}$$

6. $y = \frac{x^2}{1000}(50 - x)$

(a) (i) $x = 10\text{ m} \Rightarrow \text{height } (y) = \frac{(10)^2}{1000}(50 - 10) = \frac{1}{10}(40) = 4\text{ m}$

(ii) $x = 40\text{ m} \Rightarrow \text{height } (y) = \frac{(40)^2}{1000}(50 - 40)$

$$= \frac{1600}{1000}(10) = 16\text{ m}$$



(b) $y = \frac{x^2}{1000}(50 - x) = \frac{50}{1000}x^2 - \frac{x^3}{1000} = \frac{1}{20}x^2 - \frac{1}{1000}x^3$

$$\text{Slope} = \frac{dy}{dx} = \frac{1}{20}(2x) - \frac{1}{1000}(3x^2) = \frac{1}{10}x - \frac{3}{1000}x^2$$

(i) $x = 10\text{ m} \Rightarrow \text{slope} = \frac{1}{10}(10) - \frac{3}{1000}(10)^2 = 1 - 0.3 = 0.7$

(ii) $x = 40\text{ m} \Rightarrow \text{slope} = \frac{1}{10}(40) - \frac{3}{1000}(40)^2$

$$= 4 - 4.8 = -0.8$$

(c) (i) Height is a maximum $\Rightarrow \frac{dy}{dx} = 0$

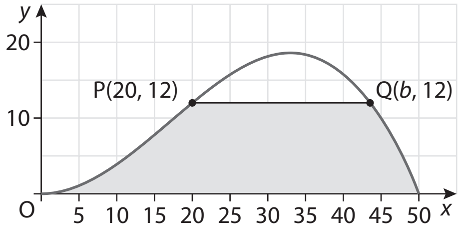
$$\Rightarrow \frac{1}{10}x - \frac{3}{1000}x^2 = 0$$

$$\Rightarrow 100x - 3x^2 = 0$$

$$\Rightarrow x(100 - 3x) = 0$$

$$\Rightarrow x = 0 \text{ (Not valid) OR } 100 = 3x$$

$$\Rightarrow x = \frac{100}{3}$$



(ii) When $x = \frac{100}{3} \Rightarrow \text{height } (y) = \frac{\left(\frac{100}{3}\right)^2}{1000}\left(50 - \frac{100}{3}\right)$

$$= \frac{10{,}000}{9000}\left(\frac{50}{3}\right)$$

$$= \frac{10}{9}\left(\frac{50}{3}\right) = \frac{500}{27}\text{ m}$$

(d) $\text{Area} = \int_0^{50}\left(\frac{1}{20}x^2 - \frac{1}{1000}x^3\right)dx$

$$= \left[\frac{1}{20}\left(\frac{x^3}{3}\right) - \frac{1}{1000}\left(\frac{x^4}{4}\right)\right]_0^{50}$$

$$= \left[\frac{1}{60}(x^3) - \frac{1}{4000}(x^4)\right]_0^{50}$$

$$= \left[\frac{1}{60}(50)^3 - \frac{1}{4000}(50)^4\right] - \left[\frac{1}{60}(0)^3 - \frac{1}{4000}(0)^4\right]$$

$$= \frac{12500}{6} - \frac{6250}{4} = \frac{3125}{6}\text{ m}^2$$

(e) (i) Q$(b,12)$ has $y = 12$

$$\Rightarrow \frac{1}{20}x^2 - \frac{1}{1000}x^3 = 12$$

$$\Rightarrow 50x^2 - x^3 = 12{,}000$$

$$\Rightarrow x^3 - 50x^2 + 12{,}000 = 0$$

P(20, 12) has $x = 20 \Rightarrow$ factor $(x - 20)$

$$\begin{array}{lrl} & & x^2 - 30x - 600 \\ & \Rightarrow\ x - 20 & \overline{)x^3 - 50x^2 + 12000} \\ \text{(subtract)} & & \underline{x^3 - 20x^2} \\ & & 0 - 30x^2 + 12000 \\ \text{(subtract)} & & \underline{-30x^2 + 600x} \\ & & 0 - 600x + 12000 \\ \text{(subtract)} & & \underline{-600x + 12000} \\ & & 0 \end{array}$$

$$x^2 - 30x - 600 = 0 \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{30 \pm \sqrt{(-30)^2 - 4(1)(-600)}}{2(1)}$$

$$= \frac{30 \pm \sqrt{3300}}{2} = 15 \pm 5\sqrt{33}$$

$$\Rightarrow Q = (15 + 5\sqrt{33}, 12)$$

(ii) $a = 20$ and $b = 15 + 5\sqrt{33}$

$$R = \int_{20}^{15+5\sqrt{33}} 12\,dx$$

$$= [12x]_{20}^{15+5\sqrt{33}}$$

$$= 12(15 + 5\sqrt{33}) - 12(20)$$

$$= 180 + 60\sqrt{33} - 240$$

$$= 60\sqrt{33} - 60$$

Area top of the mound $= \int_{20}^{15+5\sqrt{33}} \frac{x^2}{1000}(50 - x)\,dx - R$

Therefore, $a = 20$, $b = 15 + 5\sqrt{33}$ and $R = 60\sqrt{33} - 60$

7. (a) $f(x) = 1 + e^x$

$$f(-x) = 1 + e^{-x} = 1 + \frac{1}{e^x}$$

$$f(x) \times f(-x) = (1 + e^x)\left(1 + \frac{1}{e^x}\right)$$

$$= 1\left(1 + \frac{1}{e^x}\right) + e^x\left(1 + \frac{1}{e^x}\right)$$

$$= 1 + \frac{1}{e^x} + e^x + \frac{e^x}{e^x}$$

$$= 1 + \frac{1}{e^x} + e^x + 1 = 2 + e^x + \frac{1}{e^x}$$

$$f(x) + f(-x) = (1 + e^x) + \left(1 + \frac{1}{e^x}\right)$$

$$= 1 + e^x + 1 + \frac{1}{e^x}$$

$$= 2 + e^x + \frac{1}{e^x}$$

Hence $f(x) \times f(-x) = f(x) + f(-x)$

(b) $\dfrac{dy}{dx} = \dfrac{3 - e^{2x}}{e^x}$

$= \dfrac{3}{e^x} - \dfrac{e^{2x}}{e^x}$

$= 3e^{-x} - e^x$

$y = \displaystyle\int (3e^{-x} - e^x)dx$

$= 3\dfrac{e^{-x}}{-1} - e^x + c$

$= \dfrac{-3}{e^x} - e^x + c$

$y = 4$ when $x = 0 \Rightarrow \dfrac{-3}{e^0} - e^0 + c = 4$

$\Rightarrow \dfrac{-3}{1} - 1 + c = 4$

$\Rightarrow -3 - 1 + c = 4 \Rightarrow c = 8$

$\Rightarrow y = \dfrac{-3}{e^x} - e^x + 8$

(c) Shaded area $= \displaystyle\int_0^2 e^{2x}\,dx - \int_0^2 e^{-x}\,dx$

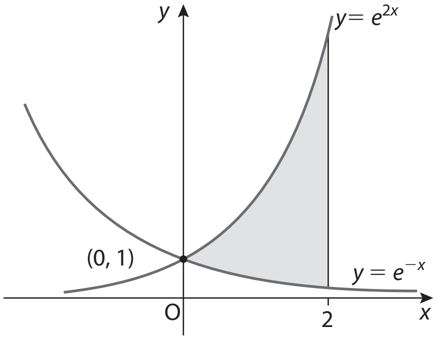
$= \left[\dfrac{e^{2x}}{2}\right]_0^2 - \left[\dfrac{e^{-x}}{-1}\right]_0^2$

$= \left[\dfrac{e^{2x}}{2} + \dfrac{1}{e^x}\right]_0^2$

$= \left[\dfrac{e^{2(2)}}{2} + \dfrac{1}{e^2}\right] - \left[\dfrac{e^{2(0)}}{2} + \dfrac{1}{e^0}\right]$

$= \left(\dfrac{e^4}{2} + \dfrac{1}{e^2}\right) - \left(\dfrac{1}{2} + \dfrac{1}{1}\right)$

$= \dfrac{e^4}{2} + \dfrac{1}{e^2} - \dfrac{3}{2}$



Chapter 4

Exercise 4.1

1. (i) $y = x^2 - 3x + 2$

$$\Rightarrow \frac{dy}{dx} = 2x - 3$$

At (1, 0) $\Rightarrow \frac{dy}{dx} = 2(1) - 3 = -1$

(ii) $y = x + \frac{1}{x} = x + x^{-1}$

$$\Rightarrow \frac{dy}{dx} = 1 - 1x^{-2} = 1 - \frac{1}{x^2}$$

At $\left(\frac{1}{2}, \frac{5}{2}\right) \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{\left(\frac{1}{2}\right)^2}$

$$= 1 - \frac{1}{\frac{1}{4}} = 1 - 4 = -3$$

2. $f(x) = 2x^2 - 4x - 5$

$\Rightarrow f'(x) = 4x - 4$

At (3, 1) $\Rightarrow f'(3) = 4(3) - 4 = 8$

$\Rightarrow$ Equation of Tangent: $y - 1 = 8(x - 3)$

$\Rightarrow y - 1 = 8x - 24$

$\Rightarrow 8x - y - 23 = 0$

3. $f(x) = x^2 - 6x$

$\Rightarrow f'(x) = 2x - 6$

Where $x = 2 \Rightarrow$ $f'(2) = 2(2) - 6 = -2$ (slope)

Where $x = 2 \Rightarrow$ $f(2) = (2)^2 - 6(2) = -8$

$\Rightarrow$ Point $(2, -8)$

$\Rightarrow$ Equation of Tangent: $y + 8 = -2(x - 2)$

$\Rightarrow y + 8 = -2x + 4$

$\Rightarrow 2x + y + 4 = 0$

4. $y = x^3 + \frac{1}{2x^2} = x^3 + \frac{1}{2}x^{-2}$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 1x^{-3} = 3x^2 - \frac{1}{x^3}$$

where $x = 1 \Rightarrow \frac{dy}{dx} = 3(1)^2 - \frac{1}{(1)^3} = 3 - 1 = 2$ (slope)

where $x = 1 \Rightarrow y = (1)^3 + \frac{1}{2(1)^2} = 1 + \frac{1}{2} = 1\frac{1}{2}$

$\Rightarrow$ Point $\left(1, 1\frac{1}{2}\right)$

$\Rightarrow$ Equation of Tangent: $y - 1\frac{1}{2} = 2(x - 1)$

$\Rightarrow y - 1\frac{1}{2} = 2x - 2$

$\Rightarrow 2y - 3 = 4x - 4$

$\Rightarrow 4x - 2y - 1 = 0$

5. $y = x^2 + kx$

$\Rightarrow \dfrac{dy}{dx} = 2x + k$

where $x = -1 \quad \Rightarrow \dfrac{dy}{dx} = 2(-1) + k = 3$

$\Rightarrow \quad -2 + k = 3$

$\Rightarrow \quad k = 5$

6. $y = x^2 + 3x - 1$

$\Rightarrow \dfrac{dy}{dx} = 2x + 3 = 5$

$\Rightarrow \quad 2x = 2$

$\Rightarrow \quad x = 1$

$\Rightarrow y = (1)^2 + 3(1) - 1 = 1 + 3 - 1 = 3$

$\Rightarrow$ Point $= (1, 3)$

7. $y = x^2 + 4x + 6$

$\Rightarrow \dfrac{dy}{dx} = 2x + 4 = -2$

$\Rightarrow \quad 2x = -6$

$\Rightarrow \quad x = -3$

$\Rightarrow y = (-3)^2 + 4(-3) + 6$

$= 9 - 12 + 6 = 3$

$\Rightarrow$ Point $= (-3, 3)$

8. $y = \dfrac{5x^2}{1 + x^2}$

Quotient Rule: $u = 5x^2$ and $v = 1 + x^2$

$\Rightarrow \dfrac{du}{dx} = 10x \qquad \Rightarrow \dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1 + x^2)(10x) - (5x^2)(2x)}{(1 + x^2)^2}$$

$$= \frac{10x + 10x^3 - 10x^3}{(1 + x^2)^2}$$

$$= \frac{10x}{(1 + x^2)^2}$$

Point $(2, 4) \Rightarrow \dfrac{dy}{dx} = \dfrac{10(2)}{[1 + (2)^2]^2} = \dfrac{20}{(5)^2} = \dfrac{20}{25} = \dfrac{4}{5}$

$\Rightarrow$ Equation of Tangent: $y - 4 = \frac{4}{5}(x - 2)$

$\Rightarrow 5y - 20 = 4x - 8$

$\Rightarrow 4x - 5y + 12 = 0$

9. $y = x^3 - 12x + 4$

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 12 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0$

$\Rightarrow x = -2 \quad$ OR $\quad x = 2$

$\Rightarrow y = (-2)^3 - 12(-2) + 4 \quad$ and $\quad y = (2)^3 - 12(2) + 4$

$= -8 + 24 + 4 = 20 \qquad = 8 - 24 + 4 = -12$

$\Rightarrow$ Points $= (-2, 20)$ and $(2, -12)$

10. $y = ax^2 + bx + 5$

$\Rightarrow \frac{dy}{dx} = 2ax + b$

Point (5, 0) $\Rightarrow \frac{dy}{dx} = 2a(5) + b = 4$

$\Rightarrow 10a + b = 4$

Point (5, 0) $\Rightarrow 0 = a(5)^2 + b(5) + 5$

$\Rightarrow 0 = 25a + 5b + 5$

$\Rightarrow 5a + b = -1$

and $\underline{-10a - b = -4}$

$\Rightarrow -5a = -5$

$\Rightarrow a = 1$

$\Rightarrow 10(1) + b = 4 \Rightarrow b = -6$

11. $y = ax^2 + b$

$\Rightarrow \frac{dy}{dx} = 2ax$

Point (2, −2) $\Rightarrow \frac{dy}{dx} = 2a(2) = 3$

$\Rightarrow 4a = 3 \Rightarrow a = \frac{3}{4}$

Hence, $y = \frac{3}{4}x^2 + b$

Point (2, −2) $\Rightarrow -2 = \frac{3}{4}(2)^2 + b$

$\Rightarrow -2 = 3 + b$

$\Rightarrow b = -5$

12. $y = \ln x + x - 2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x} + 1$

where $x = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{1} + 1 = 2$ (slope)

where $x = 1 \Rightarrow y = \ln(1) + 1 - 2 = 0 - 1 = -1$

$\Rightarrow$ Point (1, −1)

$\Rightarrow$ Equation of Tangent: $y + 1 = 2(x - 1)$

$\Rightarrow y + 1 = 2x - 2$

$\Rightarrow 2x - y - 3 = 0$

13. $y = e^{3x} \qquad \Rightarrow \frac{dy}{dx} = e^{3x} . 3 = 3e^{3x}$

Point (0, 1) $\Rightarrow \frac{dy}{dx} = 3e^{3(0)} = 3 . e^0 = 3 . 1 = 3$

$\Rightarrow$ Equation of Tangent: $y - 1 = 3(x - 0)$

$\Rightarrow y - 1 = 3x - 0$

$\Rightarrow 3x - y + 1 = 0$

14. $y = x^3 - 3x^2 - 5x + 10$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 5$

Line: $y = 4x - 7 \Rightarrow$ slope $= 4$

Hence, $3x^2 - 6x - 5 = 4$

$\Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

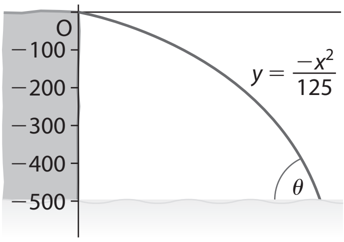
$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = -1$ OR $x = 3$

$$x = -1 \Rightarrow y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$$
$$= -1 - 3 + 5 + 10 = 11 \quad \Rightarrow \text{Point } (-1, 11)$$
$$x = 3 \quad \Rightarrow y = (3)^3 - 3(3)^2 - 5(3) + 10$$
$$= 27 - 27 - 15 + 10 = -5$$
$$\Rightarrow \text{Point } (3, -5)$$

15. (i) $y = -\frac{x^2}{125} = -500$

$$\Rightarrow -x^2 = -62\,500$$
$$\Rightarrow \quad x^2 = 62\,500$$
$$\Rightarrow \quad x = \sqrt{62\,500} = 250 \text{ m}$$



(ii) $y = -\frac{1}{125}x^2 \Rightarrow \frac{dy}{dx} = \frac{-2}{125}x$

At $x = 250 \Rightarrow \frac{dy}{dx} = \frac{-2}{125}(250) = -4$

$$\Rightarrow \tan\theta = -4$$
$$\Rightarrow \quad \theta = \tan^{-1}(-4) = 104.036 = 104°$$

The angle at which the stone enters the water is 180° − 104° = 76°.

16. (i) Curve is increasing $\Rightarrow \frac{dy}{dx} > 0$ (Positive)

(ii) Curve is decreasing $\Rightarrow \frac{dy}{dx} < 0$ (Negative)

(a) $y = x^2 - x - 6$

$$\Rightarrow \frac{dy}{dx} = 2x - 1$$

Turning point $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2x - 1 = 0$

$$\Rightarrow \quad x = \frac{1}{2}$$

Function is decreasing $\Rightarrow 2x - 1 < 0$

$$\Rightarrow \quad 2x < 1$$
$$\Rightarrow \quad x < \frac{1}{2}$$

(b) $y = x^3 + 6x^2 - 2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 12x$$

Turning points $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 3x^2 + 12x = 0$

$$\Rightarrow \quad x^2 + 4x = 0$$
$$\Rightarrow \quad x(x + 4) = 0$$
$$\Rightarrow x = 0 \quad \text{OR} \quad x = -4$$

Function is decreasing $\Rightarrow 3x^2 + 12x < 0$

$$\Rightarrow \quad x^2 + 4x < 0$$
$$\Rightarrow \quad -4 < x < 0$$

17. $f(x) = 4x^2 + 4x + 7$

(i) $\Rightarrow f'(x) = 8x + 4$

(ii) (a) $f(x)$ is increasing $\Rightarrow \quad f'(x) > 0$

$$\Rightarrow 8x + 4 > 0$$
$$\Rightarrow 2x + 1 > 0$$
$$\Rightarrow \quad 2x > -1$$
$$\Rightarrow \quad x > -\frac{1}{2}$$

(b) $f(x)$ is decreasing $\Rightarrow \quad f'(x) < 0$

$\Rightarrow 8x + 4 < 0$

$\Rightarrow 2x + 1 < 0$

$\Rightarrow \quad 2x < -1$

$\Rightarrow \quad x < -\frac{1}{2}$

18. (i) $f(x) = 4x - 3x^2$

$\Rightarrow f'(x) = 4 - 6x$

$f(x)$ is increasing $\Rightarrow \quad f'(x) > 0$

$\Rightarrow 4 - 6x > 0$

$\Rightarrow 2 - 3x > 0$

$\Rightarrow \quad -3x > -2$

$\Rightarrow \quad 3x < 2$

$\Rightarrow \quad x < \frac{2}{3}$

(ii) $f(x) = 3x^2 + 8x + 2$

$\Rightarrow f'(x) = 6x + 8$

$f(x)$ is increasing $\Rightarrow \quad f'(x) > 0$

$\Rightarrow 6x + 8 > 0$

$\Rightarrow 3x + 4 > 0$

$\Rightarrow \quad 3x > -4$

$\Rightarrow \quad x > \frac{-4}{3}$

(iii) $f(x) = 2x^3 - 15x^2 + 36x$

$\Rightarrow f'(x) = 6x^2 - 30x + 36$

$f(x)$ is increasing $\Rightarrow f'(x) > 0$

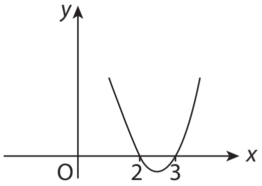
$\Rightarrow 6x^2 - 30x + 36 > 0$

$\Rightarrow \quad x^2 - 5x + 6 > 0$

Factors $\Rightarrow (x - 2)(x - 3) = 0$

Roots $\Rightarrow x = 2, 3$

Solution: $x < 2$ OR $x > 3$



19. (i) $f(x) = 3x - 5x^2$

$\Rightarrow f'(x) = 3 - 10x$

$f(x)$ is decreasing $\Rightarrow f'(x) < 0$

$\Rightarrow 3 - 10x < 0$

$\Rightarrow \quad -10x < -3$

$\Rightarrow \quad 10x > 3$

$\Rightarrow \quad x > \frac{3}{10}$ (OR $x > 0.3$)

(ii) $f(x) = 4 - 2x - x^2$

$\Rightarrow f'(x) = -2 - 2x$

$f(x)$ is decreasing $\Rightarrow f'(x) < 0$

$\Rightarrow -2 - 2x < 0$

$\Rightarrow \quad -2x < 2$

$\Rightarrow \quad 2x > -2$

$\Rightarrow \quad x > -1$

(iii) $f(x) = 2x^3 - 3x^2 - 12x$

$\Rightarrow f'(x) = 6x^2 - 6x - 12$

$f(x)$ is decreasing $\Rightarrow f'(x) < 0$

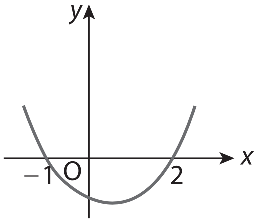
$\Rightarrow 6x^2 - 6x - 12 < 0$

$\Rightarrow x^2 - x - 2 < 0$

Factors $\Rightarrow (x + 1)(x - 2) = 0$

Roots $\Rightarrow x = -1, 2$

Solution: $-1 < x < 2$



20. $f(x) = x^3 - 6x^2 + 18x + 4$

$f'(x) = 3x^2 - 12x + 18$

$f(x)$ is increasing $\Rightarrow f'(x) > 0$

$\Rightarrow f'(x) = 3x^2 - 12x + 18 > 0$

$\Rightarrow x^2 - 4x + 6 > 0$

$\Rightarrow x^2 - 4x + 4 + 2 > 0$

$\Rightarrow (x - 2)^2 + 2 > 0$ True

21. $y = \dfrac{2x + 1}{3x + 6}$

Quotient Rule: $u = 2x + 1$ and $v = 3x + 6$

$\Rightarrow \dfrac{du}{dx} = 2 \quad \Rightarrow \quad \dfrac{dv}{dx} = 3$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(3x + 6) \,.\, (2) - (2x + 1) \,.\, (3)}{(3x + 6)^2}$$

$$= \frac{6x + 12 - 6x - 3}{(3x + 6)^2}$$

$$= \frac{9}{(3x + 6)^2}$$

$f(x)$ is increasing $\Rightarrow f'(x) > 0$

$\Rightarrow \dfrac{9}{(3x + 6)^2} > 0$ True

22. (i) $y = x^2 - 14x + 53$

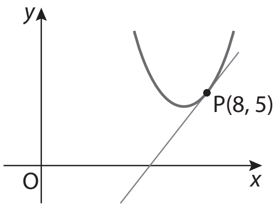
$\Rightarrow \dfrac{dy}{dx} = 2x - 14$

Point $(8, 5) \Rightarrow \dfrac{dy}{dx} = 2(8) - 14 = 2$

$\Rightarrow$ Equation of Tangent: $y - 5 = 2(x - 8)$

$\Rightarrow y - 5 = 2x - 16$

$\Rightarrow 2x - y - 11 = 0$



(ii) $y = -x^2 + 10x - 27$

$\Rightarrow \dfrac{dy}{dx} = -2x + 10 = 2$

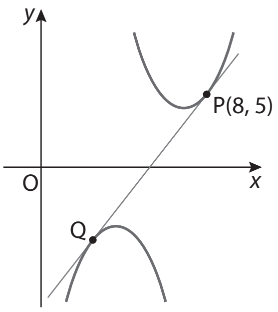
$\Rightarrow -2x = -8$

$\Rightarrow \quad x = 4$

$\Rightarrow y = -(4)^2 + 10(4) - 27$

$= -16 + 40 - 27 = -3$

$\Rightarrow Q = (4, -3)$



23. $y = 2 + 0.12x - 0.01x^3$

(i) $\Rightarrow \dfrac{dy}{dx} = 0.12 - 0.03x^2$

$$\text{At } x = 0 \Rightarrow \frac{dy}{dx} = 0.12 - 0.03(0)^2 = 0.12$$

$$\text{At } x = 3 \Rightarrow \frac{dy}{dx} = 0.12 - 0.03(3)^2$$

$$= 0.12 - 0.27 = -0.15$$

(ii) Gradient is zero $\Rightarrow 0.12 - 0.03x^2 = 0$

$$\Rightarrow \quad 4 - x^2 = 0$$

$$\Rightarrow (2 + x)(2 - x) = 0$$

$$\Rightarrow \quad x = -2, 2$$

For $0 \leqslant x \leqslant 3$ $\quad \Rightarrow x = 2$

$$\Rightarrow y = 2 + 0.12(2) - 0.01(2)^3$$

$$= 2 + 0.24 - 0.08$$

$$= 2.16 \text{ km} = \text{height}$$

24. $y = \sqrt{x + 2}$

(a) On x-axis $\Rightarrow y = 0 \Rightarrow \sqrt{x + 2} = 0$

$$\Rightarrow x + 2 = 0$$

$$\Rightarrow x = -2 \quad \Rightarrow A(-2, 0)$$

On y-axis $\Rightarrow x = 0 \Rightarrow y = \sqrt{0 + 2} = \sqrt{2}$

$$\Rightarrow B\left(0, \sqrt{2}\right)$$

(b) $y = \sqrt{x + 2} = (x + 2)^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x + 2)^{\frac{-1}{2}} = \frac{1}{2\sqrt{x + 2}}$$

(c) (i) At $x = -1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{-1 + 2}} = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$

(ii) At $x = -1 \Rightarrow y = \sqrt{-1 + 2} = \sqrt{1} = 1 \Rightarrow$ Point $(-1, 1)$

$\Rightarrow$ Equation of Tangent: $y - 1 = \dfrac{1}{2}(x + 1)$

$$\Rightarrow 2y - 2 = x + 1$$

$$\Rightarrow 2y - x = 3$$

(iii) On x-axis $\Rightarrow y = 0 \Rightarrow 0 - x = 3$

$$\Rightarrow x = -3 \quad \Rightarrow C(-3, 0)$$

On y-axis $\Rightarrow x = 0 \Rightarrow 2y - 0 = 3$

$$\Rightarrow y = 1\frac{1}{2} \quad \Rightarrow D\left(0, 1\frac{1}{2}\right)$$

$$\Rightarrow |CD| = \sqrt{(-3 - 0)^2 + \left(0 - 1\frac{1}{2}\right)^2} = \sqrt{9 + 2\frac{1}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

(d) $\dfrac{dy}{dx} < 1 \Rightarrow \dfrac{1}{2\sqrt{x + 2}} < 1$

$$\Rightarrow 2\sqrt{x + 2} > 1$$

$$\Rightarrow \sqrt{x + 2} > \frac{1}{2}$$

$$\Rightarrow x + 2 > \frac{1}{4}$$

$$\Rightarrow x > -2 + \frac{1}{4} = \frac{-7}{4}$$

$$\Rightarrow x > \frac{-7}{4}$$

Exercise 4.2

1. $y = x^2 - 4x + 9$

$\Rightarrow \dfrac{dy}{dx} = 2x - 4 = 0$

$\Rightarrow 2x = 4 \;\Rightarrow x = 2$

$\Rightarrow y = (2)^2 - 4(2) + 9 = 4 - 8 + 9 = 5 \Rightarrow$ Point (2, 5)

$\Rightarrow \dfrac{d^2y}{dx^2} = 2 > 0 \Rightarrow$ Point (2, 5) is a minimum

2. $y = 4 - 8x - 2x^2$

$\Rightarrow \dfrac{dy}{dx} = -8 - 4x = 0$

$\Rightarrow \quad -4x = 8$

$\Rightarrow \quad x = -2$

$\Rightarrow y = 4 - 8(-2) - 2(-2)^2$

$= 4 + 16 - 8 = 12 \Rightarrow$ Point $(-2, 12)$

$\Rightarrow \dfrac{d^2y}{dx^2} = -4 < 0 \Rightarrow$ Point $(-2, 12)$ is a maximum

3. $y = 3x^2 - 6x + 4$

$\Rightarrow \dfrac{dy}{dx} = 6x - 6 = 0$

$\Rightarrow \quad 6x = 6 \quad \Rightarrow \quad x = 1$

$\Rightarrow y = 3(1)^2 - 6(1) + 4 = 3 - 6 + 4 = 1 \Rightarrow$ Point (1, 1)

$\Rightarrow \dfrac{d^2y}{dx^2} = 6 > 0 \Rightarrow$ Point (1, 1) is a minimum

4. $y = x^3 - 9x^2 + 15x + 2$

$\dfrac{dy}{dx} = 3x^2 - 18x + 15 = 0$

$\Rightarrow \quad x^2 - 6x + 5 = 0$

$\Rightarrow \quad (x - 1)(x - 5) = 0$

$\Rightarrow x = 1 \qquad$ OR $\qquad x = 5$

$\Rightarrow y = (1)^3 - 9(1)^2 + 15(1) + 2$ and $y = (5)^3 - 9(5)^2 + 15(5) + 2$

$= 1 - 9 + 15 + 2 = 9$

$\Rightarrow$ Point (1, 9)

$= 125 - 225 + 75 + 2$

$= -23$

$\Rightarrow$ Point $(5, -23)$

$\dfrac{d^2y}{dx^2} = 6x - 18$

Point (1, 9) $\Rightarrow \dfrac{d^2y}{dx^2} = 6(1) - 18 = -12 < 0 \Rightarrow$ (1, 9) Maximum

Point $(5, -23) \Rightarrow \dfrac{d^2y}{dx^2} = 6(5) - 18 = +12 > 0 \Rightarrow (5, -23)$ Minimum

5. (i) $y = 2x^3 - 3x^2 - 12x + 5$

$$\Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12 = 0$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \quad \text{OR} \quad x = -1$$

$$\Rightarrow y = 2(2)^3 - 3(2)^2 - 12(2) + 5 \quad \text{and} \quad y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5$$

$$= 16 - 12 - 24 + 5 = -15 \qquad = -2 - 3 + 12 + 5 = 12$$

$\Rightarrow$ Point $(2, -15)$ Point $(-1, 12)$

$$\frac{d^2y}{dx^2} = 12x - 6$$

Point $(2, -15) \Rightarrow \frac{d^2y}{dx^2} = 12(2) - 6 = 18 > 0 \Rightarrow (2, -15)$ Minimum

Point $(-1, 12) \Rightarrow \frac{d^2y}{dx^2} = 12(-1) - 6 = -18 < 0 \Rightarrow (-1, 12)$ Maximum

(ii) $y = \frac{x^2}{x+2}$

Quotient Rule: $u = x^2$ and $v = x + 2$

$$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+2).(2x) - (x^2).(1)}{(x+2)^2}$$

$$= \frac{2x^2 + 4x - x^2}{(x+2)^2}$$

$$= \frac{x^2 + 4x}{(x+2)^2} = 0$$

$$\Rightarrow x(x+4) = 0$$

$$\Rightarrow x = 0 \quad \text{OR} \quad x = -4$$

$$\Rightarrow y = \frac{(0)^2}{0+2} = 0 \quad \text{and} \quad y = \frac{(-4)^2}{-4+2}$$

Point $(0, 0)$ $= \frac{16}{-2} = -8$

Point $(-4, -8)$

$$\frac{dy}{dx} = \frac{x^2 + 4x}{(x+2)^2}$$

Quotient Rule: $u = x^2 + 4x$ and $v = (x+2)^2$

$$\Rightarrow \frac{du}{dx} = 2x + 4 \qquad \Rightarrow \frac{dv}{dx} = 2(x+2)$$

$$\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+2)^2(2x+4) - (x^2+4x).2(x+2)}{(x+2)^4}$$

At $(0, 0) \Rightarrow \frac{d^2y}{dx^2} = \frac{(0+2)(0+4) - (0+0).2(0+2)}{(0+2)^4} = \frac{8}{16} = \frac{1}{2} > 0$

$\Rightarrow (0, 0)$ Minimum

At $(-4, -8) \Rightarrow \frac{d^2y}{dx^2} = \frac{(-4+2)^2(-8+4) - (16-16).2(-4+2)}{(-4+2)^2} = \frac{-16}{16} = -1 < 0$

$\Rightarrow (-4, -8)$ Maximum

6. $f(x) = 4x + \frac{4}{x} = 4x + 4x^{-1}$

$f'(x) = 4 - 4x^{-2} = 4 - \frac{4}{x^2} = 0$

$\Rightarrow 4x^2 - 4 = 0$

$\Rightarrow \quad x^2 - 1 = 0$

$\Rightarrow (x + 1)(x - 1) = 0$

$\Rightarrow x = -1 \quad \text{OR} \quad x = 1$

$f(-1) = 4(-1) + \frac{4}{-1} = -8$ and $f(1) = 4(1) + \frac{4}{1} = 8$

Point $(-1, -8)$ Point $(1, 8)$

$f''(x) = 0 + 8x^{-3} = \frac{8}{x^3}$

At $(-1, -8) \Rightarrow f''(-1) = \frac{8}{(-1)^3} = \frac{8}{-1} = -8 < 0$

$\Rightarrow (-1, -8)$ Maximum

At $(1, 8) \Rightarrow f''(1) = \frac{8}{(1)^3} = 8 > 0$

$\Rightarrow (1, 8)$ Minimum

7. $y = x^2 + \frac{250}{x} = x^2 + 250x^{-1}$

$\frac{dy}{dx} = 2x - 250x^{-2} = 2x - \frac{250}{x^2} = 0$

$\Rightarrow 2x^3 - 250 = 0$

$\Rightarrow x^3 - 125 = 0$

$\Rightarrow x = \sqrt[3]{125} = 5$

$\Rightarrow y = (5)^2 + \frac{250}{5} = 25 + 50 = 75 \Rightarrow$ Point $(5, 75)$

$\frac{d^2y}{dx^2} = 2 + 500x^{-3} = 2 + \frac{500}{x^3}$

At $(5, 75) \Rightarrow \frac{d^2y}{dx^2} = 2 + \frac{500}{(5)^3} = 2 + 4 = 6 > 0$

$\Rightarrow (5, 75)$ Minimum

8. $y = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{2}x^{\frac{-1}{2}} = 1 - \frac{1}{2\sqrt{x}} = 0$

$\Rightarrow 2\sqrt{x} - 1 = 0$

$\Rightarrow 2\sqrt{x} = 1$

$\Rightarrow \sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

$\Rightarrow y = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow$ Point $\left(\frac{1}{4}, \frac{-1}{4}\right)$

$\frac{d^2y}{dx^2} = 0 + \frac{1}{4}x^{\frac{-3}{2}} = \frac{1}{4x^{\frac{3}{2}}}$

At $\left(\frac{1}{4}, -\frac{1}{4}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4\left(\frac{1}{4}\right)^{\frac{3}{2}}} = \frac{1}{\frac{1}{2}} = 2 > 0$

$\Rightarrow \left(\frac{1}{4}, -\frac{1}{4}\right)$ Minimum

9. (i) $y = x^3 + 3x^2 + 1$

$\Rightarrow \frac{dy}{dx} = 3x^2 + 6x$

$\Rightarrow \frac{d^2y}{dx^2} = 6x + 6 = 0$

$\Rightarrow x + 1 = 0$

$\Rightarrow x = -1 \Rightarrow y = (-1)^3 + 3(-1)^2 + 1 = 3$

$\Rightarrow$ Point of inflection $= (-1, 3)$

(ii) $y = x^3 - 6x^2 + 9x + 2$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 9$

$\Rightarrow \frac{d^2y}{dx^2} = 6x - 12 = 0$

$\Rightarrow x - 2 = 0 \quad \Rightarrow x = 2$

$\Rightarrow y = (2)^3 - 6(2)^2 + 9(2) = 4$

$\Rightarrow$ Point of inflection $= (2, 4)$

10. $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -\cos x$

At $x = \frac{\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = -\cos\frac{\pi}{2} = -0 = 0$

Hence, point of inflection occurs at $x = \frac{\pi}{2}$

11. $y = ax^3 + bx^2 + c$

$\Rightarrow \frac{dy}{dx} = 3ax^2 + 2bx$

Point $(-1, 5) \Rightarrow \frac{dy}{dx} = 3a(-1)^2 + 2b(-1) = 0$

$\Rightarrow 3a - 2b = 0$

Point $(0, 4) \quad \Rightarrow 4 = a(0)^3 + b(0)^2 + c \Rightarrow c = 4$

Point $(-1, 5) \Rightarrow 5 = a(-1)^3 + b(-1)^2 + 4$

$\Rightarrow 5 = -a + b + 4$

$\Rightarrow a - b = -1 \Rightarrow \quad 2a - 2b = -2$

and $\quad -3a + 2b = 0$

$$\begin{array}{rl} 2a - 2b &= -2 \\ -3a + 2b &= 0 \\ \hline -a \qquad &= -2 \end{array}$$

$\Rightarrow a = 2$

$\Rightarrow 2 - b = -1$

$-b = -3 \Rightarrow b = 3$

$\Rightarrow a = 2, b = 3, c = 4$

12. $f(x) = \frac{x+1}{x-3}$

Quotient Rule $\Rightarrow \quad u = x + 1 \quad$ and $\quad v = x - 3$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = 1$$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x-3).1 - (x+1).1}{(x-3)^2}$$

$$= \frac{x - 3 - x - 1}{(x-3)^2} = \frac{-4}{(x-3)^2} = 0$$

$$\Rightarrow -4 = 0 \text{ False}$$

$\Rightarrow$ Graph has no turning points

13. (i) $y = 2x^2 - \ln x$

$$\frac{dy}{dx} = 4x - \frac{1}{x} = 4x - x^{-1}$$

At $x = 1 \Rightarrow \frac{dy}{dx} = 4(1) - \frac{1}{1} = 3$

(ii) $\frac{dy}{dx} = 0 \Rightarrow 4x - \frac{1}{x} = 0$

$$\Rightarrow 4x^2 - 1 = 0$$

$$\Rightarrow (2x + 1)(2x - 1) = 0$$

$$\Rightarrow x = \frac{-1}{2}, x = \frac{1}{2}$$

Since $x > 0 \Rightarrow x = \frac{1}{2} \Rightarrow y = 2\left(\frac{1}{2}\right)^2 - \ln\frac{1}{2}$

$$= \frac{1}{2} + \ln 2$$

Point $\left(\frac{1}{2}, \frac{1}{2} + \ln 2\right)$

$\Rightarrow \frac{d^2y}{dx^2} = 4 + x^{-2} = 4 + \frac{1}{x^2}$

$\Rightarrow$ At $x = \frac{1}{2} \Rightarrow \frac{d^2y}{dx^2} = 4 + \frac{1}{\left(\frac{1}{2}\right)^2} = 8 > 0$

$\Rightarrow$ Minimum point

14. (i) $y = e^x - x$

$\frac{dy}{dx} e^x - 1 = 0$

$\Rightarrow e^x = 1 \Rightarrow x = 0$

$\Rightarrow y = e^0 - 0 = 1 - 0 \qquad \Rightarrow$ Point (0, 1)

(ii) $\frac{d^2y}{dx^2} = e^x$

Point (0, 1) $\Rightarrow \frac{d^2y}{dx^2} = e^0 = 1 > 0$

$\Rightarrow$ Minimum point

15. (a) $y = x^3 - 9x^2 + 24x - 20$

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 18x + 24 = 0$

$\Rightarrow \quad x^2 - 6x + 8 = 0$

$\Rightarrow \quad (x - 2)(x - 4) = 0$

$\Rightarrow \quad x = 2 \quad \text{OR} \quad x = 4$

$\Rightarrow y = (2)^3 - 9(2)^2 + 24(2) - 20$ and $y = (4)^3 - 9(4)^2 + 24(4) - 20$

$= 8 - 36 + 48 - 20 = 0$ $\qquad = 64 - 144 + 96 - 20 = -4$

Point (2, 0) $\qquad$ Point (4, −4)

$\Rightarrow \dfrac{d^2y}{dx^2} = 6x - 18$

Point (2, 0) $\Rightarrow \dfrac{d^2y}{dx^2} = 6(2) - 18 = -6 < 0 \qquad \Rightarrow$ (2, 0) Maximum

Point (4, −4) $\Rightarrow \dfrac{d^2y}{dx^2} = 6(4) - 18 = 6 > 0 \qquad \Rightarrow$ (4, −4) Minimum

(b) (i) $(x - 2)^2(x - 5) = (x^2 - 4x + 4)(x - 5)$

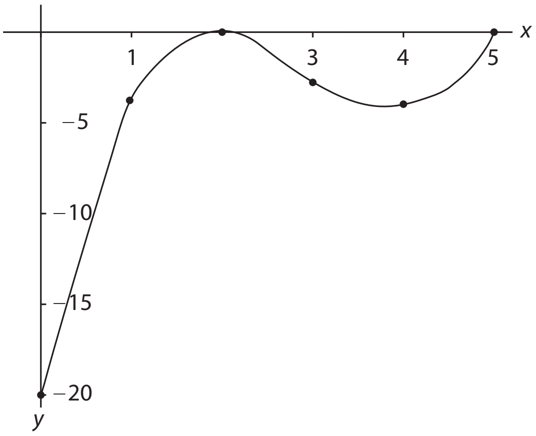
$= x^3 - 5x^2 - 4x^2 + 20x + 4x - 20$

$= x^3 - 9x^2 + 24x - 20$

(ii) $f(x) = x^3 - 9x^2 + 24x - 20$

$f(0) = -20 \qquad f(1) = -4 \qquad f(2) = 0$

$f(3) = -2 \qquad f(4) = -4 \qquad f(5) = 0$



16. (i) $f(x) = (1 + x)\log_e(1 + x)$

Product Rule: $u = 1 + x$ and $v = \log_e(1 + x)$

$\Rightarrow \dfrac{du}{dx} = 1 \qquad \dfrac{dv}{dx} = \dfrac{1}{1 + x}$

$\Rightarrow \dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = (1 + x)\,.\,\dfrac{1}{1 + x} + \log_e(1 + x)\,.\,1$

$= 1 + \log_e(1 + x) = 0$

$\Rightarrow \log_e(1 + x) = -1 \Rightarrow e^{-1} = 1 + x$

$\Rightarrow \dfrac{1}{e} = x + 1$

$\Rightarrow x = \dfrac{1}{e} - 1 = \dfrac{1 - e}{e}$

$$x = \frac{1-e}{e} \Rightarrow f\left(\frac{1-e}{e}\right) = \left(1+\frac{1}{e}-1\right)\left(\log_e\left(1+\frac{1}{e}-1\right)\right)$$

$$= \frac{1}{e} \cdot \log_e \frac{1}{e}$$

$$= \frac{1}{e} \cdot \log_e e^{-1}$$

$$= \frac{1}{e} \cdot -1 \cdot \log_e e = -\frac{1}{e}$$

$$\Rightarrow \text{Turning Point} = \left(\frac{1-e}{e}, -\frac{1}{e}\right)$$

(ii) $\dfrac{dy}{dx} = 1 + \log_e (1 + x)$

$$\Rightarrow \frac{d^2y}{dx^2} = 0 + \frac{1}{1+x} = \frac{1}{1+x}$$

$$\text{At } x = \frac{1}{e} - 1 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{1+\frac{1}{e}-1} = \frac{1}{\frac{1}{e}} = e > 0$$

$$\Rightarrow \left(\frac{1-e}{e}, \frac{-1}{e}\right) \text{ is a minimum}$$

17. $f(x) = ax^3 + bx^2 + cx + d$

Point (0, 4) $\Rightarrow f(0) = a(0)^3 + b(0)^2 + c(0) + d = 4$

$\Rightarrow d = 4$

Point (1, 0) $\Rightarrow a(1)^3 + b(1)^2 + c(1) + 4 = 0$

$\Rightarrow a + b + c = -4$

$f'(x) = 3ax^2 + 2bx + c$

$\Rightarrow f'(0) = 3a(0)^2 + 2b(0) + c = 0 \Rightarrow c = 0$

$f''(x) = 6ax + 2b$

$\Rightarrow f''(1) = 6a(1) + 2b = 0$ $\qquad\Rightarrow 3a + b = 0$

and $\quad a + b + 0 = -4$ $\qquad\Rightarrow \underline{-a - b = +4}$

$2a \quad = 4$

$\Rightarrow a \quad = 2$

$\Rightarrow 2 + b = -4$

$\Rightarrow \quad b = -6$

18. $f(x) = \dfrac{x}{x+2}$

Quotient Rule: $u = x$ $\qquad$ and $\qquad v = x + 2$

$\Rightarrow \dfrac{du}{dx} = 1$ $\qquad \Rightarrow \dfrac{dv}{dx} = 1$

$$\Rightarrow f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+2).(1) - (x).(1)}{(x+2)^2}$$

$$= \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2} = 0$$

$\Rightarrow \qquad 2 = 0$ False

$\Rightarrow f(x)$ has no turning points

$$f'(x) = \frac{2}{(x+2)^2} = 2(x+2)^{-2}$$

$$\Rightarrow f''(x) = -4(x+2)^{-3} = \frac{-4}{(x+2)^3} = 0$$

$$\Rightarrow -4 = 0 \text{ False}$$

$\Rightarrow f(x)$ has no point of inflection

19. (i) $$g(x) = x^2 + \frac{a}{x^2} = x^2 + ax^{-2}$$

$$\Rightarrow g'(x) = 2x - 2ax^{-3} = 2x - \frac{2a}{x^3}$$

$$\Rightarrow g'(2) = 2(2) - \frac{2a}{(2)^3} = 4 - \frac{2a}{8} \Rightarrow 32 - 2a = 0$$

$$\Rightarrow a = 16$$

(ii) $$g(x) = x^2 + \frac{16}{x^2} \Rightarrow g'(x) = 2x - \frac{32}{x^3} = 0$$

$$\Rightarrow \quad 2x^4 - 32 = 0$$

$$\Rightarrow \quad x^4 - 16 = 0$$

$$\Rightarrow \quad (x^2-4)(x^2+4) = 0$$

$$\Rightarrow \quad (x+2)(x-2) = 0$$

$$\Rightarrow x = -2 \quad \text{OR} \quad x = 2$$

$$g'(x) = 2x - \frac{32}{x^3} = 2x - 32x^{-3}$$

$$\Rightarrow g''(x) = 2 + 96x^{-4} = 2 + \frac{96}{x^4}$$

$$\Rightarrow g''(-2) = 2 + \frac{96}{(-2)^4} = 2 + 6 = 8 > 0$$

$\Rightarrow$ Minimum at $x = -2$

$$\Rightarrow g''(2) = 2 + \frac{96}{(2)^4} = 2 + 6 = 8 > 0$$

$\Rightarrow$ Minimum at $x = 2$

$\Rightarrow g(x)$ has no local maximum point

20. (i) $$C = \frac{1400}{v} + \frac{2v}{7} = 1400v^{-1} + \frac{2}{7}v$$

$$\Rightarrow \frac{dC}{dv} = -1400v^{-2} + \frac{2}{7} = \frac{-1400}{v^2} + \frac{2}{7} = 0$$

$$\Rightarrow \quad -9800 + 2v^2 = 0$$

$$\Rightarrow \quad 2v^2 = 9800$$

$$\Rightarrow \quad v^2 = 4900$$

$$\Rightarrow \quad v = \sqrt{4900} = 70 \text{ km/hr}$$

(ii) $$\frac{d^2C}{dv^2} = 2800v^{-3} = \frac{2800}{v^3}$$

At $v = 70 \Rightarrow \frac{d^2C}{dv^2} = \frac{2800}{(70)^3} = \frac{2800}{343\,000} > 0$

$\Rightarrow$ Minimum at $v = 70$

(iii) $v = 70 \Rightarrow C = \frac{1400}{70} + \frac{2(70)}{7} \Rightarrow C = €40$

Exercise 4.3

1. In (ii) $\Rightarrow \frac{dy}{dx}$ positive for all values of x.

2. In (i) and (iii) $\Rightarrow \frac{dy}{dx}$ negative for all values of x.

3. (i) Positive slope
(ii) $x < -2$ OR $x > 3$
(iii) $-2 < x < 3$
(iv) $f'(x) = 0 \quad \Rightarrow x = -2, 3$

4.
- Positive slope for $x < -1$
- Turning point at $x = -1$
- Negative slope for $x > 1$

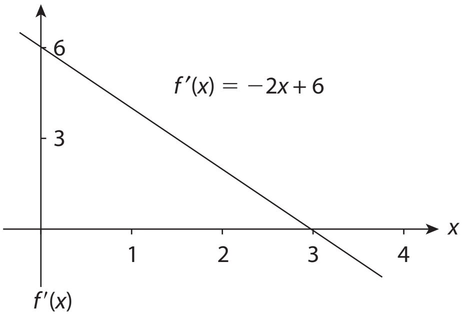
5. Answer Ⓒ because

- Positive slope for $x < 1\frac{1}{2}$
- Turning point at $x = 1\frac{1}{2}$
- Negative slope for $x > 1\frac{1}{2}$

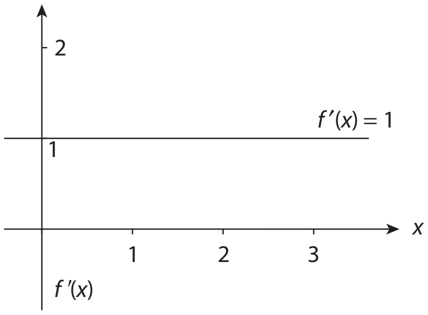
6. Answer Ⓑ because
- Positive slope for $x <$ A OR $x >$ B
- Turning point at $x =$ A and B
- Negative slope for A $< x <$ B

7. (i) $f'(x) > 0$ for $-2 < x < 1$
(ii) $f'(x) < 0$ for $x < -2$ OR $x > 1$
(iii) $f'(x) = 0$ at $x = -2, 1$

8. (i)



(ii)



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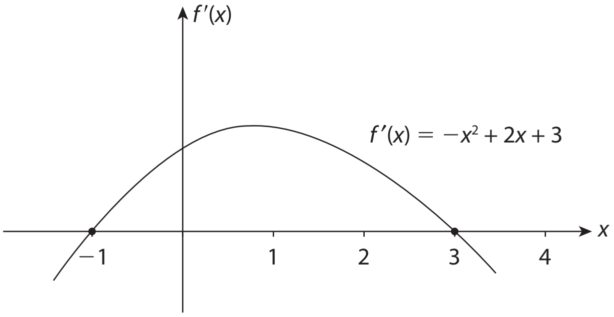


9. (i) A turning point at $x = -3$ for curve (a)

A turning point at $x = 4$ for curve (b)

(ii) Curve (a) is decreasing for $x < -3$

Curve (b) is decreasing for $x > 4$

10. (i) Curve (a) has stationary points at $x = -1, 3$

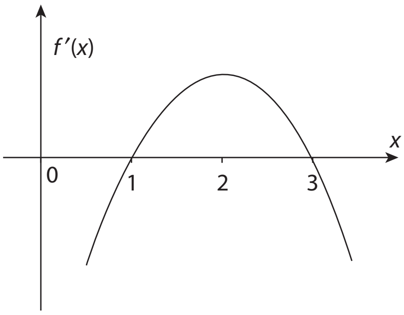
Curve (b) has stationary points at $x = -4.5, 1$

(ii) Curve (a) is increasing for $x < -1$ OR $x > 3$

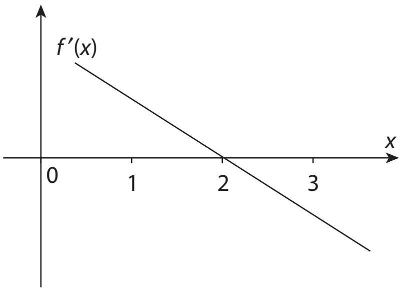
Curve (b) is increasing for $-4.5 < x < 1$

11. C. is true.

12. (i)



(ii)



13. (i) $f'(x) = k(x - a)(x - b) = k(x - 2)(x - 4) = 0$

$$\Rightarrow x = 2, x = 4$$

$\Rightarrow a = 2, b = 4$

(ii) $f'(0) = 6 \Rightarrow k(0 - 2)(0 - 4) = 6$

$$\Rightarrow 8k = 6$$

$$\Rightarrow k = \frac{6}{8} = \frac{3}{4}$$

Exercise 4.4

1. $x + y = 6$ and $A = x^2y$

$\Rightarrow y = 6 - x \quad \Rightarrow \quad A = x^2(6 - x)$

$\Rightarrow \quad A = 6x^2 - x^3$

$$\Rightarrow \frac{dA}{dx} = 12x - 3x^2 = 0$$

$\Rightarrow \quad 3x(4 - x) = 0$

$\Rightarrow x = 0 \quad \text{OR} \quad x = 4$

$$\Rightarrow \frac{d^2A}{dx^2} = 12 - 6x$$

At $x = 0 \Rightarrow \frac{d^2A}{dx^2} = 12 - 6(0) = 12 \quad \Rightarrow \quad$ Minimum

At $x = 4 \Rightarrow \frac{d^2A}{dx^2} = 12 - 6(4) = -12 < 0$

$\Rightarrow$ Maximum at $x = 4$

$\Rightarrow A = (4)^2(6 - 4) = 32$

2. (i) $P(x) = \frac{1152}{x} + 8x + 20 = 1152x^{-1} + 8x + 20$

$$\Rightarrow \frac{dP}{dx} = -1152x^{-2} + 8 = \frac{-1152}{x^2} + 8 = 0$$

$\Rightarrow -1152 + 8x^2 = 0$

$\Rightarrow \quad 8x^2 = 1152$

$\Rightarrow \quad x^2 = 144$

$\Rightarrow \quad x = 12$

$$\Rightarrow \frac{d^2P}{dx^2} = 2304x^{-3} = \frac{2304}{x^3}$$

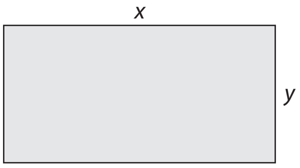
At $x = 12 \Rightarrow \frac{d^2P}{dx^2} = \frac{2304}{(12)^3} = 1\frac{1}{3} > 0 \Rightarrow$ Minimum

(ii) At $x = 12 \Rightarrow P = \frac{1152}{12} + 8(12) + 20 = 212$ cm

3. Perimeter $= 2x + 2y = 100$

$\Rightarrow \quad x + y = 50$

$\Rightarrow \quad y = 50 - x$



Area $= (x)(y) = (x)(50 - x)$

$\Rightarrow A = 50x - x^2$

$$\Rightarrow \frac{dA}{dx} = 50 - 2x = 0$$

$\Rightarrow \quad -2x = -50$

$\Rightarrow \quad x = 25$

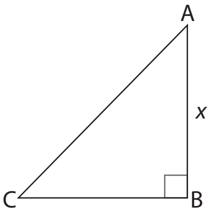
$\frac{d^2A}{dx^2} = -2 < 0 \quad \Rightarrow$ Maximum at $x = 25$

$\Rightarrow$ Area $= (25)(25) = 625$ m^2

4. (i) $|AB| + |BC| = 8$

If $|AB| = x \Rightarrow x + |BC| = 8$

$\Rightarrow \quad |BC| = 8 - x$



(ii) Area Triangle ABC $= \frac{1}{2}|AB| \cdot |BC|$

$$\Rightarrow A = \frac{1}{2}x(8 - x)$$

$$\Rightarrow A = \frac{1}{2}[8x - x^2]$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{2}[8 - 2x] = 4 - x = 0$$

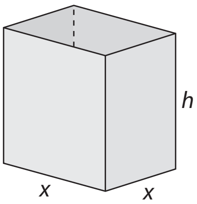
$$\Rightarrow x = 4$$

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{2}[0 - 2] = -1 < 0 \Rightarrow \text{Maximum}$$

At $x = 4 \Rightarrow$ Area $= \frac{1}{2}(4)(8 - 4)$

$$= (2)(4) = 8 \text{ cm}^2$$

5. (i) Volume $= x \,.\, x \,.\, h = x^2h = 108$



$$\Rightarrow h = \frac{108}{x^2}$$

(ii) Surface Area, $S = x \,.\, x + 4x \,.\, h$

$$\Rightarrow S = x^2 + 4x \cdot \frac{108}{x^2} = x^2 + \frac{432}{x}$$

(iii) $S = x^2 + 432x^{-1}$

$$\Rightarrow \frac{dS}{dx} = 2x - 432x^{-2}$$

$$= 2x - \frac{432}{x^2} = 0$$

$$\Rightarrow 2x^3 - 432 = 0$$

$$\Rightarrow x^3 - 216 = 0$$

$$\Rightarrow x^3 = 216$$

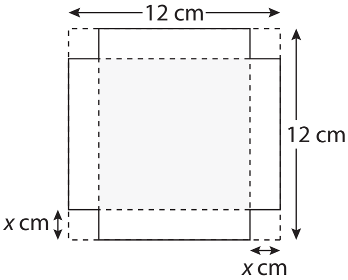
$$\Rightarrow x = \sqrt[3]{216} = 6 \quad \Rightarrow h = \frac{108}{(6)^2} = 3$$

$$\Rightarrow \frac{d^2S}{dx^2} = 2 + 864x^{-3} = 2 + \frac{864}{x^3}$$

At $x = 6 \quad \Rightarrow \frac{d^2S}{dx^2} = 2 + \frac{864}{(6)^3} = 6 > 0 \quad \Rightarrow$ Minimum

$\Rightarrow$ Dimensions: 6 m by 6 m by 3 m.

6. (i) Length of side of the box $= 12 - 2x$



$\Rightarrow$ Volume $(V) = (12 - 2x)\,(12 - 2x)\,.\,x = 4x^3 - 48x^2 + 144x$

(ii) $\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$

$\Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x - 2)(x - 6) = 0$

$\Rightarrow x = 2$ (valid) OR $x = 6$ (invalid)

$$\Rightarrow \frac{d^2V}{dx^2} = 24x - 96$$

At $x = 2 \quad \Rightarrow \frac{d^2V}{dx^2} = 24(2) - 96 = -48 < 0$

$\Rightarrow$ Maximum when $x = 2$ cm

7. (i) Total Surface Area $= 54 \text{ cm}^2$

$$\Rightarrow 2x\,.\,x + 4x\,.\,h = 54$$
$$\Rightarrow \quad x^2 + 2xh = 27$$
$$\Rightarrow \quad 2xh = 27 - x^2$$
$$\Rightarrow \quad h = \frac{27 - x^2}{2x}$$

(ii) Volume $(V) = x\,.\,x\,.\,h = x^2h$

$$\Rightarrow \quad V = x^2\frac{(27 - x^2)}{2x}$$
$$\Rightarrow \quad V = \frac{1}{2}(27x - x^3)$$

(iii) $\frac{dV}{dx} = \frac{1}{2}(27 - 3x^2) = 0$

$$\Rightarrow -3x^2 = -27$$
$$\Rightarrow \quad x^2 = 9$$
$$\Rightarrow \quad x = 3 \quad \Rightarrow h = \frac{27 - (3)^2}{2(3)} = 3$$

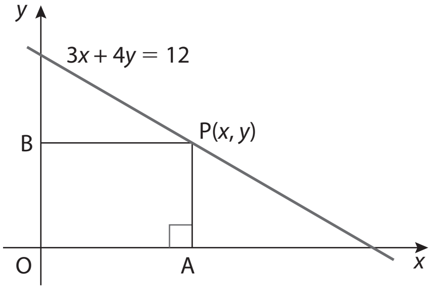
$$\Rightarrow \frac{d^2V}{dx^2} = \frac{1}{2}(0 - 6x) = -3x$$

At $x = 3 \Rightarrow \frac{d^2V}{dx^2} = -3(2) = -6 < 0 \Rightarrow$ Maximum

Hence, volume $= (3)(3)(3) = 27 \text{ cm}^3$

8. (i) $3x + 4y = 12$

$$\Rightarrow 4y = 12 - 3x \;\Rightarrow y = \frac{12 - 3x}{4}$$
$$\Rightarrow \text{P} = \left(x, \frac{12 - 3x}{4}\right)$$



(ii) Area $(A) = (x)\frac{(12 - 3x)}{4} = \frac{1}{4}(12x - 3x^2)$

(iii) $\frac{dA}{dx} = \frac{1}{4}(12 - 6x) = 0$

$$\Rightarrow -6x = -12 \Rightarrow x = 2$$
$$\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{4}(0 - 6) = \frac{-3}{2} < 0 \Rightarrow \text{Maximum}$$
$$\Rightarrow \text{Area} = \frac{1}{4}(12(2) - 3(2)^2) = 3 \text{ sq. units}$$

9. Total Surface Area $= 2\pi rh + 2\pi r^2 = 24\pi$

$$\Rightarrow rh + r^2 = 12$$
$$\Rightarrow \quad rh = 12 - r^2$$
$$\Rightarrow \quad h = \frac{12 - r^2}{r}$$

$$\Rightarrow \text{Volume} = \pi r^2h = \pi r^2\left(\frac{12 - r^2}{r}\right) = \pi r(12 - r^2)$$
$$V = 12\pi r - \pi r^3$$
$$\Rightarrow \frac{dV}{dr} = 12\pi - 3\pi r^2 = 0$$
$$\Rightarrow \quad 4 - r^2 = 0$$
$$\Rightarrow \quad r^2 = 4 \Rightarrow r = \sqrt{4} = 2$$
$$\Rightarrow h = \frac{12 - (2)^2}{2} = 4$$

$$\Rightarrow \frac{d^2V}{dr^2} = 0 - 6\pi r$$

At $r = 2 \Rightarrow \frac{d^2V}{dr^2} = -6\pi(2) = -12\pi < 0 \Rightarrow$ Maximum

10. (i) $r + h = 20 \Rightarrow h = (20 - r)$ cm

(ii) Volume $(V) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(20 - r)$

$$\Rightarrow V = \frac{\pi}{3}(20r^2 - r^3)$$

$$\Rightarrow \frac{dV}{dr} = \frac{\pi}{3}(40r - 3r^2) = 0$$

$$\Rightarrow r(40 - 3r) = 0$$

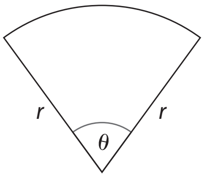
$$\Rightarrow r = 0 \text{ (invalid)} \quad \text{OR} \quad r = \frac{40}{3} \text{ (valid)}$$

$$\Rightarrow \frac{d^2V}{dr^2} = \frac{\pi}{3}(40 - 6r)$$

$$\text{At } r = \frac{40}{3} \Rightarrow \frac{d^2V}{dr^2} = \frac{\pi}{3}\left(40 - 6\left(\frac{40}{3}\right)\right) = \frac{-40\pi}{3} < 0$$

$$\Rightarrow \text{Volume is maximum when } r = \frac{40}{3} \text{ cm}$$

11. (i) Perimeter $= r\theta + 2r = 8$



$$\Rightarrow r\theta = 8 - 2r$$

$$\Rightarrow \theta = \frac{8 - 2r}{r} = \frac{8}{r} - 2$$

(ii) Area $(A) = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{8}{r} - 2\right)$

$$\Rightarrow A = 4r - r^2$$

(iii) $\Rightarrow \frac{dA}{dr} = 4 - 2r = 0$

$$\Rightarrow -2r = -4 \Rightarrow r = 2$$

$$\Rightarrow \frac{d^2A}{dr^2} = -2 < 0 \Rightarrow \text{Maximum}$$

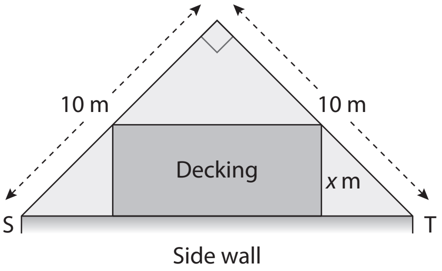
Hence, at $r = 2 \Rightarrow A = 4(2) - (2)^2 = 4$ m^2

12. (a) (i) $|ST|^2 = (10)^2 + (10)^2 = 200$

$$\Rightarrow |ST| = \sqrt{200} = 10\sqrt{2}$$

(ii) $\Rightarrow$ Length of rectangle $= 10\sqrt{2} - 2x$, width $= x$

$$\Rightarrow \text{Area } (A) = (10\sqrt{2} - 2x)x = (10\sqrt{2})x - 2x^2$$



(b) $\frac{dA}{dx} = 10\sqrt{2} - 4x = 0$

$$\Rightarrow -4x = -10\sqrt{2}$$

$$\Rightarrow \quad x = \frac{10\sqrt{2}}{4} = \frac{5\sqrt{2}}{2} \text{ (width)}$$

$$\Rightarrow \text{Length} = 10\sqrt{2} - 2\left(\frac{5\sqrt{2}}{2}\right) = 5\sqrt{2}$$

$$\Rightarrow \frac{d^2A}{dx^2} = 0 - 4 = -4 < 0 \Rightarrow \text{Maximum}$$

$$\Rightarrow \text{Dimensions: length} = 5\sqrt{2} \text{ m, width} = \frac{5\sqrt{2}}{2} \text{ m}$$

13. $r^2 + h^2 = (3)^2 = 9$

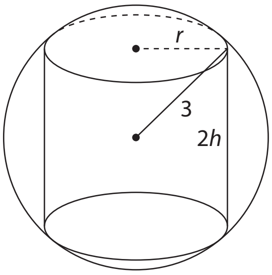
$$\Rightarrow r^2 = 9 - h^2$$

$$\Rightarrow r = \sqrt{9 - h^2}$$

Hence, Volume $(V) = \pi r^2 h$

$$= \pi(9 - h^2)\,.\,(2h)$$

$$= 2\pi h(9 - h^2) = 18\pi h - 2\pi h^3$$



$$\Rightarrow \frac{dV}{dh} = 18\pi - 6\pi h^2 = 0$$

$$\Rightarrow 3 - h^2 = 0$$

$$\Rightarrow \quad h^2 = 3 \quad \Rightarrow h = \sqrt{3}$$

$$\Rightarrow \quad r = \sqrt{9-3} = \sqrt{6}$$

$$\Rightarrow \frac{d^2V}{dh^2} = 0 - 12\pi h$$

$$\text{At} \quad h = \sqrt{3} \Rightarrow \frac{d^2V}{dh^2} = -12\pi(\sqrt{3}) = -12\sqrt{3}\pi < 0$$

$$\Rightarrow \text{ Maximum when } h = \sqrt{3}$$

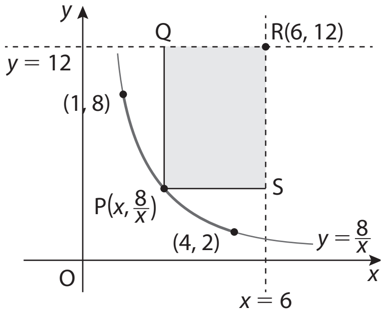
$$\Rightarrow \text{Volume} = \pi(\sqrt{6})^2 . (2\sqrt{3}) = 12\pi\sqrt{3} \text{ cm}^3$$

14. (a) (i) $|PS| = 6 - x$

$$|RS| = 12 - \frac{8}{x}$$

(ii) $\text{Area } (A) = (6 - x)\left(12 - \frac{8}{x}\right)$

$$= 72 - \frac{48}{x} - 12x + 8 = 80 - 12x - \frac{48}{x}$$



(b) $A = 80 - 12x - 48x^{-1}$

$$\Rightarrow \frac{dA}{dx} = -12 + 48x^{-2} = -12 + \frac{48}{x^2} = 0$$

$$\Rightarrow -12x^2 + 48 = 0$$

$$\Rightarrow \quad +x^2 - 4 = 0$$

$$\Rightarrow \quad x = \sqrt{4} = 2$$

$$\Rightarrow \frac{d^2A}{dx^2} = 0 - 96x^{-3} = \frac{-96}{x^3}$$

$$\text{At } x = 2 \Rightarrow \frac{d^2A}{dx^2} = \frac{-96}{(2)^3} = -12 < 0 \Rightarrow \text{Maximum}$$

$$\text{When } x = 2 \Rightarrow A = 80 - 12(2) - \frac{48}{2}$$

$$= 80 - 48 = 32 \text{ sq. units}$$

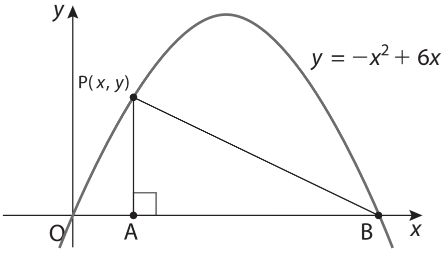
P lies between $x = 1$ and $x = 4$

$$\Rightarrow \text{At } x = 1 \Rightarrow A = 80 - 12(1) - \frac{48}{1} = 20 \text{ (minimum)}$$

$$\text{and at } x = 4 \Rightarrow A = 80 - 12(4) - \frac{48}{4} = 20 \text{ (minimum)}$$

15. (i) $y = -x^2 + 6x$

$\Rightarrow P(x, y) = (x, -x^2 + 6x)$



(ii) At B, $y = 0 \Rightarrow -x^2 + 6x = 0$

$$\Rightarrow -x(x - 6) = 0$$

$$\Rightarrow x = 0, x = 6$$

$O = (0, 0), B = (6, 0)$

$\Rightarrow |OA| = x \qquad |OB| = 6 \Rightarrow |AB| = 6 - x$

$|AP| = y = -x^2 + 6x$

$$\Rightarrow \text{Area } (A) = \frac{1}{2}|AB| \cdot |AP|$$

$$= \frac{1}{2}(6 - x)(-x^2 + 6x)$$

$$= \frac{1}{2}[-6x^2 + 36x + x^3 - 6x^2]$$

$$= \frac{1}{2}(x^3 - 12x^2 + 36x)$$

(iii) $\frac{dA}{dx} = \frac{1}{2}(3x^2 - 24x + 36) = 0$

$\Rightarrow \quad x^2 - 8x + 12 = 0$

$\Rightarrow \quad (x - 2)(x - 6) = 0$

$\Rightarrow \quad x = 2, x = 6$ (invalid)

$\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{2}(6x - 24) = 3x - 12$

At $x = 2 \Rightarrow \frac{d^2A}{dx^2} = 3(2) - 12 = -6 < 0$ (Maximum)

$\Rightarrow$ Area $= \frac{1}{2}(6 - 2)(-(2)^2 + 6(2))$

$= \frac{1}{2}(4)(8) = 16$ sq. units

16. (i) Perimeter $= 2x + 2y = 120$

$\Rightarrow \quad x + y = 60$

$\Rightarrow \quad y = 60 - x$

(ii) $S = 5x^2y = 5x^2(60 - x)$

(iii) Possible values for $x \Rightarrow 5x^2(60 - x) > 0$

$\Rightarrow 0 < x < 60$

(iv) $S = 300x^2 - 5x^3$

$\Rightarrow \frac{dS}{dx} = 600x - 15x^2 = 0$

$\Rightarrow \quad 40x - x^2 = 0$

$\Rightarrow \quad x(40 - x) = 0$

$\Rightarrow x = 0$ (invalid) OR $x = 40$ (valid)

$\Rightarrow \frac{d^2S}{dx^2} = 600 - 30x$

At $x = 40 \Rightarrow \frac{d^2S}{dx^2} = 600 - 30(40) = -600 < 0$

$\Rightarrow$ Maximum (strongest) at $x = 40$

$\Rightarrow y = 60 - 40 = 20$

(v) $x < 19$ cm $\Rightarrow$ Maximum strength $= 5(19)^2\,(60 - 19)$

$= 5(19)^2\,(41) = 74\,005$

Exercise 4.5

1. $p = 2q^3 + q$

$\Rightarrow \frac{dp}{dq} = 6q^2 + 1$

At $q = 4 \quad \Rightarrow \quad \frac{dp}{dq} = 6(4)^2 + 1 = 97$

2. (i) $y = 2x^2 + x$

$\Rightarrow \frac{dy}{dx} = 4x + 1$

When $x = 4 \quad \Rightarrow \quad \frac{dy}{dx} = 4(4) + 1 = 17$

(ii) Rate of change is 9 $\Rightarrow 4x + 1 = 9$

$\Rightarrow \quad 4x = 8$

$\Rightarrow \quad x = 2$

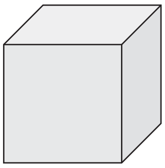
3. $A = \pi r^2$

$\Rightarrow \dfrac{dA}{dr} = 2\pi r$

(i) $r = 5\text{ cm} \quad \Rightarrow \quad \dfrac{dA}{dr} = 2\pi(5) = 10\pi\text{ cm}^2$

(ii) $r = 10\text{ cm} \quad \Rightarrow \quad \dfrac{dA}{dr} = 2\pi(10) = 20\pi\text{ cm}^2$

4. $V = x.x.x = x^3$



$\Rightarrow \dfrac{dV}{dx} = 3x^2$

(i) $x = 10\text{ cm} \quad \Rightarrow \quad \dfrac{dV}{dx} = 3(10)^2 = 300\text{ cm}^3$

(ii) $V = 125 \quad \Rightarrow \quad x^3 = 125$

$\Rightarrow \quad x = \sqrt[3]{125} = 5\text{ cm}$

Hence, $\dfrac{dV}{dx} = 3(5)^2 = 75\text{ cm}^3$

5. $P = 100(5 + t - 0.25t^2)$

$\Rightarrow \dfrac{dP}{dt} = 100(0 + 1 - 0.5t) = 100 - 50t$

When $t = 3 \quad \Rightarrow \quad \dfrac{dP}{dt} = 100 - 50(3) = -50$ people per year.

Thus the population is declining by 50 people after 3 years.

6. (i) $M = 200\,000 + 600t^2 - \dfrac{200}{3}t^3$

$\dfrac{dM}{dt} = 0 + 1200t - \dfrac{200}{3} \cdot 3t^2$

$= 1200t - 200t^2$

(ii) When $t = 3 \quad \Rightarrow \quad \dfrac{dM}{dt} = 1200(3) - 200(3)^2$

$= 3600 - 1800$

$= €1800$ per month

(iii) Rate of growth $= 0$

$\Rightarrow 1200t - 200t^2 = 0$

$\Rightarrow 200t(6 - t) = 0$

$\Rightarrow t = 0$ and $t = 6$

7. $s = t^3 - 2t^2 + 3t - 4$

(i) Speed $= \dfrac{ds}{dt} = 3t^2 - 4t + 3$

When $t = 4 \quad \Rightarrow \quad \dfrac{ds}{dt} = 3(4)^2 - 4(4) + 3$

$= 48 - 16 + 3 = 35$ m/sec.

(ii) Acceleration $= \dfrac{d^2s}{dt^2} = 6t - 4$

When $t = 4 \quad \Rightarrow \quad \dfrac{d^2s}{dt^2} = 6(4) - 4 = 20\text{ m/sec}^2$.

8. $s = t^3 - 4t^2 + 4t$

(i) Speed $= \dfrac{ds}{dt} = 3t^2 - 8t + 4$

When $t = 3 \quad \Rightarrow \quad \dfrac{ds}{dt} = 3(3)^2 - 8(3) + 4$

$= 27 - 24 + 4 = 7$ m/sec

(ii) Acceleration $= \frac{d^2s}{dt^2} = 6t - 8$

When $t = 1 \quad \Rightarrow \quad \frac{d^2s}{dt^2} = 6(1) - 8 = -2$ m/sec^2

(iii) Body momentarily at rest $\Rightarrow \frac{ds}{dt} = 0$

$\Rightarrow 3t^2 - 8t + 4 = 0$

$\Rightarrow (3t - 2)(t - 2) = 0$

$\Rightarrow t = \frac{2}{3}$ sec, $t = 2$ sec

9. $h = 600t - 5t^2$

(i) $\Rightarrow \frac{dh}{dt} = 600 - 10t = 0$

$\Rightarrow \quad -10t = -600$

$\Rightarrow \quad t = 60$ secs

(ii) When $t = 60$ secs $\quad \Rightarrow \quad h = 600(60) - 5(60)^2$

$= 36\,000 - 18\,000$

$= 18\,000$ m $= 18$ km

10. $s = t^3 - 2t^2 + 4t$

(i) At $t = 2 \quad \Rightarrow \quad s = (2)^3 - 2(2)^2 + 4(2)$

$= 8 - 8 + 8 = 8$ m

(ii) Velocity $= \frac{ds}{dt} = 3t^2 - 4t + 4 = 4$

$\Rightarrow 3t^2 - 4t = 0$

$\Rightarrow t(3t - 4) = 0$

$\Rightarrow t = 0 \quad$ OR $\quad t = \frac{4}{3}$

11. $s = 2t^3 - 5t^2 + 4t - 5$

(i) Velocity $= \frac{ds}{dt} = 6t^2 - 10t + 4 = 0$

$\Rightarrow \quad 3t^2 - 5t + 2 = 0$

$\Rightarrow (3t - 2)(t - 1) = 0$

$\Rightarrow t = \frac{2}{3} \quad$ OR $\quad t = 1$

Acceleration $= \frac{d^2s}{dt^2} = 12t - 10$

At $t = \frac{2}{3} \quad \Rightarrow \quad \frac{d^2s}{dt^2} = 12\left(\frac{2}{3}\right) - 10 = -2$

At $t = 1 \quad \Rightarrow \quad \frac{d^2s}{dt^2} = 12(1) - 10 = 2$

(ii) Acceleration $= 0 \Rightarrow 12t - 10 = 0$

$\Rightarrow 12t = 10 \quad \Rightarrow \quad t = \frac{5}{6}$

$\Rightarrow$ velocity $= \frac{ds}{dt} = 6\left(\frac{5}{6}\right)^2 - 10\left(\frac{5}{6}\right) + 4$

$= \frac{25}{6} - \frac{25}{3} + 4 = -\frac{1}{6}$

12. $x = t^3 - 11t^2 + 24t - 3$

(i) $t = 0 \quad \Rightarrow \quad x = (0)^3 - 11(0)^2 + 24(0) - 3 = -3$

$\frac{dx}{dt} = 3t^2 - 22t + 24$

at $t = 0 \quad \Rightarrow \quad$ Velocity $= \frac{dx}{dt} = 3(0)^2 - 22(0) + 24 = 24$

$\Rightarrow$ 3 cm to the left of O, moving to the right at 24 cm/sec.

(ii) Velocity $= \dfrac{dx}{dt} = 3t^2 - 22t + 24$

(iii) Particle is stationary $\Rightarrow \dfrac{dx}{dt} = 0$

$$\Rightarrow 3t^2 - 22t + 24 = 0$$
$$\Rightarrow (3t - 6)(t - 6) = 0$$
$$\Rightarrow t = \frac{4}{3} \text{ secs} \quad \text{OR} \quad t = 6 \text{ secs}$$

(iv) $t = \dfrac{4}{3} \Rightarrow x = \left(\dfrac{4}{3}\right)^3 - 11\left(\dfrac{4}{3}\right)^2 + 24\left(\dfrac{4}{3}\right) - 3$

$$= \frac{64}{27} - \frac{176}{9} + 32 - 3 = 11\frac{22}{27}$$

$t = 6 \quad \Rightarrow \quad x = (6)^3 - 11(6)^2 + 24(6) - 3$

$$= 216 - 396 + 144 - 3 = -39$$

$\Rightarrow 11\frac{22}{27}$ cm to the right of O and 39 cm to the left of O

(v) Velocity negative $= 6 - \dfrac{4}{3} = 4\dfrac{2}{3}$ secs

(vi) Acceleration $= \dfrac{d^2x}{dt^2} = 6t - 22$

(vii) Acceleration is zero $\Rightarrow \quad 6t - 22 = 0$

$$\Rightarrow \quad 6t = 22$$
$$\Rightarrow \quad t = \frac{22}{6} = \frac{11}{3} \text{ secs}$$

When $t = \dfrac{11}{3} \quad \Rightarrow \quad s = \left(\dfrac{11}{3}\right)^3 - 11\left(\dfrac{11}{3}\right)^2 + 24\left(\dfrac{11}{3}\right) - 3$

$$= \frac{1331}{27} - \frac{1331}{9} + 88 - 3 = -13\frac{16}{27} \text{ cm}$$

When $t = \dfrac{11}{3} \quad \Rightarrow \quad \dfrac{ds}{dt} = 3\left(\dfrac{11}{3}\right)^2 - 22\left(\dfrac{11}{3}\right) + 24$

$$= \frac{121}{3} - \frac{242}{3} + 24 = -16\frac{1}{3} \text{ cm/sec}$$

Hence, when $t = \dfrac{11}{3}$ secs, the particle is $13\dfrac{16}{27}$ cm left of O and moving to the left at $16\dfrac{1}{3}$ cm/sec.

13. $n = n_0 e^{0.2t}$

When $n_0 = 5 \Rightarrow n = 5e^{0.2t}$

(i) $t = 0 \quad \Rightarrow \quad n = 5e^{0.2(0)} = 5e^0 = 5.1 = 5$

$t = 10 \quad \Rightarrow \quad n = 5e^{0.2(10)} = 5e^2 = 36.945 = 37$

(ii) (0, 5), (10, 37) $\quad \Rightarrow \quad$ average rate of growth $= \dfrac{37 - 5}{10 - 0}$

$$= \frac{32}{10} = 3.2$$

(iii) $\dfrac{dn}{dt} = 5e^{0.2t}.(0.2) = (1)e^{0.2t} = e^{0.2t}$

When $t = 5 \quad \Rightarrow \quad \dfrac{dn}{dt} = e^{0.2(5)} = e^1 = e$

Exercise 4.6

1. (i) $\dfrac{dr}{dt}$ (ii) $\dfrac{dt}{dr}$ (iii) $\dfrac{ds}{dt}$

2. (i) $\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dt}$

$\Rightarrow 8 = 4 \cdot \dfrac{dr}{dt}$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{8}{4} = 2$

(ii) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$

$\Rightarrow 8 = \dfrac{dV}{dr} \cdot 2$

$\Rightarrow \dfrac{dV}{dr} = \dfrac{8}{2} = 4$

3. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= (10)\,.\,(2) = 20$

4. $A = \pi r^2$ and $\frac{dr}{dt} = 1$

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r\,.\,1 = 2\pi r$

When $r = 5 \Rightarrow \frac{dA}{dt} = 2\pi(5) = 10\pi$

5. Given $\frac{dr}{dt} = 3$ cm/sec, find $\frac{dA}{dt}$

$\Rightarrow A = \pi r^2$

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = (2\pi r)\,.\,(3) = 6\pi r$

When $r = 9$ cm $\Rightarrow \frac{dA}{dt} = 6\pi(9) = 54\pi$ cm^2/sec

6. Given $\frac{dx}{dt} = 5$ cm/sec, find $\frac{dA}{dt}$

$\Rightarrow A = x\,.\,x = x^2$

$\Rightarrow \frac{dA}{dx} = 2x$

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 2x(5) = 10x$

When $x = 10$ cm $\Rightarrow \frac{dA}{dt} = 10(10) = 100$ cm^2/sec.

7. Given $\frac{dp}{dt} = 2$, find $\frac{dM}{dt}$

$M = (2p + 3)^4 \Rightarrow \frac{dM}{dp} = 4(2p + 3)^3\,.\,2 = 8(2p + 3)^3$

$\frac{dM}{dt} = \frac{dM}{dp} \cdot \frac{dp}{dt} = 8(2p + 3)^3\,.\,2 = 16(2p + 3)^3$

When $p = 1 \Rightarrow \frac{dM}{dt} = 16[2(1) + 3]^3$

$= 16(125) = 2000$

8. Given $\frac{dV}{dt} = 6$ cm^3/sec, find $\frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\Rightarrow \quad 6 = 4\pi r^2 \cdot \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{6}{4\pi r^2} = \frac{3}{2\pi r^2}$

When $r = 3 \quad \Rightarrow \frac{dr}{dt} = \frac{3}{2\pi(3)^2} = \frac{3}{18\pi} = \frac{1}{6\pi}$ cm/sec

9. Given $\frac{dV}{dt} = 24\pi$ cm^3/sec, find $\frac{dr}{dt}$

$\Rightarrow \quad V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\Rightarrow 24\pi = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{24\pi}{4\pi r^2} = \frac{6}{r^2}$$

At $r = 6 \Rightarrow \frac{dr}{dt} = \frac{6}{(6)^2} = \frac{1}{6}$ cm/sec

10. Rectangle: length $= x$ cm and width $= w$ cm

Perimeter $= 40$ cm $\Rightarrow 2x + 2w = 40$

$\Rightarrow \quad x + w = 20$

$\Rightarrow \quad w = 20 - x$

(i) Area of rectangle $= x\,.\,w$

$\Rightarrow A = x(20 - x) = (20x - x^2)$ cm^2

(ii) $\frac{dA}{dx} = 20 - 2x$

Hence, $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = (20 - 2x)\,.\,(0.5) = 10 - x$

When $x = 3 \Rightarrow \frac{dA}{dt} = 10 - 3 = 7$ cm^2/sec

11. Given $\frac{dx}{dt} = 10\sqrt{2}$, find $\frac{dy}{dt}$

$$y = x - \frac{x^2}{40} = x - \frac{1}{40} \cdot x^2$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{40} \cdot 2x = 1 - \frac{1}{20}x$$

Hence, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$= \left(1 - \frac{1}{20}x\right)10\sqrt{2}$$

When $x = 10 \Rightarrow \frac{dy}{dt} = \left[1 - \frac{1}{20}(10)\right] \cdot 10\sqrt{2}$

$$= \left(1 - \frac{1}{2}\right) \cdot 10\sqrt{2}$$

$$= \frac{1}{2} \cdot 10\sqrt{2} = 5\sqrt{2}$$

12. (i) Given $\frac{dr}{dt} = 1$ cm/sec, find $\frac{dV}{dt}$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

Hence, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2\,.\,1 = 4\pi r^2$

When $r = 2$ m $= 200$ cm $\Rightarrow \frac{dV}{dt} = 4\pi(200)^2 = 160\,000\pi$ cm^3/sec

(ii) $\frac{dV}{dt} = 160\,000\pi$ cm^3/sec

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$$

$$= 160\,000\pi \times \frac{1}{4\pi r^2}$$

$$= 160\,000\pi \times \frac{1}{4\pi(500)^2}$$

$$= \frac{4}{25} = 0.16 \text{ cm/sec}$$

$$V = \frac{4}{3}\pi r^2$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

(iii) Area $(A) = 4\pi r^2$ $\qquad$ $A = 4\pi r^2$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \qquad \Rightarrow \frac{dA}{dr} = 8\pi r$$

$$= 8\pi r(0.16) \ldots \text{from (ii) above}$$

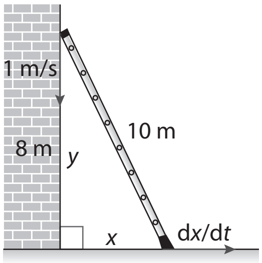
$$= 8\pi \,.\, 500(0.16)$$

$$= 640\pi \text{ cm}^2/\text{sec}$$

13. Ground to the top of the ladder $= (8 - y)$ m

Bottom of the ladder to the wall $= (6 + x)$ m

Right-angled triangle: $(8 - y)^2 + (6 + x)^2 = (10)^2$



$$\Rightarrow (6 + x)^2 = 100 - (8 - y)^2$$

$$\Rightarrow 6 + x = \sqrt{100 - (8 - y)^2}$$

$$\Rightarrow x = \left(100 - (8 - y)^2\right)^{\frac{1}{2}} - 6$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}[100 - (8 - y)^2]^{\frac{1}{2}} \cdot -2(8 - y) \cdot -1$$

$$\Rightarrow \frac{dx}{dy} = \frac{8 - y}{\sqrt{100 - (8 - y)^2}}$$

Given $\frac{dy}{dt} = 1 \Rightarrow \frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$

$$= \frac{8 - y}{\sqrt{100 - (8 - y)^2}} \cdot 1$$

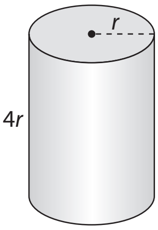
Since $8 - y = 8 \Rightarrow y = 0$

hence, $\frac{dx}{dt} = \frac{8 - 0}{\sqrt{100 - (8 - 0)^2}} = \frac{8}{\sqrt{100 - 64}}$

$$= \frac{8}{\sqrt{36}} = \frac{8}{6} = 1\frac{1}{3} \text{ m/sec}$$

14. Cylinder: Radius $= r$, height $= 4r$

Volume $= \pi r^2 h = \pi r^2(4r) = 4\pi r^3$



Given $\frac{dr}{dt} = 0.5$ cm/sec, find $\frac{dV}{dt}$

$$\Rightarrow \frac{dV}{dt} = 4\pi \,.\, 3r^2 = 12\pi r^2$$

Hence, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = (12\pi r^2) \,.\, (0.5) = 6\pi r^2$

When $r = 6$ cm $\Rightarrow \frac{dV}{dt} = 6\pi(6)^2 = 216\pi \text{ cm}^3/\text{sec}$

15. Circumference $C = 2\pi r$

$$\Rightarrow \frac{dC}{dr} = 2\pi$$

Area $= \pi r^2$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

hence, $\frac{dC}{dA} = \frac{dC}{dr} \cdot \frac{dr}{dA} = 2\pi \cdot \frac{1}{2\pi r} = \frac{1}{r}$

Given $\frac{dA}{dt} = 2 \text{ cm}^2/\text{sec}$, find $\frac{dC}{dt}$

Hence, $\frac{dC}{dt} = \frac{dC}{dA} \cdot \frac{dA}{dt}$

$$= \frac{1}{r} \cdot 2 = \frac{2}{r}$$

When $r = 3$ cm $\Rightarrow \frac{dC}{dt} = \frac{2}{3}$ cm/sec

Revision Exercise 4 (Core)

1. $y = x^2 - \frac{9}{x} = x^2 - 9x^{-1}$

$\Rightarrow \frac{dy}{dx} = 2x + 9x^{-2} = 2x + \frac{9}{x^2}$

When $x = 3 \Rightarrow \frac{dy}{dx} = 2(3) + \frac{9}{(3)^2} = 6 + 1 = 7$

When $x = 3 \Rightarrow y = (3)^2 - \frac{9}{3} = 6 \Rightarrow$ point $(3, 6)$

$\Rightarrow$ Equation of Tangent: $y - 6 = 7(x - 3)$

$\Rightarrow y - 6 = 7x - 21$

$\Rightarrow 7x - y - 15 = 0$

2. $y = x^3 - 12x + 5$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 12 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0$

$\Rightarrow x = -2$ OR $x = 2$

At $x = -2 \Rightarrow y = (-2)^3 - 12(-2) + 5$

$= -8 + 24 + 5 = 21 \Rightarrow$ point $(-2, 21)$

At $x = 2 \Rightarrow y = (2)^3 - 12(2) + 5$

$= 8 - 24 + 5 = -11 \Rightarrow$ point $(2, -11)$

$\frac{d^2y}{dx^2} = 6x$

At $(-2, 21) \Rightarrow \frac{d^2y}{dx^2} = 6(-2) = -12 < 0 \Rightarrow$ maximum

At $(2, -11) \Rightarrow \frac{d^2y}{dx^2} = 6(2) = 12 > 0 \Rightarrow$ minimum

3. $f(x) = x^3 - bx^2 - 9x + 7$

$\Rightarrow f'(x) = 3x^2 - 2bx - 9$

At $x = -1 \Rightarrow f'(-1) = 3(-1)^2 - 2b(-1) - 9 = 0$

$\Rightarrow 3 + 2b - 9 = 0$

$\Rightarrow 2b = 6$

$\Rightarrow b = 3$

4. $f(x) = x^3 + 3x^2 - 9x$

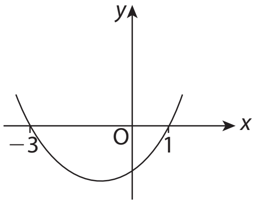
$\Rightarrow f'(x) = 3x^2 + 6x - 9 < 0$

$\Rightarrow x^2 + 2x - 3 < 0$

Factors $\Rightarrow (x + 3)(x - 1) = 0$

Roots $\Rightarrow x = -3, x = 1$

Solution: $-3 < x < 1$



5. $y = 6x^2 - x^3$

(i) $\frac{dy}{dx} = 12x - 3x^2 = 12$

$\Rightarrow 3x^2 - 12x + 12 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x - 2)(x - 2) = 0$

$\Rightarrow x = 2$

(ii) $x = 2 \Rightarrow y = 6(2)^2 - (2)^3$

$= 24 - 8 = 16 \Rightarrow$ point $(2, 16)$

Equation of Tangent: $y - 16 = 12(x - 2)$

$\Rightarrow y - 16 = 12x - 24$

$\Rightarrow 12x - y - 8 = 0$

6. $s = 2t^3 - 24t$

(i) $\Rightarrow \dfrac{ds}{dt} = 6t^2 - 24$

$$\text{At} \quad t = 4 \Rightarrow \text{speed} = \frac{ds}{dt} = 6(4)^2 - 24$$
$$= 96 - 24 = 72 \text{ m/sec}$$

(ii) Particle at rest $\Rightarrow \dfrac{ds}{dt} = 0$

$\Rightarrow 6t^2 - 24 = 0$

$\Rightarrow t^2 - 4 = 0$

$\Rightarrow (t + 2)(t - 2) = 0$

$\Rightarrow t = -2$ (invalid) OR $t = 2$ (valid)

$\Rightarrow$ Ans $= 2$ secs

7. $y = x \sin 2x$

Product rule: $u = x$ and $v = \sin 2x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = \cos 2x \,.\, 2 = 2\cos 2x$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x \,.\, 2\cos 2x + \sin 2x \,.\, 1$$
$$= 2x\cos 2x + \sin 2x$$

$$\text{At } x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = 2\left(\frac{\pi}{3}\right) \cdot \cos 2\left(\frac{\pi}{3}\right) + \sin 2\left(\frac{\pi}{3}\right)$$
$$= \frac{2\pi}{3}\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}$$
$$= \frac{2\pi}{3} \cdot \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{2} - \frac{\pi}{3}$$

8. $x + y = 100$

$\Rightarrow y = 100 - x$

$\Rightarrow P = x \,.\, y = x(100 - x) = 100x - x^2$

$\Rightarrow \dfrac{dP}{dx} = 100 - 2x = 0$

$\Rightarrow \qquad 2x = 100$

$\Rightarrow x = 50 \Rightarrow y = 100 - 50 = 50$

$\Rightarrow \dfrac{d^2P}{dx^2} = -2 < 0 \Rightarrow$ maximum

Hence, $P = (50)(50) = 2500$

9. $s(x) = -x^3 + 3x^2 + 360x + 5000$

$\Rightarrow s'(x) = -3x^2 + 6x + 360 = 0$

$\Rightarrow x^2 - 2x - 120 = 0$

$\Rightarrow (x + 10)(x - 12) = 0$

$\Rightarrow x = -10$ OR $x = 12$

$\Rightarrow s''(x) = -6x + 6$

At $x = 12 \Rightarrow s''(12) = -6(12) + 6 = -66 < 0 \Rightarrow$ maximum

Since $6 \leqslant x \leqslant 20 \Rightarrow x = 12°$ C

10. Given $\frac{dV}{dt} = 12$ cm^3/sec, find $\frac{dx}{dt}$

Volume $(V) = x \,.\, x \,.\, x = x^3$ Volume $= 125$

$\Rightarrow \frac{dV}{dx} = 3x^2$ $\Rightarrow x^3 = 125$

$\Rightarrow x = \sqrt[3]{125} = 5$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow 12 = 3x^2 \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{12}{3x^2} = \frac{4}{x^2}$$

When $x = 5 \Rightarrow \frac{dx}{dt} = \frac{4}{(5)^2} = \frac{4}{25}$ cm/sec

11. $y = x + \frac{4}{x} = x + 4x^{-1}$

$$\Rightarrow \frac{dy}{dx} = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = 0$$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0$

$\Rightarrow x = -2$ OR $x = 2$

When $x = -2 \Rightarrow y = -2 + \frac{4}{-2} = -2 - 2 = -4 \Rightarrow$ point $(-2, -4)$

When $x = 2 \Rightarrow y = 2 + \frac{4}{2} = 2 + 2 = 4 \Rightarrow$ point $(2, 4)$

Hence, $\frac{d^2y}{dx^2} = 0 + 8x^{-3} = \frac{8}{x^3}$

At $(-2, -4) \Rightarrow \frac{d^2y}{dx^2} = \frac{8}{(-2)^3} = \frac{8}{-8} = -1 < 0 \Rightarrow$ maximum

At $(2, 4) \Rightarrow \frac{d^2y}{dx^2} = \frac{8}{(2)^3} = \frac{8}{8} = 1 > 0 \Rightarrow$ minimum

For $x > 0 \Rightarrow$ curve is increasing after the minimum point $(2, 4)$; hence $x > 2$.

12. (i) $f(x) = ax^2 + bx + c$

$\Rightarrow f'(x) = 2ax + b$

$\Rightarrow f''(x) = 2a$

$f''(x) = 6 \Rightarrow 2a = 6 \Rightarrow a = 3$

Gradient at $(2, 24)$ is $22 \Rightarrow f'(2) = 2a(2) + b = 22$

$= 4a + b = 22$

$a = 3$ $\Rightarrow 4(3) + b = 22$

$\Rightarrow 12 + b = 22$

$\Rightarrow b = 10$

Point $(2, 24) \Rightarrow f(2) = 3(2)^2 + 10(2) + c = 24$

$\Rightarrow 12 + 20 + c = 24$

$\Rightarrow c = 24 - 32 = -8$

(ii) $f(x) = 3x^2 + 10x - 8$

On x-axis $\Rightarrow f(x) = 0 \Rightarrow 3x^2 + 10x - 8 = 0$

$\Rightarrow (x + 4)(3x - 2) = 0$

$\Rightarrow x = -4, x = \frac{2}{3}$

$\Rightarrow$ Points on x-axis $= (-4, 0), \left(\frac{2}{3}, 0\right)$

On y-axis $\Rightarrow x = 0 \Rightarrow f(0) = 3(0)^2 + 10(0) - 8 = -8$

$\Rightarrow$ Point on y-axis $= (0, -8)$

(iii) Turning point $\Rightarrow f'(x) = 6x + 10 = 0$

$$\Rightarrow 6x = -10$$

$$\Rightarrow x = \frac{-10}{6} = \frac{-5}{3} = -1\frac{2}{3}$$

$$\Rightarrow f\left(-1\frac{2}{3}\right) = 3\left(-1\frac{2}{3}\right)^2 + 10\left(-1\frac{2}{3}\right) - 8$$

$$= \frac{25}{3} - \frac{50}{3} - 8 = -16\frac{1}{3} \quad \Rightarrow \quad \text{Point}\left(-1\frac{2}{3}, -16\frac{1}{3}\right)$$

$f''(x) = 6 > 0 \Rightarrow$ minimum point $= \left(-1\frac{2}{3}, -16\frac{1}{3}\right)$

13. Given $\frac{dV}{dt} = 10\pi\ \text{cm}^3/\text{min}$, find $\frac{dr}{dt}$

Volume $(V) = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

Hence, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\Rightarrow \quad 10\pi = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{10\pi}{4\pi r^2} = \frac{5}{2r^2}$$

At $r = 5 \Rightarrow \frac{dr}{dt} = \frac{5}{2(5)^2} = \frac{5}{50} = \frac{1}{10}$ cm/sec

14. $s = 196t - 4.9t^2$

$\frac{ds}{dt} = 196 - 9.8t = 0$

$\Rightarrow 9.8t = 196$

$\Rightarrow t = \frac{196}{9.8} = 20$ secs

$\frac{d^2s}{dt^2} = 0 - 9.8 = -9.8 < 0 \Rightarrow$ maximum at $t = 20$

$\Rightarrow s = 196(20) - 4.9(20)^2$

$= 3920 - 1960 = 1960$ m

15. Answer = Graph Ⓒ

- $f(x)$ has no turning points $\Rightarrow$ Graph Ⓒ is above x-axis for all values of x
- $f(x)$ is strictly increasing $\Rightarrow$ Graph Ⓒ is positive for all values of x
- $f(x)$ has a point of inflection at the turning point of graph Ⓒ

Revision Exercise 4 (Advanced)

1. $f(x) = \sqrt{x^2 - 3} = (x^2 - 3)^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2}(x^2 - 3)^{-\frac{1}{2}} . 2x = \frac{x}{\sqrt{x^2 - 3}}$$

Point $(2, 1) \Rightarrow f'(2) = \frac{2}{\sqrt{(2)^2 - 3}} = \frac{2}{\sqrt{1}} = \frac{2}{1} = 2$ (slope)

Equation of Tangent: $y - 1 = 2(x - 2)$

$$\Rightarrow \quad y - 1 = 2x - 4$$

$$\Rightarrow 2x - y - 3 = 0$$

2. $y = x^2 + \ln x$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} = 3$$

$$\Rightarrow 2x^2 + 1 = 3x$$

$$\Rightarrow 2x^2 - 3x + 1 = 0$$

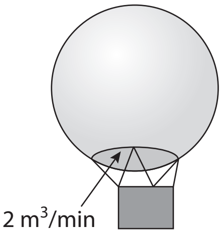
$$\Rightarrow (2x - 1)(x - 1) = 0$$

$$\Rightarrow x = \frac{1}{2}, \quad x = 1$$

$$x = \frac{1}{2} \Rightarrow y = \left(\frac{1}{2}\right)^2 + \ln\left(\frac{1}{2}\right) = \frac{1}{4} - \ln 2 \Rightarrow \text{point}\left(\frac{1}{2}, \frac{1}{4} - \ln 2\right)$$

$$x = 1 \Rightarrow y = (1)^2 + \ln(1) = 1 + 0 = 1 \Rightarrow \text{point } (1, 1)$$

3. (i) Given $\frac{dV}{dt} = 2\text{ m}^3/\text{min}$, find $\frac{dr}{dt}$



Volume $(V) = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \Rightarrow 2 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}$$

At $r = 2.5\text{ m} \Rightarrow \frac{dr}{dt} = \frac{1}{2\pi(2.5)^2} = \frac{2}{25\pi}$ m/min

(ii) Given $\frac{dr}{dt} = \frac{2}{25\pi}$, find $\frac{dA}{dt}$

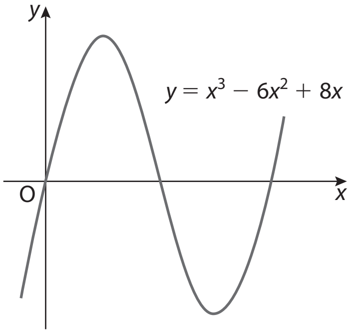
Area $(A) = 4\pi r^2$

$$\Rightarrow \quad \frac{dA}{dt} = 8\pi r$$

Hence, $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{2}{25\pi} = \frac{16}{25}r$

At $r = 2.5 \Rightarrow \frac{dA}{dt} = \frac{16}{25}(2.5) = 1.6\text{ m}^2/\text{min}$

4. (a) $y = x^3 - 6x^2 + 8x$



$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 8 = -1$$

$$\Rightarrow \quad 3x^2 - 12x + 9 = 0$$

$$\Rightarrow \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 1)(x - 3) = 0$$

$$\Rightarrow \quad x = 1 \quad \text{OR} \quad x = 3$$

$$\Rightarrow y = (1)^3 - 6(1)^2 + 8(1) = 3 \ \text{OR}\ y = (3)^3 - 6(3)^2 + 8(3) = -3$$

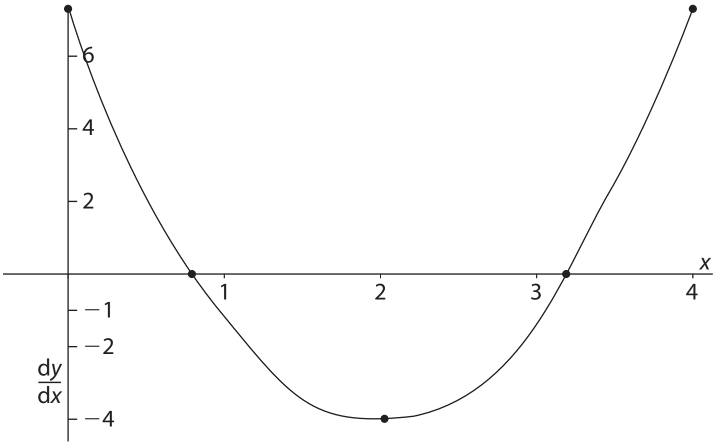
points (1, 3) OR (3, −3)

(b) Line $y = 4 - x$

$$\Rightarrow \frac{dy}{dx} = 0 - 1 = -1 \Rightarrow A = (1, 3) \text{ First Quadrant}$$

(c) Graph of $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 3x^2 - 12 + 8 = 0$$
$$\Rightarrow x = 0.85, 3.15$$
$$\frac{d^2y}{dx^2} = 6x - 12 = 0$$
$$\Rightarrow x = 2$$
$$\Rightarrow \frac{dy}{dx} = 3(2)^2 - 12(2) + 8$$
$$= -4$$

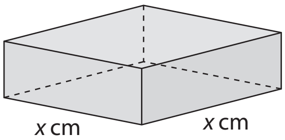


In the graph of $\frac{dy}{dx}$, there is a turning point at (2, −4).

This shows that there is a point of inflection at $x = 2$ in the graph of $y = x^3 - 6x^2 + 8x$.

5. (i) Volume $= x\,.\,x\,.\,h = 500$ cm^3

$$\Rightarrow V = x^2h = 500$$
$$\Rightarrow \quad h = \frac{500}{x^2}$$



$$\text{Area } (A) = x\,.\,x + 4x\,.\,h$$
$$= x^2 + 4xh$$
$$= x^2 + 4x\left(\frac{500}{x^2}\right)$$
$$\Rightarrow \quad A = x^2 + \frac{2000}{x}$$

(ii) $A = x^2 + 2000x^{-1}$

$$\Rightarrow \frac{dA}{dx} = 2x - 2000x^{-2} = 0$$
$$\Rightarrow 2x - \frac{2000}{x^2} = 0$$
$$\Rightarrow 2x^3 - 2000 = 0$$
$$\Rightarrow x^3 = 1000$$
$$\Rightarrow x = \sqrt[3]{1000} = 10 \text{ cm} \Rightarrow h = \frac{500}{(10)^2} = 50 \text{ cm}$$
$$\frac{d^2A}{dx^2} = 2 + 4000x^{-3} = 2 + \frac{4000}{x^3}$$
$$\text{At } x = 10 \Rightarrow \frac{d^2A}{dx^2} = 2 + \frac{4000}{(10)^3} = 6 > 0 \Rightarrow \text{minimum}$$
$$\Rightarrow \text{Area} = (10)^2 + 4(10)(5) = 300 \text{ cm}^2$$

6. (i) $C = \frac{16}{t^3} + \frac{3t^2}{4} = 16t^{-3} + \frac{3}{4}t^2$

$$\Rightarrow \frac{dC}{dt} = -48t^{-4} + \frac{3}{4}(2t) = \frac{-48}{t^4} + \frac{3}{2}t$$
$$\text{When } t = 4 \text{ hours} \Rightarrow \frac{dC}{dt} = \frac{-48}{(4)^4} + \frac{3}{2}(4)$$
$$= \frac{-3}{16} + 6 = €5\frac{13}{16} \text{ per hour}$$

(ii) Minimum $\Rightarrow \dfrac{dC}{dt} = 0$

$$\Rightarrow \frac{-48}{t^4} + \frac{3}{2}t = 0$$

$$\Rightarrow -96 + 3t^5 = 0$$

$$\Rightarrow 3t^5 = 96$$

$$\Rightarrow t^5 = 32$$

$$\Rightarrow t = \sqrt[5]{32} = 2$$

$$\Rightarrow \frac{d^2C}{dt^2} = 192t^{-5} + \frac{3}{2} = \frac{192}{t^5} + \frac{3}{2}$$

At $t = 2 \Rightarrow \dfrac{d^2C}{dt^2} = \dfrac{192}{(2)^5} + 1\dfrac{1}{2} = 7\dfrac{1}{2} > 0 \quad \Rightarrow$ Minimum

Hence, $C = \dfrac{16}{(2)^3} + \dfrac{3(2)^2}{4} = 2 + 3 =$ €5

7. $y = x \,.\, e^x$

Product Rule: $u = x$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x\,.\,e^x + e^x\,.\,1$$

$$= e^x(x + 1)$$

Turning point $\Rightarrow \dfrac{dy}{dx} = 0$

$$\Rightarrow e^x(x + 1) = 0$$

$$\Rightarrow x = -1$$

$$\Rightarrow y = (-1)e^{-1} = \frac{-1}{e} \Rightarrow \text{Point} \left(-1, \frac{-1}{e}\right)$$

$$\frac{dy}{dx} = e^x(x + 1)$$

Product Rule: $u = e^x$ and $v = x + 1$

$$\Rightarrow \frac{du}{dx} = e^x \qquad \Rightarrow \frac{dv}{dx} = 1$$

Hence, $\dfrac{d^2y}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = e^x\,.\,(1) + (x + 1)\,.\,e^x$

$$= e^x\,(x + 2)$$

At $x = -1 \Rightarrow \dfrac{d^2y}{dx^2} = e^{-1}(-1 + 2) = \dfrac{1}{e} > 0 \;\Rightarrow$ Minimum

8. (A, 2), (B, 4), (C, 1), (D, 3)

9. $y = x^3 + ax^2 + bx + c$

$$\frac{dy}{dx} = 3x^2 + 2ax + b$$

At $x = -1 \Rightarrow \dfrac{dy}{dx} = 3(-1)^2 + 2a(-1) + b = 0$

$$\Rightarrow 3 - 2a + b = 0$$

$$\Rightarrow -2a + b = -3$$

At $x = 3 \quad \Rightarrow \dfrac{dy}{dx} = 3(3)^2 + 2a(3) + b = 0$

$$\begin{aligned}
&\Rightarrow 27 + 6a + b = 0\\
&\Rightarrow \quad 6a + b = -27\\
&\Rightarrow \quad \underline{2a - b = 3}\\
&\quad \Rightarrow \quad 8a \quad = -24\\
&\quad \Rightarrow \quad a \quad = -3\\
&\Rightarrow -2(-3) + b = -3\\
&\Rightarrow \quad 6 + b = -3\\
&\Rightarrow \quad b = -9
\end{aligned}$$

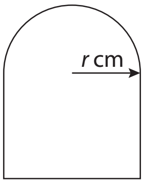
Point $(1, 1) \Rightarrow 1 = (1)^3 - 3(1)^2 - 9(1) + c$

$\Rightarrow 1 = 1 - 3 - 9 + c$

$\Rightarrow 12 = c$

$\Rightarrow a = -3, b = -9, c = 12$

10. Rectangular base $\Rightarrow$ length $= 2r$ and width $= w$



$\Rightarrow$ Perimeter $= \pi r + 2r + 2w = 40$

$$\Rightarrow \quad 2w = 40 - \pi r - 2r$$

$$\Rightarrow \quad w = \frac{1}{2}(40 - \pi r - 2r)$$

Hence, Area $= \dfrac{\pi r^2}{2} + 2r \,.\, w$

$$= \frac{\pi r^2}{2} + 2r \cdot \frac{1}{2}(40 - \pi r - 2r)$$

$$= \frac{\pi r^2}{2} + 40r - \pi r^2 - 2r^2$$

$$\Rightarrow \quad A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

$$\frac{dA}{dr} = 40 - 4r - \frac{\pi}{2}(2r)$$

$$\begin{aligned}
&\Rightarrow 40 - 4r - \pi r = 0\\
&\Rightarrow \quad 4r + \pi r = 40\\
&\Rightarrow \quad r(4 + \pi) = 40\\
&\Rightarrow \quad r = \frac{40}{4 + \pi}
\end{aligned}$$

$$\frac{d^2A}{dr^2} = 0 - 4 - \pi = -4 - \pi < 0 \quad \Rightarrow \text{Maximum}$$

Hence, Area $= 40\left(\dfrac{40}{4 + \pi}\right) - 2\left(\dfrac{40}{4 + \pi}\right)^2 - \dfrac{\pi}{2}\left(\dfrac{40}{4 + \pi}\right)^2$

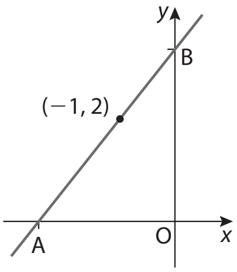
$$= \frac{1600}{4 + \pi} - \frac{3200}{(4 + \pi)^2} - \frac{800\pi}{(4 + \pi)^2}$$

$$= \frac{1600(4 + \pi) - 3200 - 800\pi}{(4 + \pi)^2}$$

$$= \frac{6400 + 1600\pi - 3200 - 800\pi}{(4 + \pi)^2}$$

$$= \frac{3200 + 800\pi}{(4 + \pi)^2} = \frac{800(4 + \pi)}{(4 + \pi)^2} = \frac{800}{4 + \pi}$$

11. (i) Point $(-1, 2)$ Slope $= m$



$\Rightarrow$ Equation of line: $y - 2 = m(x + 1)$

$$\Rightarrow y - 2 = mx + m$$

$$\Rightarrow y \quad = mx + m + 2$$

x-axis $\Rightarrow y = 0 \quad \Rightarrow mx + m + 2 = 0$

$$\Rightarrow mx = -m - 2$$

$$\Rightarrow \quad x = \frac{-m - 2}{m} \Rightarrow A = \left(\frac{-m - 2}{m}, 0\right)$$

y-axis $\Rightarrow x = 0 \Rightarrow \quad y = m(0) + m + 2$

$= m + 2 \Rightarrow B = (0, m + 2)$

$$\text{Hence, Area } = \frac{1}{2}\left|\left(\frac{-m-2}{m}\right)(m+2) - (0)(0)\right|$$

$$\Rightarrow A = \frac{(m+2)^2}{2m}$$

(ii) Quotient Rule: $u = (m + 2)^2$ and $v = 2m$

$$\Rightarrow \frac{du}{dm} = 2(m+2) \qquad \Rightarrow \frac{dv}{dm} = 2$$

$$\text{Hence, } \frac{dA}{dm} = \frac{u\frac{dv}{dm} - v\frac{du}{dm}}{v^2}$$

$$= \frac{(m+2)^2 \,.\, 2 - 2m \,.\, 2(m+2)}{(2m)^2} = 0$$

$= 2m^2 + 8m + 8 - 4m^2 - 8m = 0$

$\Rightarrow -2m^2 + 8 = 0$

$\Rightarrow m^2 - 4 = 0$

$\Rightarrow (m + 2)(m - 2) = 0$

$\Rightarrow m = -2$ (not valid) OR $m = 2$ (valid)

$$\text{When } m = 2 \Rightarrow A = \frac{(2+2)^2}{2(2)} = \frac{16}{4} = 4 \text{ sq. units}$$

12. Answer D

13. $f(x) = x^3 + 3kx^2 + 32$

$\Rightarrow f'(x) = 3x^2 + 6kx = 0$

$\Rightarrow 3x(x + 2k) = 0$

$\Rightarrow x = 0, x = -2k$

When $x = 0 \Rightarrow f(0) = (0)^3 + 3k(0)^2 + 32 = 32 \Rightarrow$ point $= (0, 32)$

When $x = -2k \Rightarrow f(-2k) = (-2k)^3 + 3k(-2k)^2 + 32$

$= -8k^3 + 12k^3 + 32 = 4k^3 + 32$

$\Rightarrow$ point $= (-2k, 4k^3 + 32)$

Two roots are equal $\Rightarrow$ Point $(-2k, 4k^3 + 32)$ lies on the x-axis $\Rightarrow 4k^3 + 32 = 0$

$\Rightarrow 4k^3 = -32$

$\Rightarrow k^3 = -8$

$\Rightarrow k = \sqrt[3]{-8} = -2$

14. Length of pipe $= 4 - x + \sqrt{9 + x^2}$

Cost $= 10(4 - x) + 25\sqrt{9 + x^2}$

$$\Rightarrow C = 40 - 10x + 25(9 + x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dC}{dx} = -10 + 25 \cdot \frac{1}{2}(9 + x^2)^{\frac{-1}{2}} \,.\, 2x = 25x\,(9 + x^2)^{\frac{-1}{2}} - 10$$

$$= -10 + \frac{25x}{\sqrt{9 + x^2}} = 0$$

$$\Rightarrow \frac{25x}{\sqrt{9 + x^2}} = 10$$

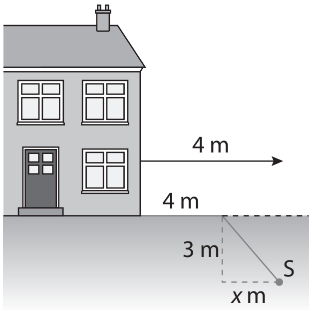
$\Rightarrow 25x = 10\sqrt{9 + x^2}$

$\Rightarrow 5x = 2\sqrt{9 + x^2}$

$\Rightarrow 25x^2 = 4(9 + x^2)$

$\Rightarrow 25x^2 = 36 + 4x^2$

$\Rightarrow 21x^2 = 36$



$$\Rightarrow x^2 = \frac{36}{21} = 1.7143$$

$$\Rightarrow x = \sqrt{1.7143} = 1.309 = 1.3$$

$$\frac{d^2C}{dx^2} = 25x \cdot \frac{-1}{2}(9 + x^2)^{\frac{-3}{2}} . 2x + (9 + x^2)^{\frac{-1}{2}} . 25$$

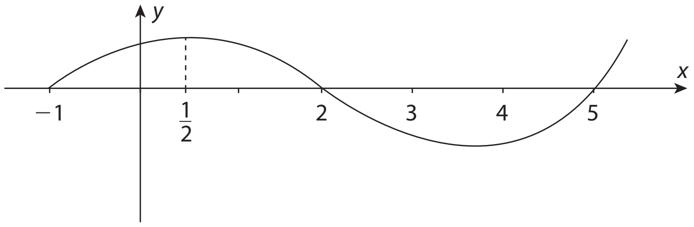
$$= \frac{-25x^2}{(9 + x^2)^{\frac{3}{2}}} + \frac{25}{\sqrt{9 + x^2}}$$

At $x = 1.3 \Rightarrow \frac{d^2C}{dx^2} = \frac{-25(1.3)^2}{[9 + (1.3)^2]^{\frac{3}{2}}} + \frac{25}{\sqrt{9 + (1.3)^2}} = 6.4 > 0$

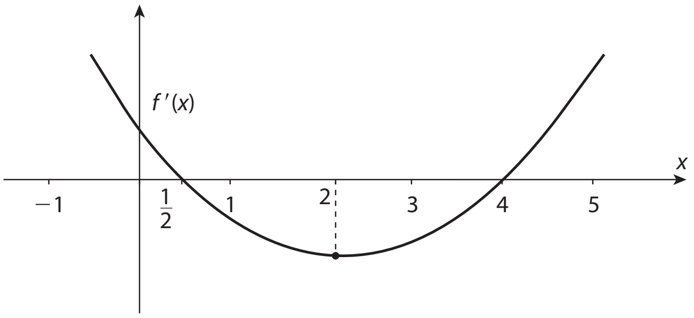
$\Rightarrow$ Minimum cost

Hence, length $= 4 - 1.3 = 2.7$ m

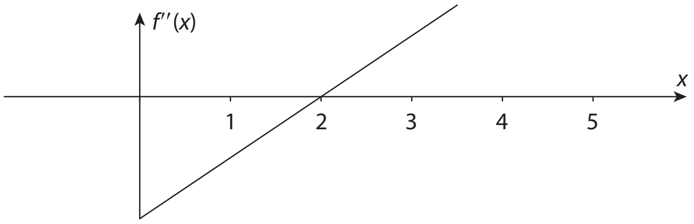
15. (i) Graph of $f(x)$



Graph of $f'(x)$



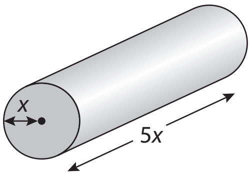
(ii) Graph of $f''(x)$



Point of inflection occurs where $x = 2$

Revision Exercise 4 (Extended-Response Questions)

1. (a) Given $\frac{dA}{dt} = 0.032$ cm²/sec, find $\frac{dx}{dt}$



Area (A) of cross-section $= \pi x^2$

$$\Rightarrow \frac{dA}{dx} = 2\pi x$$

Hence, $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$

$$\Rightarrow 0.032 = 2\pi x \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{0.032}{2\pi x}$$

When $x = 2$ cm $\Rightarrow \frac{dx}{dt} = \frac{0.032}{2\pi(2)} = \frac{0.032}{4\pi}$

$$= 0.00254 = 0.003 \text{ cm/sec}$$

(b) Volume $(V) = \pi x^2 . 5x = 5\pi x^3$

$\Rightarrow \dfrac{dV}{dx} = 15\pi x^2$

$\dfrac{dV}{dt} = \dfrac{dV}{dx} \cdot \dfrac{dx}{dt} = 15\pi x^2 . (0.003)$

$= 0.045\pi x^2$

When $x = 2$ cm $\Rightarrow \dfrac{dV}{dt} = 0.045\pi . (2)^2$

$= 0.56548 = 0.5655$ cm^3/sec

2. (a) $8x + 8x + 4h = 20$

$\Rightarrow 16x + 4h = 20$

$\Rightarrow 4x + h = 5 \quad \Rightarrow h = 5 - 4x$

(b) Volume $(V) = (x) . (3x) . (h)$

$\Rightarrow V = 3x^2 (5 - 4x) = 15x^2 - 12x^3$

(c) $V = 0 \Rightarrow 15x^2 - 12x^3 = 0$

$\Rightarrow 5x^2 - 4x^3 = 0$

$\Rightarrow x^2(5 - 4x) = 0$

$\Rightarrow x = 0, x = \dfrac{5}{4}$

Hence, domain is $0 < x < \dfrac{5}{4}$

(d) $\dfrac{dV}{dx} = 30x - 36x^2$

(e) $\dfrac{dV}{dx} = 0 \quad \Rightarrow 30x - 36x^2 = 0$

$\Rightarrow 5x - 6x^2 = 0$

$\Rightarrow x(5 - 6x) = 0$

$\Rightarrow x = 0 \quad \text{OR} \quad x = \dfrac{5}{6}$

Hence, $\dfrac{d^2V}{dx^2} = 30 - 72x$

At $x = 0 \Rightarrow \dfrac{d^2V}{dx^2} = 30 - 72(0) = 30 > 0 \Rightarrow$ minimum

At $x = \dfrac{5}{6} \Rightarrow \dfrac{d^2V}{dx^2} = 30 - 72\left(\dfrac{5}{6}\right) = -30 < 0 \Rightarrow$ maximum

At $x = \dfrac{5}{6} \Rightarrow V = 15\left(\dfrac{5}{6}\right)^2 - 12\left(\dfrac{5}{6}\right)^3 = 3\dfrac{17}{36}$ cm^3

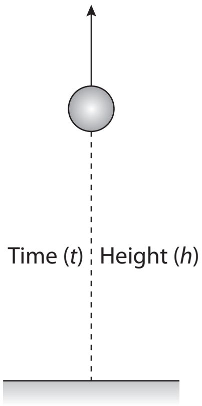
3. (i) $h = 2 + 40t - 5t^2 \Rightarrow \dfrac{dh}{dt} = 40 - 10t$

(a) When $t = 2 \Rightarrow \dfrac{dh}{dt} = 40 - 10(2)$

$= 40 - 20 = 20$ m/sec

(b) When $t = 2.5 \Rightarrow \dfrac{dh}{dt} = 40 - 10(2.5)$

$= 40 - 25 = 15$ m/sec



(ii) $\dfrac{dh}{dt} = 0 \Rightarrow 40 - 10t = 0$

$\Rightarrow 10t = 40 \quad \Rightarrow t = 4$ secs

(iii) When $t = 4 \Rightarrow h = 2 + 40(4) - 5(4)^2$

$= 162 - 80 = 82$ m

(iv) When $t = 6 \Rightarrow \dfrac{dh}{dt} = 40 - 10(6) = -20$

$\Rightarrow$ Ball is falling towards the ground at 20 m/sec after 6 seconds

(v) When $t = 0 \Rightarrow \frac{dh}{dt} = 40 - 10(0) = 40$ m/sec

(vi) When the ball hits the ground, then $h = 0$

$\Rightarrow 2 + 40t - 5t^2 = 0$

$$\Rightarrow t = \frac{-40 \pm \sqrt{(40)^2 - 4(2)(-5)}}{2(-5)}$$

$$= \frac{-40 \pm \sqrt{1640}}{-10}$$

$$= \frac{-40 - 40.497}{-10}, \frac{-40 + 40.497}{-10} \text{ (not valid)}$$

$$= 8.0497 = 8.05$$

$$\Rightarrow \frac{dh}{dt} = 40 - 10(8.05) = 40 - 80.5 = -40.5$$

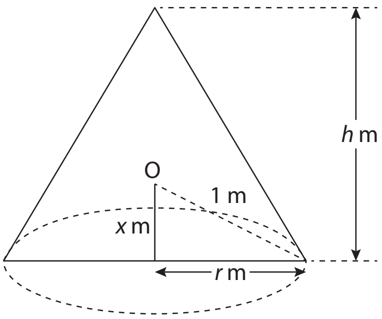
Hence, speed of ball is 40.5 m/sec as it hits the ground after 8.05 secs.

4. (a) (i) $x^2 + r^2 = 1^2$

$\Rightarrow r^2 = 1 - x^2$

$\Rightarrow r = \sqrt{1 - x^2}$

(ii) $h = 1 + x$



(b) $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\sqrt{1 - x^2}\right)^2 . (1 + x)$$

$$= \frac{1}{3}\pi(1 - x^2)(1 + x) = \frac{\pi}{3}(1 + x - x^2 - x^3)$$

(c) $0 < x < 1$

(d) (i) $\frac{dV}{dx} = \frac{\pi}{3}(1 - 2x - 3x^2)$

(ii) $\frac{dV}{dx} = 0 \quad \Rightarrow \frac{\pi}{3}(1 - 2x - 3x^2) = 0$

$\Rightarrow (1 + x)(1 - 3x) = 0$

$\Rightarrow x = -1$ (not valid) OR $x = \frac{1}{3}$ (valid)

(iii) $\frac{d^2V}{dx^2} = \frac{\pi}{3}(0 - 2 - 6x)$

At $x = \frac{1}{3} \quad \Rightarrow \frac{\pi}{3}\left(-2 - 6\left(\frac{1}{3}\right)\right) = \frac{-4\pi}{3} < 0$

$\Rightarrow$ Maximum at $x = \frac{1}{3}$

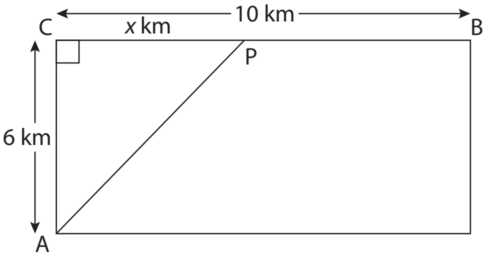
$$\Rightarrow \text{Volume} = \frac{\pi}{3}\left(1 + \frac{1}{3} - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3\right)$$

$$= \frac{\pi}{3}\left(\frac{4}{3} - \frac{1}{9} - \frac{1}{27}\right) = \frac{32\pi}{81} \text{ m}^3$$

5. (i) $|AP|^2 = (6)^2 + x^2 = 36 + x^2$

$\Rightarrow |AP| = \sqrt{36 + x^2}$ km

and $|PB| = (10 - x)$ km



(ii) Time $(T) = \frac{|AP|}{5} + \frac{|PB|}{13}$

$$\Rightarrow T = \frac{1}{5}\sqrt{36 + x^2} + \frac{1}{13}(10 - x)$$

$$\Rightarrow T = \frac{1}{5}(36 + x^2)^{\frac{1}{2}} + \frac{10}{13} - \frac{1}{13}x$$

$$\frac{dT}{dx} = \frac{1}{5} \cdot \frac{1}{2}(36 + x^2)^{-\frac{1}{2}} \cdot 2x + 0 - \frac{1}{13} = 0$$

$\Rightarrow \dfrac{x}{5\sqrt{36+x^2}} = \dfrac{1}{13}$

$\Rightarrow 13x = 5\sqrt{36+x^2}$

$\Rightarrow 169x^2 = 25(36+x^2)$

$\Rightarrow 169x^2 = 900 + 25x^2$

$\Rightarrow 144x^2 = 900$

$\Rightarrow x^2 = \dfrac{900}{144} = \dfrac{25}{4}$

$\Rightarrow x = \sqrt{\dfrac{25}{4}} = \dfrac{5}{2} = 2.5$ km

(iii) Time taken to go from A to B is a minimum at $x = 2.5$

Hence, $T = \dfrac{1}{5}\sqrt{36+(2.5)^2} + \dfrac{1}{13}(10-2.5)$

$= \dfrac{1}{5}\sqrt{40.25} + \dfrac{1}{13}(7.5)$

$= \dfrac{6.5}{5} + \dfrac{7.5}{13} = 1.8769$ hours

= 1 hour 52.615 mins

= 1 hour 53 mins

6. $P = 10 + 40r - 20r^2$

(a) $r = 0 \Rightarrow P = 10 + 40(0) - 20(0)^2 = 10$

Ans = 10 000

(b) $P = 0 \Rightarrow 10 + 40r - 20r^2 = 0 \Rightarrow 1 + 4r - 2r^2 = 0$

$\Rightarrow r = \dfrac{-4 \pm \sqrt{(4)^2 - 4(-2)(1)}}{2(-2)}$

$= \dfrac{-4 \pm \sqrt{24}}{-4}$

$= \dfrac{-2 \pm 2\sqrt{6}}{-2} = 1 \pm \sqrt{6}$

$\Rightarrow$ domain: $0 \leqslant x \leqslant 1 + \dfrac{\sqrt{6}}{2}$

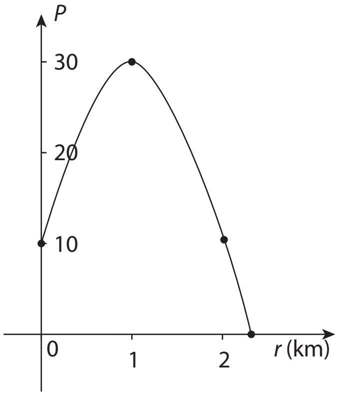
(c) Points for graph $P = f(r)$

$f(0) = 10 \Rightarrow (0, 10)$

$f(1) = 10 + 40(1) - 20(1)^2 = 30 \Rightarrow (1, 30)$

$f(2) = 10 + 40(2) - 20(2)^2 = 10 \Rightarrow (2, 10)$

$f\left(1 + \dfrac{\sqrt{6}}{2}\right) = f(2.2) = 0 \Rightarrow (2.2, 0)$



(d) $\dfrac{dP}{dr} = 40 - 40r$

(e) $r = 0.5 \Rightarrow \dfrac{dP}{dr} = 40 - 40(0.5) = 20$

$r = 1 \Rightarrow \dfrac{dP}{dr} = 40 - 40(1) = 0$

$r = 2 \Rightarrow \dfrac{dP}{dr} = 40 - 40(2) = -40$

(f) $\dfrac{dP}{dr} = 0 \Rightarrow 40 - 40r = 0$

$\Rightarrow 40r = 40 \Rightarrow r = 1$

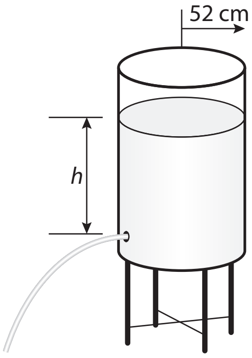
Greatest since $\dfrac{dP}{dr} = -40$ i.e. < 0

Population $= 10 + 40(1) - 20(1)^2$

$= 10 + 40 - 20$

$= 30$

$= 30\,000$



7. (a) $h = \left(10 - \frac{t}{200}\right)^2$

At $t = 0 \Rightarrow h = \left(10 - \frac{0}{200}\right)^2 = (10)^2 = 100$ cm

(b) $h = 64$ cm $\Rightarrow \left(10 - \frac{t}{200}\right)^2 = 64$

$$\Rightarrow 10 - \frac{t}{200} = \sqrt{64} = 8$$

$$\Rightarrow -\frac{t}{200} = -2$$

$$\Rightarrow t = 200(2) = 400 \text{ secs}$$

(c) Volume $(V) = \pi r^2 h = \pi(52)^2 . h = 2704\pi h$

$$\Rightarrow \frac{dV}{dh} = 2704\pi$$

$$h = \left(10 - \frac{t}{200}\right)^2$$

$$\Rightarrow \frac{dh}{dt} = 2\left(10 - \frac{t}{200}\right)^1 \cdot \frac{-1}{200} = \frac{-1}{100}\left(10 - \frac{t}{200}\right)$$

When $h = 64 \Rightarrow t = 400 \Rightarrow \frac{dh}{dt} = \frac{-1}{100}\left(10 - \frac{400}{200}\right)$

$$= -\frac{1}{100}(8) = \frac{-2}{25}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = 2704\pi \cdot \frac{-2}{25} = -216.32\pi = -679.58 = -680$$

Hence, volume of water is decreasing at the rate of 680 cm^3 s^{-1}

(d) Hole is a circle, $r = 1 \Rightarrow$ Area $(A) = \pi(1)^2 = \pi$

$$\frac{dV}{dt} = \text{Area} \times \text{speed}$$

$$\Rightarrow \text{speed} = \frac{\frac{dV}{dt}}{\text{Area}} = \frac{216.32\pi}{\pi} = 216.32 \text{ cm s}^{-1}$$

(e) $h = \left(10 - \frac{t}{200}\right)^2$

$$\Rightarrow \sqrt{h} = 10 - \frac{t}{200}$$

Speed of the water = Rate at which volume of water is decreasing divided by the area of the hole

$$= \frac{\frac{dV}{dt}}{\pi}$$

$$= \frac{2704\pi \cdot \frac{1}{100}\left(10 - \frac{t}{200}\right)}{\pi}$$

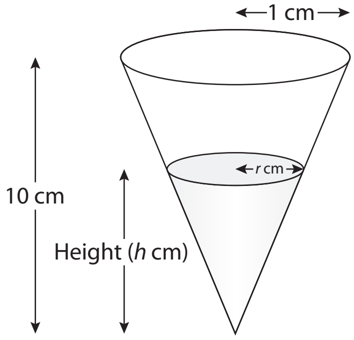
$$= 27.04 . \sqrt{h} \text{ cm/sec}$$

(f) $V = c\sqrt{1962h}$

$$\Rightarrow 27.04\sqrt{h} = c\sqrt{1962} . \sqrt{h}$$

$$\Rightarrow c = \frac{27.04}{\sqrt{1962}} = 0.61 = 0.6$$

8. (a) Volume $(V) = \frac{1}{3}\pi r^2 h$



(b) $\frac{r}{h} = \frac{1}{10} \Rightarrow 10r = h$

$\Rightarrow r = \frac{h}{10}$

(c) $V = \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 \cdot h = \frac{\pi h^3}{300}$

(d) $\frac{dV}{dt} = 0.1 \text{ cm}^3\text{ s}^{-1}$

(e) $V = \frac{\pi h^3}{300}$

$\Rightarrow \frac{dV}{dh} = \frac{3\pi h^2}{300} = \frac{\pi h^2}{100}$

At $h = \frac{1}{2}(10) = 5 \quad \Rightarrow \frac{dV}{dh} = \frac{\pi 5^2}{100} = \frac{1}{4}\pi$

Hence, $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\Rightarrow 0.1 = \frac{1}{4}\pi \cdot \frac{dh}{dt}$

$\Rightarrow 0.4 = \pi \cdot \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{0.4}{\pi} = \frac{2}{5\pi} \text{ cm s}^{-1}$

(f) Area $(A) = \pi r^2 = \pi\left(\frac{h}{10}\right)^2 = \frac{\pi h^2}{100}$

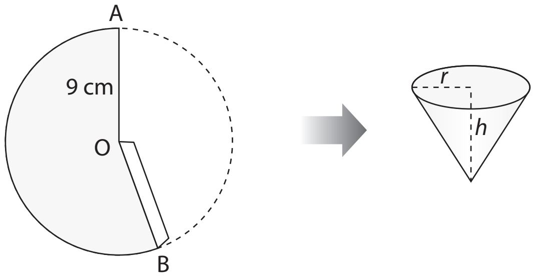
$\Rightarrow \frac{dA}{dh} = \frac{2\pi h}{100} = \frac{\pi h}{50}$

When $h = 5 \Rightarrow \frac{dA}{dh} = \frac{\pi(5)}{50} = \frac{\pi}{10}$

Hence, $\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$

$= \frac{\pi}{10} \cdot \frac{2}{5\pi} = \frac{1}{25} \text{ cm}^2\text{ s}^{-1}$

9. (a) Slant height $(h) = 9$ cm



$\Rightarrow r^2 + h^2 = 9^2 = 81$

$\Rightarrow \quad r^2 = 81 - h^2$

Hence, Volume $(V) = \frac{1}{3}\pi r^2 h$

$= \frac{\pi}{3}h(81 - h^2)$

(b) Capacity = Volume $(V) = \frac{154\pi}{3}$

$\Rightarrow \quad \frac{\pi}{3}h(81 - h^2) = \frac{154\pi}{3}$

$\Rightarrow \quad 81h - h^3 = 154$

$\Rightarrow h^3 - 81h + 154 = 0$

When $h = 2 \Rightarrow f(2) = (2)^3 - 81(2) + 154$

$= 8 - 162 + 154 = 0$

Hence, $h = 2$ is an integer root

$\Rightarrow (h - 2)$ is a factor

$$\Rightarrow \quad \begin{array}{rl}
 & \quad h^2 + 2h - 77 \\
h - 2 & \overline{)\, h^3 - 81h + 154} \\
 & \underline{h^3 - 2h^2} \\
 & \qquad 2h^2 - 81h \\
 & \qquad \underline{2h^2 - 4h} \\
 & \qquad\qquad -77h + 154 \\
 & \qquad\qquad \underline{-77h + 154} \\
 & \qquad\qquad\qquad\quad 0
\end{array}$$

Solve $h^2 + 2h - 77 = 0$

$$\begin{aligned}
\Rightarrow h &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-77)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{312}}{2} \\
&= \frac{-2 + \sqrt{312}}{2}, \frac{-2 - \sqrt{312}}{2} \\
&= 7.831, -9.831 \quad \text{(not valid)} \\
&= 7.83 \text{ non-integer root}
\end{aligned}$$

(c) $V = \frac{\pi}{3}h(81 - h^2)$

$$V = \frac{\pi}{3}[81h - h^3]$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3}[81 - 3h^2] = 0$$

$$\Rightarrow \quad 81 - 3h^2 = 0$$

$$\Rightarrow \quad 27 - h^2 = 0$$

$$\Rightarrow \quad h^2 = 27$$

$$\Rightarrow \quad h = \sqrt{27} = 3\sqrt{3} = 5.20 \text{ cm}$$

$$\Rightarrow \frac{d^2V}{dh^2} = \frac{\pi}{3}(0 - 6h) = -2\pi h$$

At $h = 3\sqrt{3} \Rightarrow \frac{d^2V}{dh^2} = -2\pi 3\sqrt{3} = -6\pi\sqrt{3} < 0$

$\Rightarrow$ Maximum volume

$$\begin{aligned}
\text{Hence, Volume} &= \frac{\pi}{3}\left[81\left(3\sqrt{3}\right) - \left(3\sqrt{3}\right)^3\right] \\
&= \frac{\pi}{3}\left[243\sqrt{3} - 81\sqrt{3}\right] \\
&= \frac{\pi}{3}162\sqrt{3} = 54\sqrt{3}\,\pi \text{ cm}^3 = 294 \text{ cm}^3
\end{aligned}$$

(d)

	Cups in part (*b*)		**Cup in part (*c*)**
Radius (*r*)	8.77 cm	4.44 cm	7.35 cm
Height (*h*)	2 cm	7.83 cm	5.20 cm
Capacity (*V*)	$\frac{154\pi}{3} = 161 \text{ cm}^3$	$\frac{154\pi}{3} = 161 \text{ cm}^3$	294 cm^3

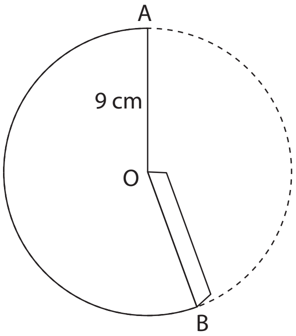
(e) A conical cup with radius = 4.44 cm and height = 7.83 cm is the most reasonable shape because it is well proportioned and it is easy to handle.

(f) In part (e), $r = 7.35$ cm and $\ell = 9$ cm

Curved Surface Area $= \pi r\ell$

$= \pi(4.44)(9)$

$= 125.538 \text{ cm}^2$



Area of sector of a circle $= \pi r^2 \dfrac{\theta}{360°}$

$\Rightarrow \pi(9)^2 \dfrac{\theta}{360°}$

$= (0.70686)\theta = 125.538 \text{ cm}^2$

$\Rightarrow \quad \theta = \dfrac{125.538}{0.70686}$

$= 177.5995$

$= 178°$

10. (a) $P = 150 + 300e^{-0.05t}$

$t = 0 \Rightarrow P = 150 + 300e^{-0.05(0)}$

$= 150 + 300e^0$

$= 150 + 300\,.\,(1) = 450$ birds

(b) $\dfrac{dP}{dt} = 0 + 300e^{-0.05t}\,.\,(-0.05)$

$= -15e^{-0.05t}$

When $t = 10 \Rightarrow \dfrac{dP}{dt} = -15e^{-0.05(10)}$

$= -15e^{-0.5}$

$= -9.098$

Hence, a decreasing rate of change

(c) Limiting value of the bird population occurs as t approaches infinity

Hence, as $t \to \infty$ then $\lim\limits_{t \to \infty} e^{-0.05t} = 0$

$\Rightarrow$ limiting value for $P = 150 + 300(0)$

$= 150$ birds

(d) $150 + 300e^{-0.05t} < 200$

$\Rightarrow \quad 300e^{-0.05t} < 50$

$\Rightarrow \quad e^{-0.05t} < \dfrac{50}{300} = \dfrac{1}{6} = 0.16666667$

Hence, ln $e^{-0.05t} <$ ln (0.16666667)

$\Rightarrow \quad -0.05t < -1.79176$

$\Rightarrow \quad t > \dfrac{1.79176}{0.05} = 35.8$

Hence, species will be eligible after 36 years.

Chapter 5

Exercise 5.1

1. $P = €3000$

$t = 10$ years

$r = 3\% = 0.03$

$$\begin{aligned}\text{Future value} &= €P(1 + i)^t \\ &= €3000(1 + 0.03)^{10} \\ &= €4031.75\end{aligned}$$

2. $P = €5000$

$t = 8$ years

$r = 2.5\% = 0.025$

$$\begin{aligned}\text{Future value} &= €P(1 + i)^t \\ &= €5000(1 + 0.025)^8 \\ &= €6092.014 \\ &= €6092.01\end{aligned}$$

$$\begin{aligned}\text{Interest paid} &= \text{Future value} - \text{sum invested } (€P) \\ &= €6092.01 - €5000 \\ &= €1092.01\end{aligned}$$

3. $(1 + r)^{12} = 1 + i$

$$1 + r = (1 + i)^{\frac{1}{12}}$$

$$r = (1 + i)^{\frac{1}{12}} - 1$$

4. (i) $i = 6\% = 0.06 =$ annual rate

$r =$ monthly rate

$\Rightarrow (1 + r)^{12} = 1 + 0.06 = 1.06$

$$1 + r = (1.06)^{\frac{1}{12}}$$

$$\begin{aligned}r &= (1.06)^{\frac{1}{12}} - 1 = 0.004868 \\ &= 0.49\%\end{aligned}$$

(ii) $2.5\% = 0.025 =$ annual rate

$r =$ monthly rate

$\Rightarrow (1 + r)^{12} = 1 + 0.025 = 1.025$

$$1 + r = 1.025^{\frac{1}{12}}$$

$$\begin{aligned}r &= 1.025^{\frac{1}{12}} - 1 \\ &= 0.0020598 \\ &= 0.21\%\end{aligned}$$

(iii) $4\% = 0.04 =$ annual rate

$r =$ monthly rate

$\Rightarrow (1 + r)^{12} = 1 + 0.04 = 1.04$

$$1 + r = 1.04^{\frac{1}{12}}$$

$$\begin{aligned}r &= 1.04^{\frac{1}{12}} - 1 \\ &= 0.0032737 \\ &= 0.33\%\end{aligned}$$

5. $P = €4500$

F.V. $= €5607.82$

$t = 5$ years

$\Rightarrow$ F.V. $= P(1 + i)^t$

$€5607.82 = €4500(1 + i)^5$

$\Rightarrow (1 + i)^5 = \dfrac{5607.82}{4500} = 1.24618$

$1 + i = (1.24618)^{\frac{1}{5}}$

$i = (1.24618)^{\frac{1}{5}} - 1$

$= 0.045$

$= 4.5\%$

6. $P = €15\,000$

$r = 3.5\% = 0.035$

$I_1 = 15\,000 \times 0.035 = 525$

$I_2 = 15\,525 \times 0.035 = 543.83$

$I_3 = 16\,068.38 \times 0.035 = 562.39$

$I_4 = 16\,630.77 \times 0.035 = 582.07$

$I_5 = 17\,212.85 \times 0.035 = 602.45$

Year	Principal	Interest
1	15 000	525
2	15 525	543.83
3	16 068.38	562.39
4	16 630.77	582.07
5	17 212.85	602.45

7. $i = 4\% = 0.04$

$\Rightarrow (1 + r)^2 = 1 + i$

$(1 + r)^2 = 1 + 0.04 = 1.04$

$1 + r = (1.04)^{\frac{1}{2}}$

$r = (1.04)^{\frac{1}{2}} - 1$

$= 0.0198 = 1.98\%$

8. $P = €6500$

$r = 1.932\% = 0.01932$

$t = 6$ years 4 months

$= 76$ months

F.V. $= P(1 + r)^t$

$= €6500(1 + 0.01932)^{76}$

$= €27\,830.10$

9. $P = €12\,000$

$i = 3.5\% = 0.035$

(i) $t = 5$ years 3 months

$= 63$ months

$\therefore (1 + r)^{12} = 1 + i$

$(1 + r)^{12} = 1 + 0.035 = 1.035$

$r = (1.035)^{\frac{1}{12}} - 1$

$= 0.0028709$

$= 0.28709\%$

F.V. $= P(1 + r)^t$

F.V. $= 12\,000(1 + 0.0028709)^{63}$

$= €14\,375.34$

(ii) 8 years 2 months

$= 98$ months

F.V. $= 12\,000(1.0028709)^{98}$

$= €15\,892.57$

(iii) 10 years 6 months

$= 126$ months

$\text{F.V.} = 12\,000(1.0028709)^{126}$

$= €17\,220.86$

10. $i = 4.2\% = 0.042$

$\text{F.V.} = €10\,000$

$t = 10$ years

$$\text{Present value} = \frac{\text{Future value}}{(1 + i)^t}$$

$$= \frac{€10\,000}{(1 + 0.042)^{10}}$$

$$= €6627.09$$

11. $\text{F.V.} = €25\,000$

$t = 9$ years

$i = 4.5\% = 0.045$

$$\text{Present value} = \frac{\text{F.V.}}{(1 + i)^t}$$

$$= \frac{25\,000}{(1 + 0.045)^9}$$

$$= €16\,822.61$$

12. $P = €50\,000$

$\text{F.V.} = €100\,000$

$i = 3.5\% = 0.035$

$\text{F.V.} = P(1 + i)^t$

$\Rightarrow \quad €100\,000 = €50\,000(1 + 0.035)^t$

$\Rightarrow \quad 2 = (1.035)^t$

$\Rightarrow \quad \log 2 = t \log 1.035$

$$t = \frac{\log 2}{\log 1.035}$$

$= 20.15$ years

13. $P = €175\,000$

$i = 4.5\% = 0.045$

$t = 20$ years

$\text{F.V.} = P(1 + i)^t$

$\text{F.V.} = 175\,000(1 + 0.045)^{20}$

$= €422\,049.95$

14. $P = 1130$

$F = 3000$

$i = 0.05$

Then

$F = P(1 + i)^t$

$3000 = 1130(1.05)^t$

$$(1.05)^t = \frac{3000}{1130} = 2.654867$$

$t = \log_{1.05} 2.654867$

$t = 20$

It will take 20 years.

15. $i = 6\% = 0.06$

$r =$ monthly rate

(i) $\Rightarrow (1 + r)^{12} = 1 + i$

$$(1 + r)^{12} = 1 + 0.06$$

$$1 + r = (1.06)^{\frac{1}{12}}$$

$$r = (1.06)^{\frac{1}{12}} - 1$$

$$= 0.0048676$$

$$= 0.48676\%$$

$$= 0.4868\%$$

(ii) $r = \frac{i}{12} = \frac{0.06}{12}$

$$= 0.005$$

$$= 0.5\%$$

$t = 3$ years

$= 36$ months

$\therefore$ F.V. $= 10\,000(1 + 0.005)^{36}$ F.V. $= 10\,000(1.00486)^{36}$

$= €11\,966.81$ $= €11\,910.35$

$\Rightarrow$ Difference $= €56.46$

(iii) $r = 0.0048676$

Correct to 3 places of decimals $= 0.005$

Correct to 4 places of decimals $= 0.0049$

$\therefore$ A minimum of 4 places of decimals are needed to create a difference.

16. $P = €15\,000$

$i = 3\% = 0.03$

$t = 2$ years

F.V. $= €P(1 + i)^t$

$= 15\,000(1 + 0.03)^2$

$= €15\,913.50$

Withdraws €2000 $\Rightarrow$ $P = €13\,913.50$

$t = 3$ years F.V. $= 13\,913.5(1.03)^3$

$= €15\,206.66$

Exercise 5.2

1. Present value (P) $= €30\,000$

Depreciation (i) $= 15\% = 0.15$

(i) $t = 5$ years F.V. $= P(1 - i)^t$

$= €30\,000(1 - 0.15)^5$

$= €13\,311.16$

(ii) $t = 10$ years F.V. $= €30\,000(1 - 0.15)^{10}$

$= €5906.23$

2. $P = €1400$

$i = 8\% = 0.08$ per month

$t = 15$ months

F.V. $= P(1 - i)^t$

$= 1400(1 - 0.08)^{15}$

$= €400.82$

3. $P = €44\,000$

$i = 20\% = 0.2$ for first year

$i = 15\% = 0.15$ after first year.

(i) 3 years $\Rightarrow$ $\text{F.V.} = P(1 - i)^t$

$= €44\,000(1 - 0.2)^1$

$= €35\,200$ after 1 year

$\Rightarrow$ $\text{F.V.} = €35\,200(1 - 0.15)^2$

$= €25\,432$

(ii) 6 years $\Rightarrow$ $\text{F.V.} = €35\,200(1 - 0.15)^5$

$= €15\,618.43$

4. $P = €140\,000$

$i = 20\% = 0.2$

$t = 4$ years

(a) (i) $\Rightarrow$ $\text{F.V.} = P(1 - i)^t$

$= €140\,000(1 - 0.2)^4$

$= €57\,344$

(ii) $P = €25\,000$ $\text{F.V.} = P(1 + i)^t$

$i = 3.5\% = 0.035$ $= 25\,000(1 + 0.035)^4$

$= €28\,688.08$

(b) Inflation 2% per annum

(i) $\text{F.V.} = 140\,000(1 + 0.02)^4$

$= €151\,540.50$

(ii) Funds shortfall $= €151\,540.50 - €(28\,688.08 + 57\,344)$

$= €65\,508.42$

5. $P = 175\,000$

$F = 73\,187.09$

$i = 0.16$

Then

$$F = P(1 - i)^t$$

$$73\,187.09 = 17\,500(1 - 0.16)^t$$

$$(0.84)^t = \frac{73\,187.09}{175\,000} = 0.418212$$

$$t = \log_{0.84} 0.418212$$

$$t = 5$$

6. $\text{P.V.} = 60\,000$ kg

$i = 15\% = 0.15$ per month

$t =$ end of Jan 2004

beginning of April 2005

$= 14$ months

$\text{F.V.} = \text{P.V.}(1 - i)^t$

$\text{F.V.} = 60\,000(1 - 0.15)^{14}$

$= 6166$ kg

7. $\text{P.V.} = €180\,000$

$\text{F.V.} = €80\,000$

$t = 10$ years

$\text{F.V.} = \text{P.V.}(1 - i)^t$

$€80\,\cancel{0}\cancel{0}\cancel{0} = 180\,\cancel{0}\cancel{0}\cancel{0}\,(1 - i)^{10}$

(i) $\Rightarrow \quad 0.4\dot{4}\dot{4} = (1 - i)^{10}$

$1 - i = (0.4\dot{4}\dot{4})^{\frac{1}{10}}$

$1 - i = 0.9221$

$\Rightarrow \quad i = 0.0778$

$= 7.8\%$

(ii) F.V. = €60 000 $\quad \Rightarrow \quad 6\cancel{0}\,\cancel{0}\cancel{0}\cancel{0} = 18\cancel{0}\,\cancel{0}\cancel{0}\cancel{0}\,(1 - 0.0778)^t$

$0.3\dot{3} = (0.922)^t$

$\Rightarrow \quad \log(0.3\dot{3}) = t\log(0.922)$

$\Rightarrow \quad t = \dfrac{\log(0.3\dot{3})}{\log(0.922)}$

$= 13.53$ years

8. $P =$ €2500

(a) €550 per year

(b) 35% = 0.35 per year

$t = 4$ years

(a) €550 per year for 4 years = €2200

$\Rightarrow \quad$ F.V. = €2500 − €2200 = €300

(b) F.V. = P.V.$(1 - i)^t$

F.V. = $2500(1 - 0.35)^4$

F.V. = €446.27

9. P.V. = €23 500

$i = 28\% = 0.28$

(i) $t = 2$ years $\quad$ F.V. = P.V.$(1 - i)^t$

$= 23\,500(1 - 0.28)^2$

$=$ €12 182.40

(ii) $t = 5$ years $\quad$ F.V. = $23\,500(1 - 0.28)^5$

$=$ €4547.06

(iii) $t = 7$ years $\quad$ F.V. = $23\,500(1 - 0.28)^7$

$=$ €2357.19

10. P.V. = €8000

F.V. = €1

$t = 20$ years

(i) $\Rightarrow \quad$ F.V. = P.V.$(1 - i)^t$

$1 = 8000(1 - i)^{20}$

$\Rightarrow \quad \left(\dfrac{1}{8000}\right)^{\frac{1}{20}} = (1 - i)$

$i = 0.3619$

$i = 36.2\%$

(ii) For F.V. = 0 $\quad \Rightarrow \quad (1 - i)^t = 0$

$\therefore$ Once i is not equal to 1, i.e. 100%.

$(1 - i)^t \neq 0$ for all values of t.

(iii) Slope $= \dfrac{-8000}{5} =$ €1600 per year

(iv) $t = 4.2$ years : $\quad$ F.V. = €1300

(v) $t = 5$ years $\quad$: $\quad$ F.V. = $8000(1 - 0.362)^5$

$=$ €845.66

(vi) Reducing balance method.

Once a system is still operating, it has value, no matter what its age.

Exercise 5.3

1. $P = €20$

$i = 0.5\% = 0.005$ per month

$t = 36$ months

Future value $= 20(1.005)^1 + 20(1.005)^2 + \dots 20(1.005)^{36}$

instalment left in for 1 month

instalment left in for 36 months

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r} \qquad a = 20(1.005)$$

$$r = 1.005$$

$$n = 36$$

$$= 20(1.005)\,\frac{(1 - (1.005)^{36})}{(1 - (1.005))}$$

$$= €790.66$$

Total interest $= €790.66 - (36 \times €20)$

$= €70.66.$

2. $P = €30.00$ per month

$i = 4\%$ per annum

$= 0.04$

(i) $(1 + r)^{12} = 1 + i$

$(1 + r)^{12} = 1 + 0.04$

$1 + r = (1.04)^{\frac{1}{12}}$

$r = (1.04)^{\frac{1}{12}} - 1 = 0.00327$

$= 0.33\%.$

(ii) 18 years → 21 years

$= 36$ months

$= 30(1.0033) + 30(1.0033)^2 + \dots 30(1.0033)^{36}$

$$\Rightarrow \quad \text{F.V.} = 30(1.0033)\left[\frac{1 - (1.0033)^{36}}{1 - 1.0033}\right]$$

$= €1148.55$

3. $P = €2000$

$i = 4\% = 0.04$

$t = 5$ years

F.V. $= 2000(1.04) + 2000(1.04)^2 + \dots 2000(1.04)^5$

$$= 2000(1.04)\left[\frac{1 - (1.04)^5}{1 - (1.04)}\right]$$

$= €11\,265.95$

4. n payments of $€P$

$i\,\%$

F.V. $= P(1 + i)^1 + P(1 + i)^2 + \dots P(1 + i)^n$

i.e. the sum of a geometric series with $a = P(1 + i)$, $r = (1 + i)$

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

$$= P(1 + i)\left[\frac{1 - (1 + i)^n}{1 - (1 + i)}\right]$$

$$= P(1 + i)\left[\frac{1 - (1 + i)^n}{-i}\right]$$

$$= P(1 + i)\left[\frac{(1 + i)^n - 1}{i}\right]$$

5. n payments of €P

i %

$$\text{P.V.} = \frac{€P}{1+i} + \frac{P}{(1+i)^2} + \dots \frac{P}{(1+i)^n}$$

i.e. the sum of a geometric series with $a = \left(\frac{P}{1+i}\right)$, $r = \left(\frac{1}{1+i}\right)$

$$S_n = a\left[\frac{1-r^n}{1-r}\right]$$

$$\text{P.V.} = \frac{P}{(1+i)}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\left(\frac{1}{1+i}\right)}\right]$$

$$= \frac{P}{(1+i)}\left[\frac{1-\frac{1}{(1+i)^n}}{\frac{\cancel{1}+i-\cancel{1}}{1+i}}\right]$$

$$= \frac{P}{\cancel{(1+i)}}\left(\cancel{\frac{1+i}{i}}\right)\left[\frac{(1+i)^n-1}{(1+i)^n}\right]$$

$$= \frac{P}{(1+i)^n}\left[\frac{(1+i)^n-1}{i}\right]$$

6. F.V. = €6523.33

$i = 9\% = 0.09$

$t = 5$ years

(i) $\text{F.V.} = A(1.09) + A(1.09)^2 + \dots A(1.09)^5$

(ii) $€6523.33 = a\left(\frac{1-r^n}{1-r}\right)$

$$= A(1.09)\left(\frac{1-(1.09)^5}{1-1.09}\right)$$

$€6523.33 = A(6.52333)$

$\Rightarrow \quad A = €1000$

7. (i) $r = 0.09$ (annual rate)

Let i be the equivalent monthly rate, as a decimal. Then

$(1+i)^{12} = 1 + r$

$1 + i = (1.09)^{\frac{1}{12}} = 1.007207$

$i = 0.007207$, or 0.7207%.

(ii) $F = P(1+i)\left[\frac{(1+i)^t-1}{i}\right]$

$$F = 200(1.007207)\left[\frac{(1.007202)^{24}-1}{0.007207}\right]$$

$F = €5257.31$

8. F.V. = €10 000

$t = 7$ years

$i = 8.5\% = 0.085$

$P = ?$

$$\text{F.V.} = P(1.085) + P(1.085)^2 + \dots P(1.085)^7$$

$$\therefore €10\,000 = P(1.085)\left[\frac{1-(1.085)^7}{1-(1.085)}\right]$$

$10\,000 = P(9.8306)$

$\Rightarrow \quad P = €1017.23$

9. F.V. $= €5000$

$t = 3$ years $= 12$ quarters

$i = 7.2\% = 0.072$

$P = ?$

$$(1 + r)^4 = 1 + i$$

$$(1 + r)^4 = 1.072$$

$$r = (1.072)^{\frac{1}{4}} - 1$$

$$= 0.01753$$

$$\text{F.V.} = €5000 = P(1.01753) + P(1.01753)^2 + \dots + P(1.01753)^{12}$$

$$= P(1.01753)\left[\frac{1 - (1.01753)^{12}}{1 - (1.01753)}\right]$$

$$= P(13.4592)$$

$$\Rightarrow \quad P = €371.49$$

10. Let €P be the instalment at the start of each period at i%.

$\therefore 1 + i\% = 1.0i$

$$\Rightarrow \text{Present value} = P + \frac{P}{1.0i} + \frac{P}{(1.0i)^2} + \frac{P}{(1.0i)^3} + \dots \frac{P}{(1.0i)^n}$$

$$= P\left(\frac{1 - \left(\frac{1}{1.0i}\right)^n}{1 - \frac{1}{1.0i}}\right) = P\left[\frac{\frac{(1.0i)^n - 1}{(1.0i)^n}}{\frac{1.0i - 1}{1.0i}}\right]$$

$$= P\left[\frac{1.0i^n - 1}{(1.0i)^n}\right] \cdot \frac{1.0i}{0.0i}$$

$$= \frac{P}{(1.0i)^{n-1}}\left[\frac{1.0i^n - 1}{0.0i}\right]$$

$$\text{Future value} = P(1.0i) + P(1.0i)^2 + \dots + P(1.0i)^n$$

$$= P(1.0i)\left[\frac{1 - (1.0i)^n}{1 - 1.0i}\right]$$

$$= P(1.0i)\left[\frac{1 - (1.0i)^n}{-0.0i}\right]$$

$$= P(1.0i)\left[\frac{(1.0i)^n - 1}{0.0i}\right]$$

$$\Rightarrow \quad \frac{\text{Future value}}{(1 + i)^n} = \frac{P(1.0i)}{(1.0i)^n}\left(\frac{1.0i^n - 1}{0.0i}\right)$$

$$= \frac{P}{(1.0i)^{n-1}}\left(\frac{1.0i^n - 1}{0.0i}\right) = \text{P.V.}$$

11. $P = €3000$

$t = 6$ years

$i = 8\% = 0.08,$

Assuming payment is made at the start of each year.

(i) $$\text{P.V.} = 3000 + \frac{3000}{1.08} + \frac{3000}{(1.08)^2} + \dots \frac{3000}{(1.08)^5}$$

$$= 3000\left[\frac{1 - \left(\frac{1}{1.08}\right)^5}{1 - \frac{1}{1.08}}\right]$$

$$= €14\,978.13$$

(ii) $$\text{F.V.} = 3000(1.08) + 3000(1.08)^2 + \dots 3000(1.08)^6$$

$$= 3000(1.08)\left[\frac{1 - (1.08)^6}{1 - 1.08}\right]$$

$$= €23\,768.41$$

(iii) $A = P(1 + i)^n$

$= €14\,978(1 + 0.08)^6$

$= €23\,768.41$

Exercise 5.4

1. Mortgage €200 000

$t = 30$ years $= 360$ months

$i = 6\% = 0.06$ per year

$(1 + r)^{12} = 1 + i = 1.06$

$1 + r = (1.06)^{\frac{1}{12}}$

$r = (1.06)^{\frac{1}{12}} - 1$

$= 0.004867$

$$\text{Payment} = \frac{M(i)(1 + i)^n}{(1 + i)^n - 1}$$

$$= \frac{200\,000(0.004867)(1.004867)^{36}}{(1.004867)^{36} - 1}$$

$= €1178.66$

2. $t = 20$ years $= 240$ months

$i = 8\% = 0.08$ per year

Repayment $=$ €850 per month

$(1 + r)^{12} = 1 + i = 1.08$

$1 + r = (1.08)^{\frac{1}{12}}$

$r = (1.08)^{\frac{1}{12}} - 1$

$= 0.006434$

$$\text{Payment} = \frac{M(i)(1 + i)^n}{(1 + i)^n - 1}$$

$$€850 = \frac{M(0.006434)(1.006434)^{240}}{(1.006434)^{240} - 1}$$

$€850 = M(0.0081914)$

$\Rightarrow \quad M = €103\,766$

$= €103\,800$

3. €75 000

$i = 8\% = 0.08$

(a) $t = 20$ years

$= 240$ months

$(1 + r)^{12} = 1 + i = 1.08$

$1 + r = (1.08)^{\frac{1}{12}}$

$r = (1.08)^{\frac{1}{12}} - 1$

$= 0.006434$

$$P = \frac{75\,000(0.006434)(1.006434)^{240}}{(1.006434)^{240} - 1}$$

$= €614$

$I = (€614 \times 240) - €75\,000 = €72\,360$

(b) $t = 25$ years

$= 25 \times 12 = 300$ months

$$P = \frac{75\,000(0.006434)(1.006434)^{300}}{(1.006434)^{300} - 1}$$

$= €565$

$I = (€565 \times 300) - €75\,000 = €94\,500$

(c) $t = 30$ years

$= 30 \times 12 = 360$ months

$$P = \frac{75\,000(0.006434)(1.006434)^{360}}{(1.006434)^{360} - 1}$$

$= €536$

$I = (€536 \times 360) - €75\,000 = €117\,960$

4. $M = €15\,000$

Plan A 10% discount, 9% for 5 years.

10% discount $= 15\,000 \times (0.1) = €1500$

$\Rightarrow M = €13\,500$

$$\text{Repayments} = \frac{M(i)(1+i)^t}{(1+i)^t - 1}$$

$$= \frac{13\,500(0.09)(1.09)^5}{(1.09)^5 - 1}$$

$= €3469$ per year

Plan B 3% for 5 years.

$$\text{Repayments} = \frac{15\,000(0.03)(1.03)^5}{(1.03)^5 - 1}$$

$= €3275$ per year

∴ *Plan B* is better

5. Fund $= €250\,000$

$t = 25$ years

$i = 5\% = 0.05$

$$\text{Payment} = \frac{M(i)(1+i)^t}{(1+i)^t - 1}$$

$$= \frac{250\,000(0.05)(1.05)^{25}}{(1.05)^{25} - 1}$$

$= €17\,738.11$

6. Option 1: €200 000 invested for 25 years at 5%

$A = P(1+i)^t$

$= 200\,000(1 + 0.05)^{25}$

$= €677\,270$

Option 2:

$A = 15\,000(1.05) + 15\,000(1.05)^2 + \ldots 15\,000(1.05)^{25}$

$$A = 15\,000(1.05)\left[\frac{1 - (1.05)^{25}}{1 - 1.05}\right]$$

$= €751\,701.81$

Option 2 is better.

7. $i = 6.6\% = 0.066$
$t = 3$ years $= 36$ months
Payment $=$ €400

$$(1 + r)^{12} = 1 + i = 1.066$$
$$1 + r = (1.066)^{\frac{1}{12}}$$
$$r = (1.666)^{\frac{1}{12}} - 1$$
$$= 0.005340$$
$$\therefore \text{ Payment} = \frac{\text{Mortgage}(i)(1 + i)^t}{(1 + i)^t - 1}$$
$$400 = \frac{M(0.00534)(1.00534)^{36}}{(1.00534)^{36} - 1}$$
$$= M(0.030607)$$
$$\Rightarrow \qquad M = €13\,068.78$$

Revision Exercise 5 (Core)

1. Instalment $=$ €1000
$i = 8\% = 0.08$
$t = 5$ years

$$\text{F.V.} = €1000(1.08) + 1000(1.08)^2 + \ldots 1000(1.08)^5$$
$$= €1000(1.08)\left[\frac{1 - (1.08)^5}{1 - (1.08)}\right]$$
$$= 1000(1.08)\left[\frac{1.08^5 - 1}{0.08}\right]$$
$$= €6335.93$$

2. Instalment $=$ €300
$i = 6\% = 0.06$ per year
$t = 8$ years $= 96$ months

$$(1 + r)^{12} = 1 + i = 1.06$$
$$1 + r = (1.06)^{\frac{1}{12}}$$
$$r = (1.06)^{\frac{1}{12}} - 1$$
$$= 0.004867$$
$$\text{F.V.} = 300(1.004867)\left[\frac{(1.004867)^{96} - 1}{0.004867}\right]$$
$$= €36\,778.58$$

3. Car loan $=$ €20 000
$i = 2\% = 0.02$
$t = 25$ instalments

$$\text{Payment} = \frac{M(i)(1 + i)^t}{(1 + i)^t - 1}$$
$$= \frac{€20\,000(0.02)(1.02)^{25}}{(1.02)^{25} - 1}$$
$$= €1024$$

4. $t = 3$ years
$= 36$ months
Payment $=$ €600
$M = ?$
$i = 4\% = 0.04$

$$\begin{aligned}
(1+r)^{12} &= 1+i = 1.04\\
1+r &= (1.04)^{\frac{1}{12}}\\
r &= (1.04)^{\frac{1}{12}} - 1\\
&= 0.003274\\
P &= \frac{M(i)(1+i)^t}{(1+i)^t - 1}\\
€600 &= \frac{M(0.003274)(1.003274)^{36}}{(1.003274)^{36} - 1}\\
€600 &= M(0.0294923)\\
M &= €20\,344.29
\end{aligned}$$

5. Introductory: $1.25\% = 0.0125$ per months

$$(1+r)^{12} = 1+i \qquad r\% \text{ per month}$$
$$i\% \text{ per year}$$

$$\begin{aligned}
\Rightarrow \quad 1+i &= (1+0.0125)^{12}\\
i &= (1.0125)^{12} - 1\\
&= 0.1607\\
&= 16.07\%\\
&= 16.1\%
\end{aligned}$$

Regular: $2.5\% = 0.025$ per month

$$\begin{aligned}
1+i &= (1.025)^{12}\\
i &= (1.025)^{12} - 1\\
&= 34.5\%
\end{aligned}$$

6. $\text{Savings} = 200(1.0075) + 200(1.0075)^2 + 200(1.0075)^3 + 200(1.0075)^4 + 200(1.0075)^5$

$a = 200(1.0075)$

$r = 1.0075$

$$S_n = 200(1.0075)\left[\frac{1-(1.0075)^5}{1-(1.0075)}\right]$$

7. $$\text{F.V.} = P(1+i)\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\left.\begin{aligned}
P &= €1600\\
i &= 6\% = 0.06\\
n &= 5 \text{ years}
\end{aligned}\right\} \quad \begin{aligned}
\text{F.V.} &= €1600(1.06)\left[\frac{(1.06)^5 - 1}{0.06}\right]\\
&= €9560.51
\end{aligned}$$

8. (i)

$$\begin{aligned}
\text{1st:} \quad & F = 3000(1.073)^8\\
\text{2nd:} \quad & F = 3000(1.073)^7\\
\text{Last:} \quad & F = 3000(1.073)\\
\text{Total} &= 3000(1.073) + \dots + 3000(1.073)^7 + 3000(1.073)^8\\
&= \frac{3000(1.073)[1-(1.073)^8]}{1-(1.073)}\\
&= €33\,385.23
\end{aligned}$$

(ii)

$$\begin{aligned}
P &= \frac{F}{(1+i)^t}\\
P &= \frac{33\,385.23}{1.073^8}\\
P &= €19\,000.03
\end{aligned}$$

(iii)

$$\begin{aligned}
F &= P(1+i)^t\\
F &= 19\,000.13(1.073)^8\\
F &= €33\,385.23
\end{aligned}$$

Revision Exercise 5 (advanced)

1. $t = 10$ years

F.V. $=$ €500 000

$i = 9\% = 0.09$

P.V. $=$?

(i) $$\text{P.V.} = \frac{\text{F.V.}}{(1+i)^t} = \frac{€500\,000}{(1+0.09)^{10}} = €211\,205.40$$

(ii) $$\text{F.V.} = P + P(1+i) + P(1+i)^2 + \ldots + P(1+i)^{10}$$
$$€500\,000 = P + P(1.09) + P(1.09)^2 + \ldots + P(1.09)^{10}$$
$$= P\left[\frac{1-(1.09)^{10}}{1-(1.09)}\right]$$
$$€500\,000 = P(15.1929)$$
$$\Rightarrow \quad P = \frac{€500\,000}{15.1929} = €32\,910$$

2. $t = 20$ years

$= 240$ months

$i = 12\% = 0.12$

$(1+r)^{12} = 1 + i = 1.12$

$1 + r = (1.12)^{\frac{1}{12}}$

$r = (1.12)^{\frac{1}{12}} - 1$

$= 0.00948$

(i) €100 000 invested $\Rightarrow$ F.V. $= P(1+i)^t$

$= €100\,000\,(1.00948)^{240}$

$= €964\,629.31$

(ii) €1000 per month

$$\text{F.V.} = P\,(1+i)\left[\frac{(1+i)^t - 1}{i}\right]$$
$$= 1000(1.00948)\left[\frac{1.00948^{240} - 1}{0.00948}\right]$$
$$= €919\,857.64$$

$\therefore$ Option (i) : Invest €100 000 at 12% AER for 20 years.

3. *Account D* : €638.14 $=$ €500 $(1+i)^5$

$\Rightarrow (1+i)^5 = 1.27628$

$1 + i = (1.27628)^{\frac{1}{5}}$

$i = (1.27628)^{\frac{1}{5}} - 1$

$= 4.99\%$

$= 5\%$

Account C has money invested each year (€500).

Account D has one investment only (€500).

$$\text{F.V.} = P(1+i)\left[\frac{(1+i)^n - 1}{i}\right]$$
$$= 500(1.05)\left[\frac{(1.05)^6 - 1}{i}\right]$$

Note: After 5 years, there has been 6 payments, the first payment was at year 0.

$= €3571.$

4. $i = 10\% = 0.10$

$t = 20 \text{ years} = 240 \text{ months}$

Repayments = €700

$$(1 + r)^{12} = 1 + i = 1.1$$

$$1 + r = (1.1)^{\frac{1}{12}}$$

$$r = (1.1)^{\frac{1}{12}} - 1$$

$$= 0.007974$$

$$\text{Repayments} = \frac{€M(i)[(1 + i)^t]}{(1 + i)^t - 1}$$

$$\Rightarrow \qquad M = \frac{\text{repayment}\,[(1 + i)^t - 1]}{i[(1 + i)^t]}$$

$$= \frac{700[(1.007974)^{240} - 1]}{(0.007974)(1.007974)^{240}}$$

$$= €74\,736$$

5. $t = 25 \text{ years}$

$= 300 \text{ months}$

$€M = €100\,000$

Repayments = €800

$$\Rightarrow 800 = \frac{100\,000(i)(1 + i)^{300}}{(1 + i)^{300} - 1}$$

let $i = 0.008$: $\dfrac{100\,000(0.008)(1.008)^{300}}{(1.008)^{300} - 1} = €881 > 800$

let $i = 0.007$: $\dfrac{100\,000(0.007)(1.007)^{300}}{(1.007)^{300} - 1} = €798 < 800$

let $i = 0.0072$: $\dfrac{100\,000(0.0072)(1.0072)^{300}}{(1.0072)^{300} - 1} = €815 > 800$

let $i = 0.0071$: $\dfrac{100\,000(0.0071)(1.0071)^{300}}{(1.0071)^{300} - 1} = €807 > 800$

let $i = 0.00702$: $\dfrac{100\,000(0.00702)(1.00702)^{300}}{(1.00702)^{300} - 1} = €800 = €800$

$\therefore i_m = 0.00702$ per month.

$$(1 + i_m)^{12} = 1 + i_{\text{year}}$$

$$\Rightarrow \quad (1.00702)^{12} = 1 + i$$

$$\Rightarrow \qquad i = 0.0875$$

$$= 8.75\%$$

6.

Year	Pension fund	Interest added	Payment
One	€127 953	€3838.59	€15 000
Two	€116 791.59	€3503.75	€15 000
Three	€105 295.38	€3158.86	€15 000
Four	€93 454.24	€2803.63	€15 000
Five	€81 257.87	€2437.74	€15 000
Six	€68 695.61	€2060.87	€15 000
Seven	€55 756.48	€1672.69	€15 000
Eight	€42 429.17	€1272.58	€15 000
Nine	€28 702.01	€861.06	€15 000
Ten	€14 563.07	€436.87	€15 000

Revision Exercise 5 (Extended-Response Questions)

1.

Year	Principal	Interest	Payment	Year	Principal	Interest	Payment
1	100 000	+5000	−6000	1	100 000	+5000	−12 000
2	99 000	+4950	−6000	2	93 000	+4650	−12 000
3	97 950	+4897.5	−6000	3	85 650	+4282.5	−12 000
4	96 847.5	+4842.38	−6000	4	77 932.5	+3896.63	−12 000
5	95 689.88	+4784.49	−6000	5	69 829.13	+3491.46	−12 000
6	94 474.37	+4723.72	−6000	6	61 320.59	+3066.03	−12 000
7	93 198.09	+4659.90	−6000	7	52 386.62	+2619.33	−12 000
8	91 857.99	+4592.90	−6000	8	43 005.95	+2150.3	−12 000
9	90 450.89	+4522.54	−6000	9	33 156.25	+1657.81	−12 000
10	88 973.43	+4448.67	−6000	10	22 814.06	+1140.70	−12 000
€87 422.10				€11 954.76			

2. (i) 65 years → 100 years = 35 years

Pension of €20 000 at the end of each of the 35 years

$$= \text{Present value} = \frac{\text{Pension}}{(1+i)^n}\left(\frac{(1+i)^n - 1}{i}\right)$$

$4\% = 0.04$

$$= \frac{20\,000}{(1.04)^{35}}\left(\frac{(1.04)^{35} - 1}{0.04}\right)$$

$= €373\,292.26$ pension cost

$€300\,000.00$ relocation cost

$€673\,292.26$ Total

(ii) €40 000 Invested for 30 years at 5%

$$\text{F.V.} = \text{P.V.}\,(1+i)^t$$
$$= €40\,000(1+0.05)^{30}$$
$$= €172\,877.69$$

∴ Amount needed to be saved $= (673\,292.26 - 172\,877.69)$

$= €500\,414.57.$

Let instalment = €P

⇒ Future value in 30 years of €P yearly instalments at 5%

$$\Rightarrow \quad €500\,414.57 = P(1+0.05) + P(1+0.05)^2 + \ldots P(1+0.05)^{30}$$

$$= P(1.05)\left(\frac{1-(1.05)^{30}}{1-(1.05)}\right)$$

$$= P(1.05)\left(\frac{(1.05)^{30}-1}{(1.05)}\right)$$

$€500\,414.57 = €P(69.7608)$

⇒ Instalment €P per year = €7173.30

(iii) ∴ A total of €673 292.26 needed in 25 yrs at 5%.

$$€673\,292.26 = \frac{P(1.05)(1.05^{25} - 1)}{0.05}$$

$€673\,292.26 = P(50.1135)$

$\Rightarrow \quad P = €13\,435.36$

3. (i) $€P = \dfrac{€M(i)(1+i)^n}{(1+i)^n - 1}$

€P = repayments on loan

€M = value of loan/mortgage

i = interest rate

n = number of repayments (months/years)

(ii) $i = 9\% = 0.09$

$$\Rightarrow \quad (1 + r)^{12} = 1 + i = 1.09$$

$$1 + r = (1.09)^{\frac{1}{12}}$$

$$r = (1.09)^{\frac{1}{12}} - 1$$

$$= 0.007207$$

$$= 0.72\% \text{ monthly.}$$

(iii) $€M = €100\,000$

$€P = €800$

$i = 9\%$ per annum

$= 0.72\%$ per month

$n =$ number of payments

$$\therefore \quad P = \frac{M(i)(1+i)^n}{(1+i)^n - 1}$$

$$800 = \frac{100\,000(0.007207)(1.007207)^n}{(1.007207)^n - 1}$$

$$\frac{800}{720.7} = \frac{(1.007207)^n}{(1.007207)^n - 1}$$

$$1.11((1.007207)^n - 1) = (1.007207)^n$$

$$1.11(1.007207)^n - (1.007207)^n = 1.11$$

$$0.11(1.007207)^n = 1.11$$

$$(1.007207)^n = \frac{1.11}{0.11}\ 10.\dot{0}\dot{9}$$

$$\Rightarrow \quad n\log(1.007207) = \log(10.0909)$$

$$n = \frac{\log(10.0909)}{\log(1.007207)} = 321$$

(iv) 26 years, 9 months (i.e. 321 monthly repayments)

4. (i) Note: (i) Formula for $B2$: $= A2 * 0.006628$

Since $(1 + r)^{12} = 1 + i$ where $i = 8.25\% = 0.0825$

$$\Rightarrow \quad r = 0.006628$$

Note: (ii) Formula for $D2$: $= A2 + B2 - C2$

Note: (iii) Payment must be constant.

$\therefore$ Formula for $C3$: $= C2$

Note: (iv) Formula for $A3$: $= D2$

After 5 years (60 payments)

Mortgage interest payment balance.

140 450.2	930.9037	1127	140 254.1

$\therefore$ Balance remaining $= €140\,254.1$

(ii) Using the mortgage formula: $€P = \dfrac{€M(i)(1+i)^n}{(1+i)^n - 1}$

$$\Rightarrow 1127 = \frac{150\,000(0.006628)(1.006628)^n}{1.006628^n - 1}$$

$$\Rightarrow 1.1335747 = \frac{1.006628^n}{1.006628^n - 1}$$

$$\Rightarrow (1.1335747)(1.006628^n) - 1.1335747 = 1.006628^n$$

$$\Rightarrow 0.1335747(1.006628)^n = 1.1335747$$

$$(1.006628)^n = \frac{1.1335747}{0.1335747} = 8.4864477$$

$$\Rightarrow \quad n = \frac{\log 8.4864477}{\log 1.006628}$$

$$= 323.7 \text{ months.}$$

$$= 324 \text{ months.}$$

(iii) Mortgage remaining $= €140\,254.1 - €40\,000$

$= €100\,254.1$

Using mortgage formula

$$1127 = \frac{100\,254.1(0.006628)(1.006628)^n}{1.006628^n - 1}$$

$$\Rightarrow 1.6960524 = \frac{(1.006628)^n}{1.006628^n - 1}$$

$$\Rightarrow \quad (1.6960524)(1.006628)^n - 1.6960524 = (1.006628)^n$$

$$(0.6960524)(1.006628)^n = 1.6960524$$

$$1.006628^n = 2.4366734$$

$$n = \frac{\log 2.4366734}{\log 1.006628} = 134.8$$

$$= 135 \text{ months.}$$

5. (i) 1st payment $= A$

2nd payment $= A(1.04)$

3rd payment $= A(1.04)^2$

4th payment $= A(1.04)^3$

(ii) $S_{26} = A + A(1.04) + A(1.04)^2 + A(1.04)^3 + \ldots A(1.04)^{25} = \$21\,500\,000.00$

(iii) number of terms

$$S_{26} = \frac{A(1-(1.04)^{26})}{(1-1.04)}$$

$$= \frac{A((1.04)^{26}-1)}{0.04}$$

$$\Rightarrow \$21\,500\,000.00 = A(44.311)$$

$$\Rightarrow \quad A = \$485\,199$$

(iv)

Payment	1	2	3	4
Actual amount	\$485 199	\$504 607	\$524 791	\$545 783

(v)

Payment	1	2	3	4
Present value	\$485 199	\$481 587	\$478 002	\$474 443.85

(vi) $\text{P.V.} = \dfrac{\text{F.V.}}{(1+i)^{n-1}}$: $(n-1)$ is used since payments were paid immediately.

$$\Rightarrow = \frac{\$485\,199(1.04)^{n-1}}{(1+0.0478)^{n-1}}$$

(vii) $\Rightarrow \text{P.V.} = 485\,199 + \dfrac{485\,199(1.04)}{(1.0478)} + \dfrac{485\,199(1.04)}{1.0478} + \ldots \dfrac{485\,199(1.04)^{25}}{(1.0478)^{25}}$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$= 485\,199\left[\frac{1-\left(\frac{1.04}{1.0478}\right)^{26}}{1-\left(\frac{1.04}{1.0478}\right)}\right]$$

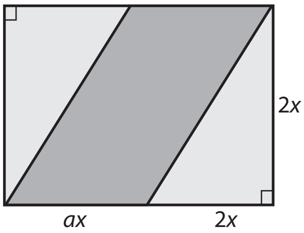
$$= \$11\,500\,000.00$$

(viii) Tax $= \$11.5$ m $- \$7.9$ m $= \$3.6$ m

$$\Rightarrow \quad \frac{3.6 \text{ m}}{11.5 \text{ m}} \times \frac{100}{1} = 31.3\% \text{ tax.}$$

Chapter 6

Exercise 6.1



1. (i) Area of parallelogram $= (ax)(2x)$

$= 2ax^2$

Area of rectangle $= (2x + ax)(2x)$

$= 2x^2(2 + a)$

Fraction $= \dfrac{\cancel{2}a\cancel{x^2}}{\cancel{2x^2}(2 + a)} = \dfrac{a}{2 + a}$

(ii) Area of parallelogram $= \dfrac{4}{5}$ Area of rectangle

$2ax^2 = \dfrac{4}{5}((2x)(2x + ax))$

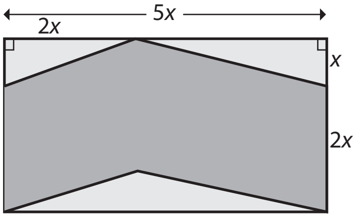
$\Rightarrow \quad 10ax^2 = 16x^2 + 8ax^2$

$\Rightarrow \quad 2ax^2 - 16x^2 = 0$

$\Rightarrow \quad 2x^2(a - 8) = 0$

$\Rightarrow \quad a - 8 = 0$

$\therefore \quad a = 8$



2. (i) Area of dark parallelograms $= 2x \times 3x = 6x^2$

$+ 2x \times 2x = 4x^2$

$= 10x^2$

(ii) Area of lighter section

$=$ Area of rectangle $- 10x^2$

$= 5x.3x - 10x^2 = 15x^2 - 10x^2$

$= 5x^2$

(iii) Dark to Light

$10x^2 : 5x^2$

$2 : 1$

3. Height $= (x - 5)$ cm

Base $= x$ cm

Area $= 52$ cm^2

$\Rightarrow \quad \dfrac{1}{2}(x)(x - 5) = 52$

$\Rightarrow \quad x^2 - 5x = 104$

$x^2 - 5x - 104 = 0$

$(x - 13)(x + 8) = 0$

$\therefore \quad x = 13$ or $x = -8$

Since $x > 0 \therefore x = 13$ cm

$\therefore$ base $= 13$ cm

$\Rightarrow$ height $= 13 - 5 = 8$ cm.

4. $\therefore a^2 + (49 - a)^2 = 41^2$

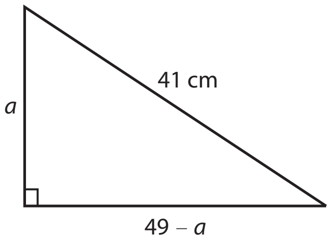
$a^2 + 2401 - 98a + a^2 = 1681$

$2a^2 - 98a + 720 = 0$

$a^2 - 49a + 360 = 0$

$(a - 40)(a - 9) = 0$

$\Rightarrow a = 40$ cm or $a = 9$ cm



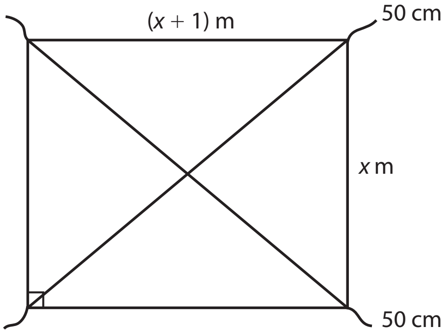
5. $\therefore 2x + 2(x + 1) = 14$

$$\begin{aligned} 4x + 2 &= 14 \\ x &= 3 \\ x + 1 &= 4 \end{aligned}$$

$\therefore$ Diagonal $= \sqrt{4^2 + 3^2} = 5$

$\therefore$ Steel cable $= 2 \times 5 + 4 \times (0.5)$

$= 12$ m



6. Let $A + B + C = 180°$

and $A > B > C$

$\therefore C = \frac{2}{3}B$

and $B = \frac{3}{7}A$

$\Rightarrow C = \frac{2}{3}\left(\frac{3}{7}\right)A$

$= \frac{6}{21}A$

$\therefore A + B + C = 180°$

$\Rightarrow A + \frac{3}{7}A + \frac{6}{21}A = 180°$

$\Rightarrow 21A + 9A + 6A = 180°(21)$

$36A = 180°(21)$

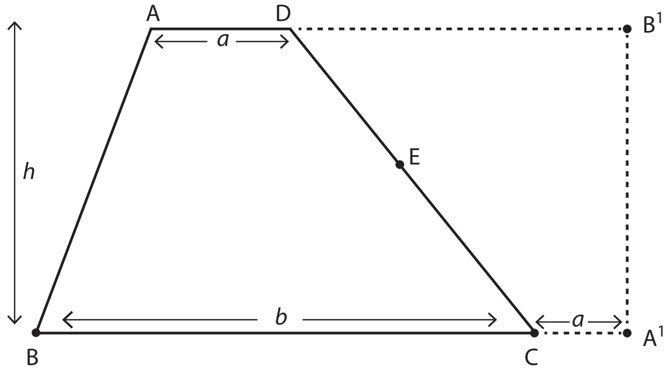
$A = \frac{180°(21)}{36}$

$= 105°$

$\Rightarrow B = \frac{3}{7}(105) = 45°$

$C = \frac{6}{21}(105) = 30°$

7.



(i) A parallelogram

(ii) Area of parallelogram $= (a + b)(h) = (|AD| + |BC|) \times h$

(iii) The trapezium $= \frac{1}{2}$ the area of parallelogram

$= \frac{1}{2}(a + b) \,.\, h$

$= \left(\frac{a + b}{2}\right) . \, h$ = formula for the area of a trapezium.

8. Three times width exceeds twice length by 3 cm.

Let x = width and y = length

$\Rightarrow 3x = 2y + 3$

$\Rightarrow 3x - 2y = 3$ .... A

Four times its length is 12 cm more than its perimeter.

$\Rightarrow 4y = 2(x + y) + 12$

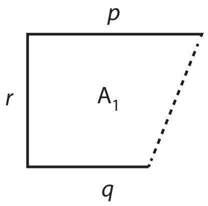
$\Rightarrow 4y = 2x + 2y + 12$

$\Rightarrow 2x - 2y = -12$ .... B

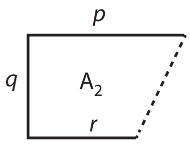
$$\begin{aligned} \text{A: } 3x - 2y &= 3 \\ \text{B: } 2x - 2y &= -12 \\ \text{A} - \text{B: } \quad x \quad &= 15 \text{ cm} \end{aligned}$$

$$\begin{aligned} \Rightarrow 3(15) - 2y &= 3 \\ -2y &= -42 \\ y &= 21 \text{ cm} \end{aligned}$$

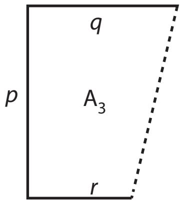
9.



$$A_1 = \left[\frac{|p| + |q|}{2}\right] \cdot |r| = \frac{|p||r| + |q||r|}{2}$$



$$A_2 = \left[\frac{|p| + |r|}{2}\right] \cdot |q| = \frac{|p||q| + |q||r|}{2}$$



$$A_3 = \left[\frac{|q| + |r|}{2}\right] \cdot |p| = \frac{|p||q| + |p||r|}{2}$$

$$|p| > |q| > |r|$$

$$\Rightarrow |p||p| > |p||q| > |p||r| \quad \Rightarrow \quad |p||q| > |p||r|$$

Compare A_1 and A_2, i.e. $\frac{|p||r| + |q||r|}{2}$ and $\frac{|p||q| + |q||r|}{2}$

Since $|p||q| > |p||r|$

$$\Rightarrow \frac{|p||q| + |q||r|}{2} > \frac{|p||r| + |q||r|}{2}$$

$$\Rightarrow A_2 > A_1$$

Compare A_3 and A_2, i.e. $\frac{|p||q| + |p||r|}{2}$ and $\frac{|p||q| + |q||r|}{2}$

$$|p| > |q|$$

$$\Rightarrow |p||r| > |q||r|$$

$$\Rightarrow \frac{|p||q| + |p||r|}{2} > \frac{|p||q| + |q||r|}{2}$$

$$\Rightarrow A_3 > A_2$$

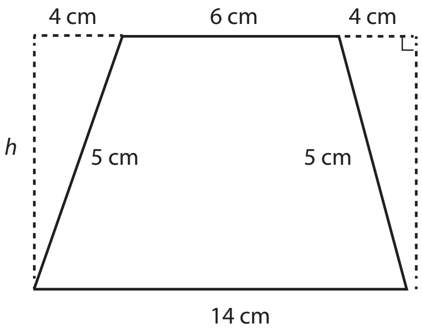
$$\Rightarrow A_3 > A_2 > A_1$$

10. $\Rightarrow h = \sqrt{5^2 - 4^2}$

$= 3$ cm

$\Rightarrow \text{Area} = \left(\frac{6 + 14}{2}\right) \times 3$

$= 30 \text{ cm}^2$



11. (i) $(2a)^2 = a^2 + h^2$

$4a^2 = a^2 + h^2$

$h = \sqrt{3}a$

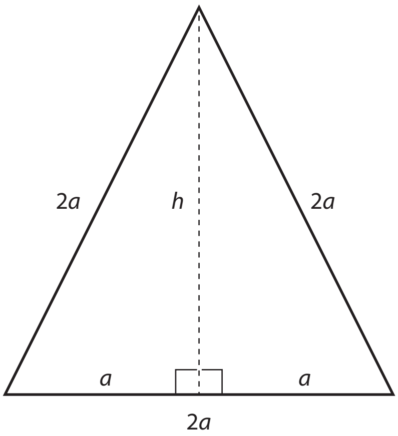
$$\text{Area} = \frac{1}{2} \,.\, 2a \,.\, h$$

$$= a \,.\, \sqrt{3}a = \sqrt{3}a^2$$

$$\Rightarrow \quad \sqrt{3}a^2 = 173$$

$$\Rightarrow \quad a = \sqrt{\frac{173}{\sqrt{3}}} = 9.994 \text{ cm}$$

$$\Rightarrow \quad 2a = 19.99 \text{ cm.}$$



(ii) $h^2 = 10.75^2 - 5.275^2$

$$h = \frac{43\sqrt{3}}{8} = 9.31 \text{ cm}$$

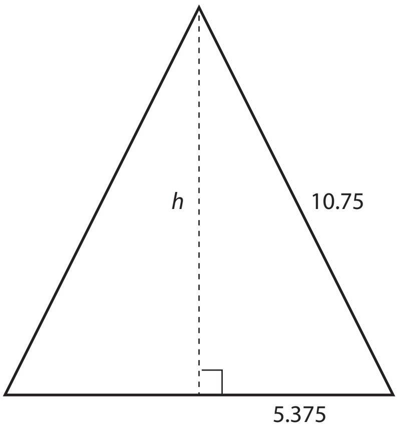
$$A = \frac{1}{2}(10.75) \cdot \frac{43\sqrt{3}}{8}$$

$$= 50.04 \text{ cm}^2$$

$$A = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}(10.75)(10.75)(\sin 60°)$$

$$= 50.04 \text{ cm}^2$$



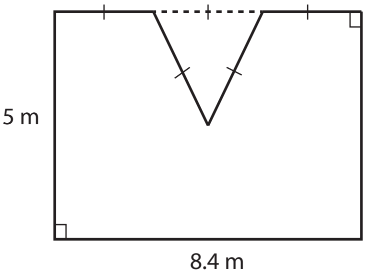
12. Area of rectangle $= (8.4 \times 5) \text{ m}^2$

$$= 42 \text{ m}^2$$

$$\text{Area of triangle} = \frac{1}{2}(2.8)(2.8) \sin 60°$$

$$= 3.3948$$

$$= 38.605 \text{ m}^2$$



13. (i) $|\angle EOD| = \frac{360°}{6} = 60°$

(ii) $|\angle ODE| = |\angle OED|$ (equal radii)

$|\angle ODE| + |\angle OED| = 120°$

$\Rightarrow \quad |\angle ODE| = 60°$

(iii) $\text{Area } \angle EOD = \frac{1}{2}(a)(b) \sin \theta = \frac{1}{2}(5)(5) \sin 60°$

$$= 10.825 \text{ cm}^2$$

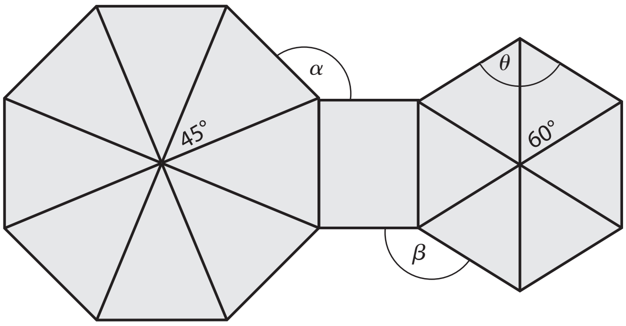
$\Rightarrow$ Area of hexagon ABCDEFA $= 6 \times \text{Area } \Delta EOD$

$$= (6 \times 10.825) \text{ cm}^2$$

$$= 64.95 \text{ cm}^2$$

14. Pentagon − square − hexagon

(i)



Pentagon: internal angle $= \dfrac{360°}{8} = 45°$

Base angles $= \dfrac{180° - 45°}{2} = 67.5°$

$\alpha = 360° - 90° - 2(67.5°)$

$\quad = 135°$

Hexagon: internal angle $= \dfrac{360°}{6} = 60°$

base angles $= \dfrac{180° - 60°}{2} = 60°$

$\theta = 2 \times 60° = 120°$

Square: $\beta = 360° - 90° - 2(60°)$

$= 150°$

(ii) Area: square $= 4 \times 4 = 16\text{ cm}^2$

hexagon $= 6 \times \dfrac{1}{2}(4)(4)\sin 60°$

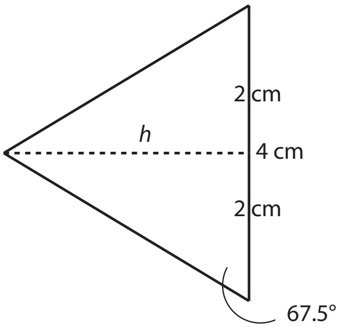
$= 24\sqrt{3}\text{ cm}^2$

pentagon: $h = 2\tan 67.5°$

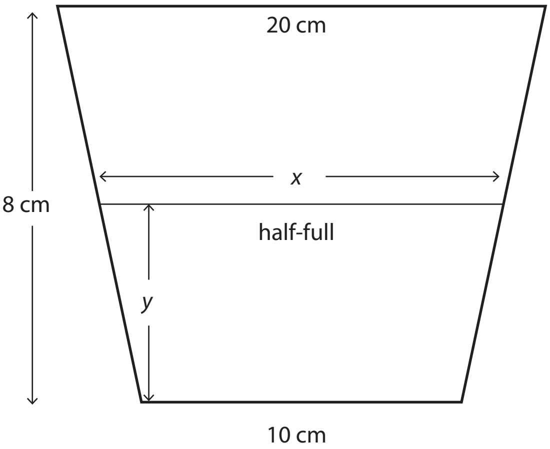
$\Rightarrow$ Area $= 8 \times \dfrac{1}{2}\,.\,4\,.\,(2\tan 67.5°)$

$= 77.25$

$\Rightarrow$ Total area $= 134.8\text{ cm}^2$



15. Note: The skip has a constant depth ⇒ it will be half-full when the two areas on the side are equal.



Area of the top, $A_t = \left(\dfrac{x + 10}{2}\right) \cdot y$

Area of the bottom, $A_b = \left(\dfrac{20 + x}{2}\right)(8 - y)$

Total area $= \left(\dfrac{20 + 10}{2}\right)8 = 120\text{ cm}^2$

$\therefore$ A: $\left(\dfrac{x + 10}{2}\right)y = 60$

$\Rightarrow \quad xy + 10y = 120 \quad \Rightarrow \quad xy = 120 - 10y$

$x = \dfrac{120 - 10y}{y}$

and B: $\left(\dfrac{20 + x}{2}\right)(8 - y) = 60$

$\Rightarrow 160 - 20y + 8x - xy = 120$

$\Rightarrow 160 - 20y + 8\left(\dfrac{120 - 10y}{y}\right) - \left(\dfrac{120 - 10y}{\cancel{y}}\right)\cancel{y} = 120$

$\Rightarrow -20y^2 + 960 - 80y - 120y + 10y^2 = -40y$

$\Rightarrow y^2 + 16y - 96 = 0$

$a = 1, b = 16, c = -96$

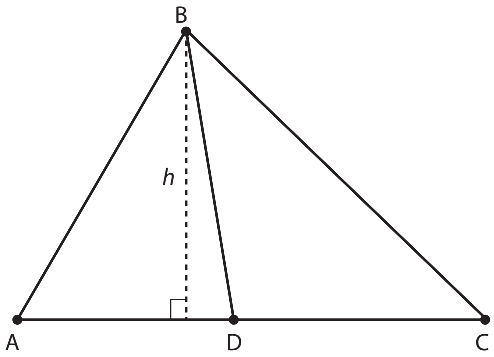
$\therefore\ y = \dfrac{-16 \pm \sqrt{16^2 - 4(1)(-96)}}{2(1)} = -8 \pm 4\sqrt{10}$

$\therefore\ y = 4.65 \text{ cm}$

$\Rightarrow\ x = \dfrac{120 - 10(4.65)}{4.65}$

$= 15.8 \text{ cm}$

16.



(i) $\text{Area } \Delta ABD : \text{Area } \Delta CBD = \frac{1}{2}|AD| \,.\, h : \frac{1}{2}|DC| \,.\, h$

$= |AD| : |DC|$

(ii) ABCD is a trapezium

$\Rightarrow$ AB is parallel to DC.

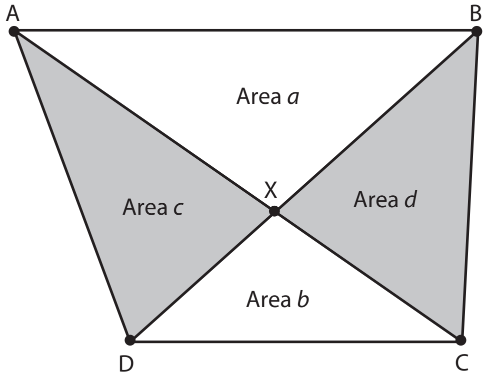
$=$ Area of ΔADC = Area ΔBDC (same base and same perpendicular height)

$\Rightarrow$ Area c + Area b = Area d + Area b

$\Rightarrow$ Area c = Area d

$\Rightarrow$ $c = d$

(iii) Area of trapezium $= a + b + c + d$



But $\dfrac{c}{a} = \dfrac{DX}{XB} = \dfrac{b}{d}$

$\Rightarrow\ ab = cd$

$\Rightarrow\ ab = c^2$

$\Rightarrow\ c = \sqrt{ab}$

$\Rightarrow$ Area of trapezium $= a + b + c + d$

$= a + b + 2c$

$= a + b + 2\sqrt{ab}.$

Exercise 6.2

1. (i) Perimeter (curved) $= \dfrac{80}{360}(2 \,.\, \pi \,.\, 7) + \dfrac{80}{360}(2 \,.\, \pi \,.\, 10)$

$= 23.73 \text{ cm}$

Total perimeter $= 23.73 + 6 \text{ cm}$

$= 29.7 \text{ cm}$

(ii) Area $= \dfrac{80}{360}(\pi \,.\, 10^2) - \dfrac{80}{360}(\pi \,.\, 7^2)$

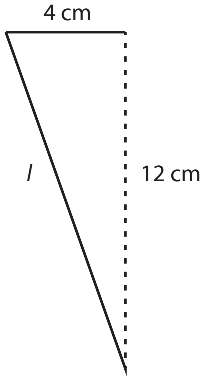
$= 35.6 \text{ cm}^2$

2. (i) Area $= \frac{1}{2} . 8 . 12 + \frac{1}{2} . \pi . 4^2$

$= 73 \text{ cm}^2$

(ii) Perimeter $= 2\sqrt{160} + \pi . 4$

$= 38 \text{ cm}$



$l = \sqrt{12^2 + 4^2}$

$= \sqrt{160}$

3. (a) Area $= \dfrac{\pi . r^2}{2}$

(b) Area $= \pi R^2 - \pi r^2$

$= \pi(R^2 - r^2)$

(c) Area = Area of square of side $(x + 2a)$ − Area of circle of radius (a)

Area $= (x + 2a)^2 - \pi a^2$

(d) Area = Area of rectangle + Area of circle

$= ab + \pi\left(\frac{a}{2}\right)^2$

$= ab + \dfrac{\pi a^2}{4}$

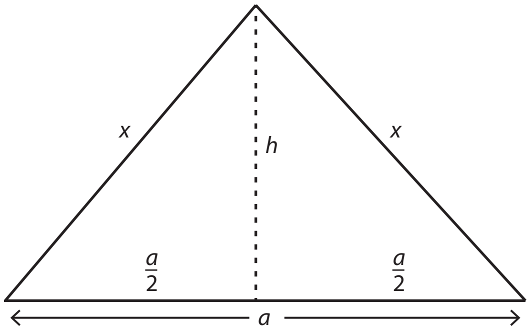
(e) Area $= \frac{1}{2} \times a \times h$

$= \frac{1}{2}(a)\left(\sqrt{x^2 - \frac{a^2}{4}}\right)$

(f) Area $= \frac{1}{2}(a)(a) \sin 60°$

$= \dfrac{a^2}{2} \cdot \dfrac{\sqrt{3}}{2}$

$= \dfrac{\sqrt{3}a^2}{4}$

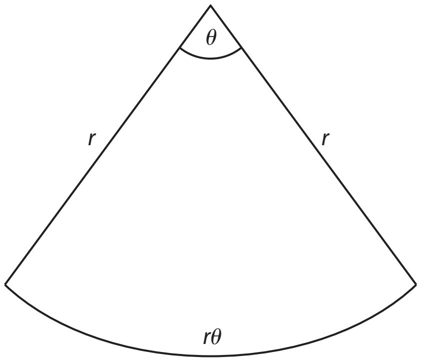


$h = \sqrt{x^2 - \frac{a^2}{4}}$

4. $P = 2r + r\theta$

$= r(2 + \theta)$

$\Rightarrow r = \dfrac{P}{2 + \theta}$



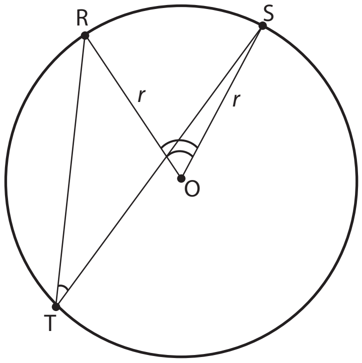
5. $|\angle RTS| = 0.4$ radians

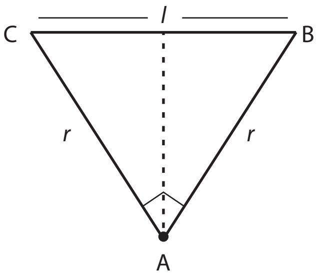
$\Rightarrow |\angle ROS| = 2|\angle RTS|$

$= 0.8$ radians

$\Rightarrow |RS| = 8.5 \times (0.8)$

$= 6.8$ cm





6. $l^2 = r^2 + r^2$

$l = \sqrt{2}r$

(i) $\Rightarrow$ Area of square BCDE $= l^2$

$= 2r^2$

(ii) Area of shaded section $= \dfrac{\text{Area of circle} - \text{Area of square}}{4}$

$= \dfrac{\pi r^2 - 2r^2}{4}$

$= \dfrac{r^2}{4}(\pi - 2)$

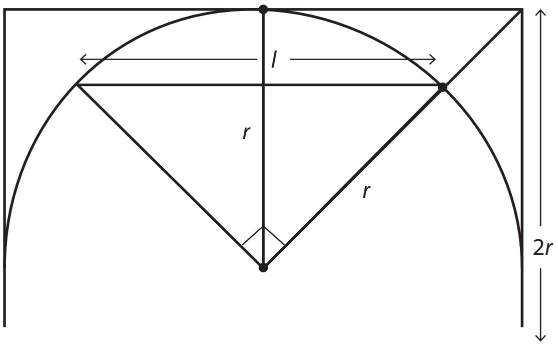
7. $80 = 2\pi r$

$\Rightarrow \quad r = \dfrac{80}{2\pi} = 12.73 \text{ cm}$

Area $= \pi . r^2 = \pi . (12.73)^2$

$= 509.30 \text{ cm}^2.$

Because the radius depends on π, i.e. an irrational number that cannot be expressed as a faction.



8. (i) Let circle radius $= r$.

$\Rightarrow$ Area of outer square $= (2r)(2r) = 4r^2$

$l^2 = r^2 + r^2 = 2r^2$

$l = \sqrt{2}r$

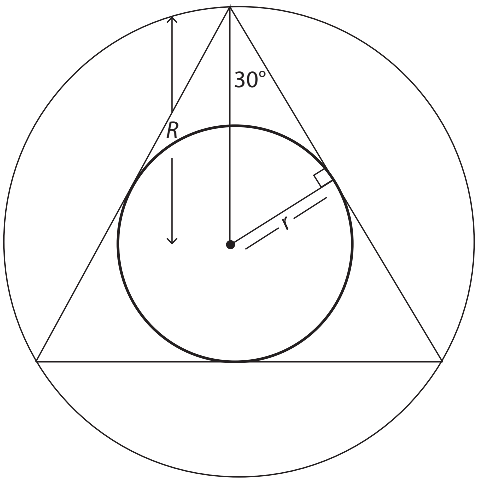
$\Rightarrow$ Area of inner square

$= (l)(l)$

$= (\sqrt{2}r)(\sqrt{2}r) = 2r^2$

$\therefore$ $\text{Area}_{\text{inner}} : \text{Area}_{\text{outer}} = 2r^2 : 4r^2$

$= 1 : 2$



(ii) $\sin 30° = \dfrac{r}{R}$

$\Rightarrow r = R . \sin 30°$

$= R\left(\dfrac{1}{2}\right)$

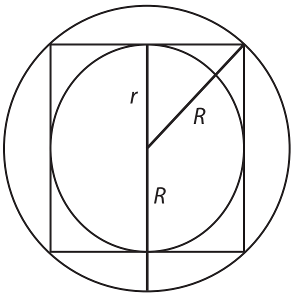
$= \dfrac{R}{2}$

$\Rightarrow \text{Area}_{\text{circumcircle}} : \text{Area}_{\text{incircle}} = \pi R^2 : \pi r^2$

$= \pi R^2 : \dfrac{\pi R^2}{4}$

$= 1 : \dfrac{1}{4}$

$= 4 : 1$



9. Area of Trapezium $= \left(\dfrac{2 + 5}{2}\right)h.$

where $h = 2R$.

Given $\quad 12 = 2\pi r$

$r = \dfrac{6}{\pi}$

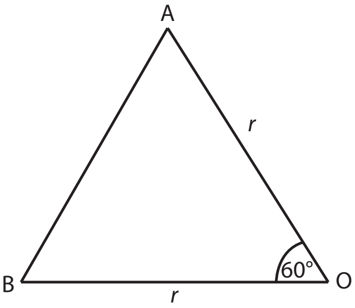
$R^2 = \left(\dfrac{6}{\pi}\right)^2 + \left(\dfrac{6}{\pi}\right)^2 = \dfrac{2 \times 36}{\pi^2} = \dfrac{72}{\pi^2}$

$\Rightarrow \quad R = \sqrt{\dfrac{72}{\pi^2}} = \dfrac{6\sqrt{2}}{\pi}$

$\therefore$ $\text{Area}_{\text{trapezium}} = \left(\dfrac{7}{\cancel{2}}\right) . \cancel{2} . \dfrac{6\sqrt{2}}{\pi}$

$= \dfrac{42\sqrt{2}}{\pi}$

10. (i) $\angle AOB = \pi - \frac{2\pi}{3} = \frac{\pi}{3} = 60°$



(ii) $\angle OAB = \angle ABO$

$\Rightarrow \angle OAB = 60° = \angle ABO$

$\Rightarrow |AB| = r.$

$\therefore$ Circumference $= |AB| + 2|BO| + \text{sector}$

$$= 2 + 2(2) + 2\left(\frac{2\pi}{3}\right)$$

$$= 6 + \frac{4\pi}{3}$$

(iii) Area $= \text{Area}_{\text{triangle}} + A_{\text{sector}}$

$$= \frac{1}{2} \cdot (2)(2)\sin 60° + \frac{1}{2}(2)(2)\left(\frac{2\pi}{3}\right)$$

$$= 2 \cdot \frac{\sqrt{3}}{2} + \frac{4\pi}{3}$$

$$= \sqrt{3} + \frac{4\pi}{3}$$

(iv) $\text{Area}_{\text{non-shaded}} = \text{Area}_{\text{semi-circle}} - \left(\sqrt{3} + \frac{4\pi}{3}\right)$

$$= \frac{\pi(2)^2}{2} - \left(\sqrt{3} + \frac{4\pi}{3}\right)$$

$$= 2\pi - \frac{4\pi}{3} - \sqrt{3}$$

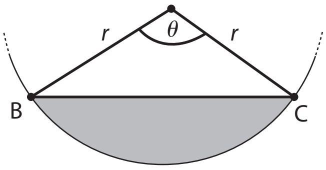
$$= \frac{2\pi}{3} - \sqrt{3}$$

11. $\text{Area}_{\text{shaded}} = \text{Area}_{\text{sector}} - \text{Area}_{\text{triangle}}$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}(r)(r)\sin\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta).$$

At $\theta = \frac{\pi}{2}$:



$\text{Area}_{\text{minor segment}} = \frac{1}{2}r^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$$= \frac{r^2}{2}\left(\frac{\pi}{2} - 1\right)$$

$\text{Area}_{\text{major segment}} = \pi r^2 - \text{Area}_{\text{minor segment}}$

$$= \pi r^2 - \frac{r^2}{2}\left(\frac{\pi}{2} - 1\right)$$

$$= \pi r^2 - \frac{\pi r^2}{4} + \frac{r^2}{2}$$

$$= \frac{3\pi r^2}{4} + \frac{r^2}{2}$$

$$= r^2\left(\frac{3\pi}{4} + \frac{1}{2}\right)$$

Ratio: $\text{Area}_{\text{major segment}} : \text{Area}_{\text{minor segment}} = r^2\left(\frac{3\pi}{4} + \frac{1}{2}\right) : \frac{r^2}{2}\left(\frac{\pi}{2} - 1\right)$

$$= \frac{3\pi + 2}{4} : \frac{\pi - 2}{4}$$

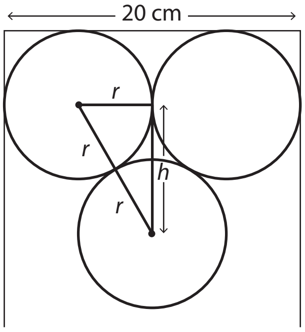
$$= 3\pi + 2 : \pi - 2$$

12. Radius of disc $= \frac{20}{4} = 5$

Area of 5 discs $= 5 \times (\pi 5^2)$
$= 125\pi$

$$\begin{aligned}\text{Height of frame} &= 2(h + r)\\ &= 2(\sqrt{3}r + r)\\ &= 2r(\sqrt{3} + 1)\\ &= 10(\sqrt{3} + 1)\end{aligned}$$

$$\begin{aligned}h &= \sqrt{(2r)^2 - r^2}\\ &= \sqrt{3}r\end{aligned}$$



$$\begin{aligned}\text{Area of frame} &= 20 \times 10(\sqrt{3} + 1)\\ &= 546.41\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of remaining space} &= 546.41 - 125\pi\\ &= 153.71\ \text{cm}^2\end{aligned}$$

13. $\text{Area}_{\text{sector}} = \frac{1}{2}r^2\theta$ $\qquad$: $\text{Area}_{\text{perimeter}} = 2r + r\theta$

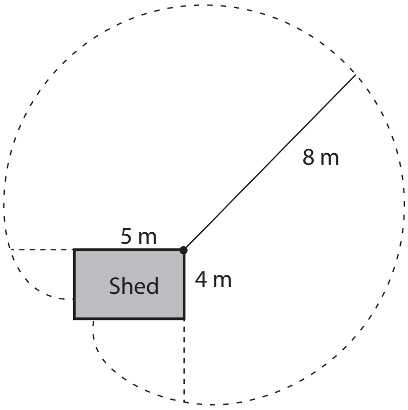
$48 = \frac{1}{2}(r^2)\theta$ $\qquad$ $28 = r(2 + \theta)$

$\Rightarrow \quad \theta = \frac{96}{r^2}$ $\qquad$ $28 = r\left(2 + \frac{96}{r^2}\right)$

$$\begin{aligned}28r^2 &= 2r^3 + 96r\\ 2r^3 - 28r^2 + 96r &= 0\\ 2r^2 - 28r + 96 &= 0\\ (2r - 12)(r - 8) &= 0\end{aligned}$$

$\Rightarrow r = 6$ cm or $r = 8$ cm

14. (i) and (ii)



(iii) Area (I) $= \frac{1}{4}(\pi 4^2) = 4\pi$

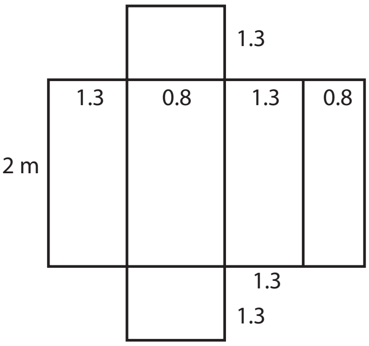
Area (II) $= \frac{3}{4}(\pi 8^2) = 48\pi$

Area (III) $= \frac{1}{4}(\pi 3^2) = \frac{9\pi}{4}$

$$\begin{aligned}\text{Area}_{\text{TOTAL}} &= 170.43\\ &= 170\ \text{m}^2\end{aligned}$$

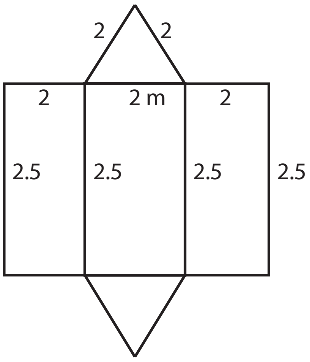
Exercise 6.3

1. (a)



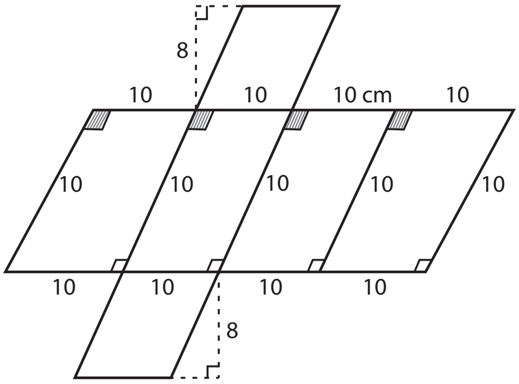
$$\begin{aligned}\text{Area} &= 2(1.3 \times 2) &&= 5.2\\ &+2(0.8 \times 2) &&= 3.2\\ &+2(0.8 \times 1.3) &&= \underline{2.08}\\ & && \ 10.48\\ & &&= 10.5\ \text{m}^2\end{aligned}$$

(b)



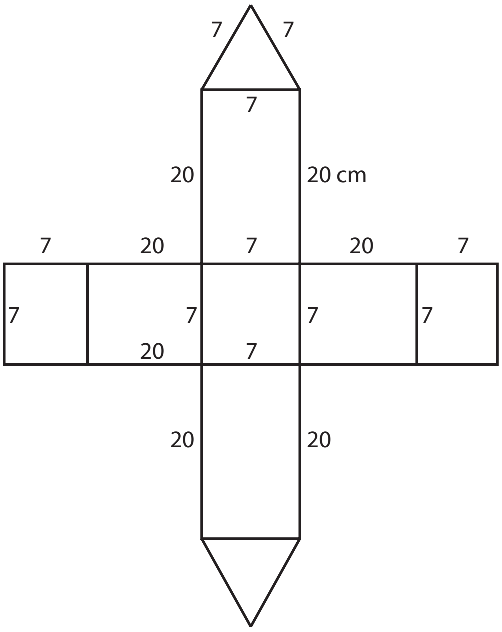
$$\text{Area of Triangle} = \frac{1}{2}(2)(2)\sin 60^\circ$$
$$= 2 \cdot \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}$$
$$\text{Area of rectangles} = 2(2.5) = 5$$
$$\text{Total area} = 2\sqrt{3} + 3(5)$$
$$= 18.46 \text{ m}^2$$
$$= 18.5 \text{ m}^2$$

(c)



$$\text{Area of square} = 10 \times 10$$
$$= 100 \text{ cm}^2$$
$$\text{Area of parallelogram} = 8 \times 10 = 80$$
$$\text{Area (Total)} = 4(100) + 2(80)$$
$$= 560 \text{ cm}^2$$

(d)



$$\text{Area}_{\text{triangle}} = \frac{1}{2}(7)(7)\sin 60^\circ$$
$$= \frac{49\sqrt{3}}{4}$$
$$\text{Area}_{\text{rectangle}} = 7 \times 20 = 140$$
$$\text{Area}_{\text{square}} = 7 \times 7 = 49$$
$$\text{Area}_{\text{TOTAL}} = \left(2 \times \frac{49\sqrt{3}}{4}\right) + (4 \times 140) + (3 \times 49)$$
$$= 749.43$$
$$= 749.4 \text{ cm}^2$$

2. (i) $\text{Area }(a) = 4\pi r^2$

$= 4\pi(10^2) = 1257 \text{ mm}^2$

$\text{Area }(b) = \pi rl + \pi r^2$

$= \pi(10)(30) + \pi(10)^2$

$= 1257 \text{ mm}^2$

$\text{Area }(c) = 2\pi r^2 + 2\pi rh$

$= 2\pi 3^2 + 2\pi(3)(12)$

$= 282.74$

$= 283 \text{ mm}^2$

$\text{Area }(d) = 2\pi r^2 + \pi r^2$

$= 3\pi r^2$

$= 3\pi(15)^2$

$= 2120.57$

$= 2121 \text{ mm}^2$

$$\text{Area } (e) = \frac{1}{4}(4\pi r^2) + \frac{1}{2}(\pi r^2) + \frac{1}{2}(\pi r^2)$$
$$= 2\pi r^2$$
$$= 2\pi(12)^2$$
$$= 904.77$$
$$= 905 \text{ mm}^2$$

$$\text{Area } (f) = \frac{1}{2}(2\pi rh) + 2\left(\frac{\pi r^2}{2}\right) + 2rh$$
$$= \pi rh + \pi r^2 + 2rh$$
$$= \pi(12)(32) + \pi(12)^2 + 2(12)(32)$$
$$= 2426.76$$
$$= 2427 \text{ mm}^2$$

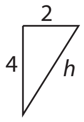
$$\text{Area } (g) = \frac{1}{2}(\pi r^2) + \frac{1}{2}(\pi rl) + \frac{1}{2}(2r \times h)$$
$$h = \sqrt{26^2 - 10^2}$$
$$= 24 \text{ mm}$$
$$\text{Area} = \frac{\pi\,.\,(10)^2}{2} + \frac{\pi(10)(20)}{2} + 10 \times 24$$
$$= 805.49 \text{ mm}^2$$
$$= 805 \text{ mm}^2$$

$$\text{Area } (h) = 2\left(\frac{1}{2}\right)(16)(12) + (11)(16) + (12)(11) + 11(l)$$
$$= 192 + 176 + 132 + 220$$

$\left[\text{Note: } l = \sqrt{16^2 + 12^2} = 20\right]$ $= 720 \text{ mm}^2$

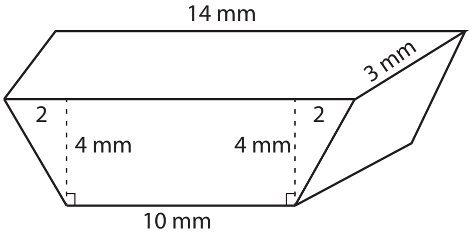
$$\text{Area } (i) = 2\left(\frac{10 + 14}{2}\right)4 + (10)(3) + 14(3) + 2(3)(h)$$

Note:



$\Rightarrow h = \sqrt{16 + 4}$
$h = \sqrt{20}$

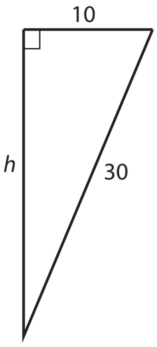
$$= 96 + 30 + 42 + 6\sqrt{20}$$
$$= 194.83$$
$$= 195 \text{ mm}^2$$



(i) $f, d, \frac{b}{a}, e, g, h, c, i$.

(ii) $\text{Volume } (a) = \frac{4}{3}\pi r^3 = \frac{4}{3} \cdot \pi \cdot 10^3$
$$= 4188.79$$
$$= 4189 \text{ mm}^3$$

$$\text{Volume } (b) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(10)^2\left(20\sqrt{2}\right)$$
$$= 2961.9$$
$$= 2962 \text{ mm}^3$$



$h = \sqrt{30^2 - 10^2}$
$= 20\sqrt{2}$

$$\text{Volume } (c) = \pi r^2 h$$
$$= \pi\,.\,3^2\,.\,12$$
$$= 339 \text{ mm}^3$$

$$\text{Volume } (d) = \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)\pi r^3$$
$$= \frac{2}{3}\pi \cdot (15)^3$$

$= 7068.58$

$= 7069 \text{ mm}^3$

Volume (e) $= \frac{1}{4}\left(\frac{4}{3}\right)\pi r^3$

$= \frac{1}{3} \cdot \pi \cdot 12^3$

$= 1809.55$

$= 1810 \text{ mm}^3$

Volume (f) $= \frac{1}{2}(\pi r^2 h)$

$= \frac{1}{2} \cdot \pi \cdot (12)^2 \cdot 32$

$= 7238.229$

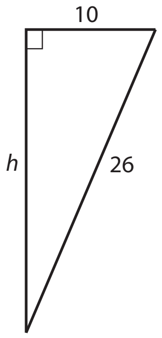
$= 7238 \text{ mm}^3$

Vol (g) $= \frac{1}{2}\left(\frac{1}{3}\right)\pi r^2 h$

$= \frac{1}{6} \cdot \pi \cdot 10^2 \cdot 24$

$= 1256.63$

$= 1257 \text{ mm}^3$



$h = \sqrt{26^2 - 10^2}$

$= 24$

Vol (h) $= \frac{1}{2}(12 \times 16) \times 11$

$= 1056 \text{ mm}^3$

Vol (i) $= \left(\frac{14 + 10}{2}\right) \times 4 \times 3$

$= 144 \text{ mm}^2$

(ii) $f, d, a, b, e, g, h\ c, i$

3. (i) Vol $= \text{Area}_{\text{trapezium}} \times \text{Depth}$

$= \left(\frac{2.5 + 1.6}{2}\right) \times 1.2 \times 2$

$= 4.92 \text{ m}^3$

(ii) Volume pick-up = €4.92 × 80 = €393.60

Weight pick-up = 1.3 tonnes

= 1300 kg

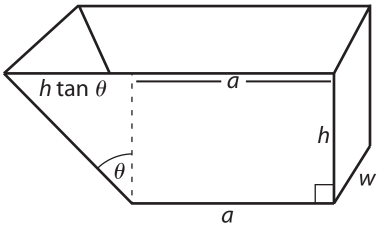
= €13(×100) × 30

= €390

Weight pick-up option is better value.

(iii) $\text{Volume}_{\text{skip}} = \left(\frac{a + a + h \tan\theta}{2}\right) . h . w$

$= \left(\frac{2a + h \tan\theta}{2}\right) . h . w$



(iv) $\theta = 45°$ Volume $= \left(\frac{2a + 1.2 \tan 45°}{2}\right) \times 1.2 \times 2$

$\therefore$ $4.92 = \left(\frac{2a + 1.2}{2}\right) \times 2.4$

$\therefore$ $2a = 2.9$

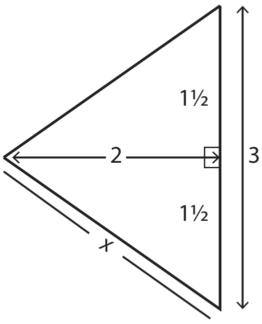
$\therefore$ $a = 1.45 \text{ m}$

$\Rightarrow$ bottom $= 1.45 \text{ m} = 1.5 \text{ m}$

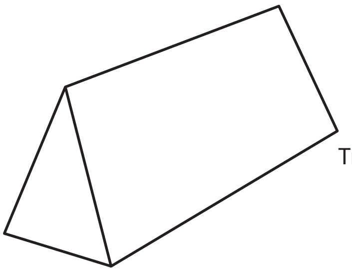
$$\begin{aligned}\text{Top} &= a + h\tan\theta \\ &= 1.45 + 1.2\tan 45^\circ \\ &= 2.65\text{ m} = 2.7\text{ m}\end{aligned}$$

4. (i) $x = \sqrt{2^2 + \left(\frac{3}{2}\right)^2}$

$= 2.5$ cm



(ii)

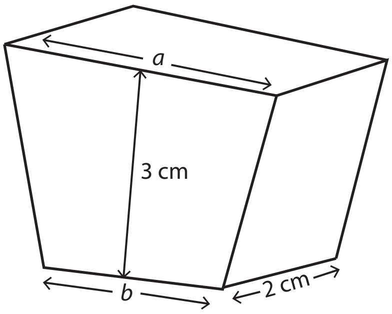


(iii) Volume $= \text{Area}_{\text{triangle}} \times \text{Length}$

$= \frac{1}{2}(3 \times 2) \times 8$

$= 24\text{ cm}^3$

(iv)



Let height = 3 cm.

Let width = 2 cm.

$\Rightarrow$ Area of Trapezium $= \frac{24}{2} = 12$

$\Rightarrow$ Sum of lengths of parallel sides $= a + b$

$\Rightarrow \left(\frac{a+b}{2}\right) \times 3 = 12$

$a + b = 8$

Let $a = 5, b = 3$

$\Rightarrow$ Base = 3 cm, Top = 5 cm, Height = 3 cm, Width = 2 cm

5. (i) Volume of sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \cdot \pi \cdot 5^3$

$= 523.599\text{ cm}^3$

Volume of cube $= 10 \times 10 \times 10$

$= 1000\text{ cm}^3$

Volume taken away $= 1000 - 523.599$

$= 476.4\text{ cm}^3$

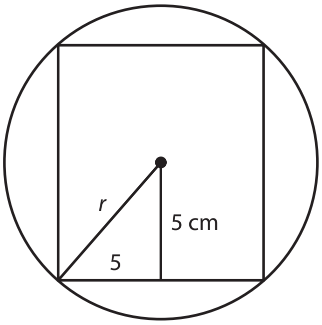
(ii) Volume of sphere enclosing cube

$= \frac{4}{3}\pi . r^3$

$= \frac{4}{3}\pi . (\sqrt{50})^3$

$= 1480.96$

$= 1481.0\text{ cm}^3$



$r = \sqrt{5^2 + 5^2}$

$= \sqrt{50}$

6. 8 litres per minute

$= 8000 \text{ cm}^3$ per minute

(i) Shortest time ⇒ smallest tank and fastest drain-rate

$\Rightarrow 1.0 \text{ m} \longrightarrow 0.95 \text{ m}$

$1.5 \text{ m} \longrightarrow 1.45 \text{ m}$

$1.3 \text{ m} \longrightarrow 1.25 \text{ m}$

$$\Rightarrow \quad \text{volume} = (0.95) \times (1.45) \times (1.25)$$

$$= 1.721875 \text{ m}^3$$

$$= 1721875 \text{ cm}^3$$

Drain-rate $= 8.49l$ per minute

$= 8490 \text{ cm}^3$ per minute

$$\therefore \text{Time} = \frac{1721875}{8490} \text{ min}$$

$$= 202.81 \text{ min}$$

$$= 3 \text{ hrs } 23 \text{ min}$$

(ii) Longest time ⇒ biggest time and slowest drain-rate

$1.0 \text{ m} \longrightarrow 1.04 \text{ m}$

$1.5 \text{ m} \longrightarrow 1.54 \text{ m}$

$1.3 \text{ m} \longrightarrow 1.34 \text{ m}$

$$\text{Volume} = (1.04) \times (1.54) \times (1.34)$$

$$= 2.146144 \text{ m}^3$$

$$= 2146144 \text{ cm}^3$$

Drain-rate $= 7.5$ litres per minute

$= 7500 \text{ cm}^3$ per minute

$$\therefore \text{Time} = \frac{2146144}{7500}$$

$$= 286.15$$

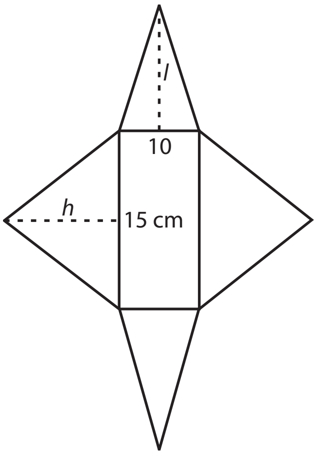
$$= 4 \text{ hrs } 46 \text{ minutes}$$

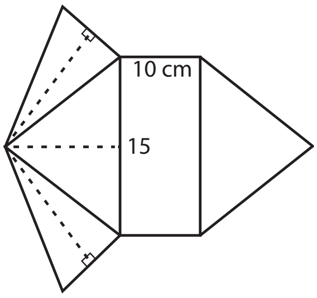
7. (i) $\text{Volume} = \frac{1}{3}\text{Area}_{\text{base}} \times \text{height}$

$$= \frac{1}{3}(10 \times 15) \times 14$$

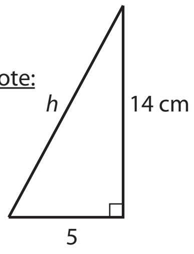
$$= 700 \text{ cm}^3$$

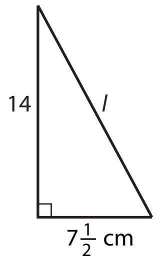
(ii)





Note:





$$h = \sqrt{5^2 + 14^2} \quad : \quad l = \sqrt{14^2 + \left(7\frac{1}{2}\right)^2}$$

$$= 14.866 \qquad l = 15.88$$

$$\text{Area} = 15 \times 10 + 2\left(\frac{1}{2} \times 15 \times 14.86\right) + 2\left(\frac{1}{2} \times 10 \times 15.88\right)$$

$$= 150 + 222.99 + 158.8$$

$$= 531.79$$

$$= 532 \text{ cm}^2$$

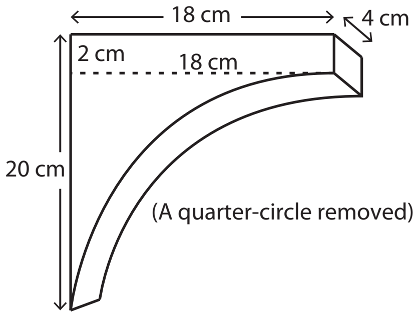
8. Note: radius of circle = 18 cm

Volume = volume$_{(\text{cuboid})}$ + volume$_{\left(\text{cuboid} - \frac{1}{4}\text{cylinder}\right)}$

$$= (18 \times 4 \times 2) + \left(18 \times 18 \times 4 - \frac{1}{4}\pi \, . \, 18^2 \, . \, 4\right)$$

$$= 422.124 \text{ cm}^3$$

$$= 422 \text{ cm}^3$$



$$\text{Surface Area} = 18 \times 4 + 20 \times 4 + 2 \times 4 + \frac{1}{4}(2\pi(18)(4))$$

$$+ 2(18 \times 2) + 2\left((18)(18) - \frac{1}{4}\pi(18)^2\right)$$

$$= 484.16$$

$$= 484 \text{ cm}^3$$

9. $x, y, z \rightarrow$ lengths

$\pi, a \rightarrow$ numbers.

(a) $\pi x^2 + \pi y^2 + \pi z^2$

$= \pi(x^2 + y^2 + z^2)$

x^2 = length × length = area

y^2 = length × length = area

z^2 = length × length = area

$\Rightarrow \pi(x^2 + y^2 + z^2)$ = area

(b) $ax + \pi y$

= length + length = length

(c) axz = number × length × length

= area.

(d) $a\pi \, . \, y$ = number × number × length

= length.

(e) $axy + \pi az$ = number × length × length

+ number × number × length

= Area + Length

(f) $ax + xy$ = number × length

+ length × length

= Length + Area

(g) $axyz$ = number × length × length × length

= volume.

(h) $x^2y + y^2z + z^2x$ = length^2 × length + length^2 × length

+ length^2 × length = volume

10. (i) $A \,.\, x = z^3$

$\Rightarrow (\text{length})^2 \,.\, \text{length} = (\text{length})^3$, consistent.

(ii) $x = \dfrac{V}{Ay}$

$\Rightarrow \text{length} = \dfrac{(\text{length})^3}{(\text{length})^2 \,.\, \text{length}}$, inconsistent.

(iii) $V = xy + z$

$= (\text{length})^3 = (\text{length})(\text{length}) + (\text{length})$, inconsistent.

$= (\text{length})^2 + \text{length}$

(iv) $A = x^2 + y^2 + z^2$

$\Rightarrow (\text{length})^2 = (\text{length})^2 + (\text{length})^2 + (\text{length})^2$

$= 3(\text{length})^2$, consistent.

(v) $V = A(x + y + z)$

$(\text{length})^3 = (\text{length})^2 (\text{length} + \text{length} + \text{length})$

$= (\text{length})^2 \,.\, 3 \text{ length}$

$= 3(\text{length})^3$, consistent.

(vi) $A = \dfrac{V}{x} + y$

$(\text{length})^2 = \dfrac{(\text{length})^3}{\text{length}} + \text{length}$

$= (\text{length})^2 + \text{length}$, inconsistent.

(vii) $x = y + z$

Length = length + length

= 2 length, consistent.

11. (i) True volume $= 122 \times 75 \times 53 = 484950$

Error $= 485000 - 484950$

$= 50$

% Error $= \dfrac{50}{484950} \times 100\%$

$= 0.01\%$

(ii) Least possible volume $= 121.5 \times 74.5 \times 52.5$

$= 475216.875$

Greatest possible volume $= 122.5 \times 75.5 \times 53.5$

$= 494808.125$

The tolerance interval is then from 475216.875 cm^3 to 494808.875 cm^3.

The midpoint of this interval is

$$\frac{475216.875 + 494808.125}{2} = 485012.5 \text{ cm}^3$$

The tolerance interval can then be written

$(485012.5 \pm 9795.625) \text{ cm}^3$.

12. Minimum possible distance = (55.5 km/h) × (2.35 hours)

= 130.425 km

Maximum possible distance = (56.5 km/h) × (2.45 hours)

= 138.425 km

13. (i) $V_p = \frac{1}{3}(a \times a) \times h$

$= \frac{1}{3}a^2 h$

If $a = 6$ cm and $h = 7$ cm,

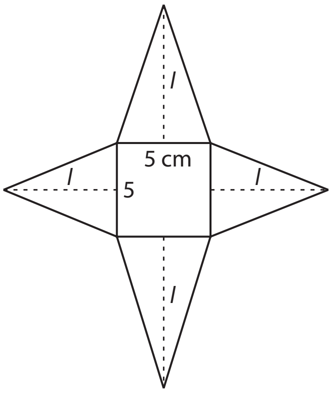
$\Rightarrow V = \frac{1}{3}(6)^2(7)$

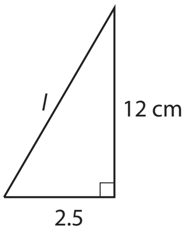
$= 84 \text{ cm}^3$

(ii) If V = 100 cm^3, a = 5 cm.

$$\Rightarrow 100 = \frac{1}{3}(5)^2 . h$$

$$\Rightarrow \quad h = 12 \text{ cm}$$





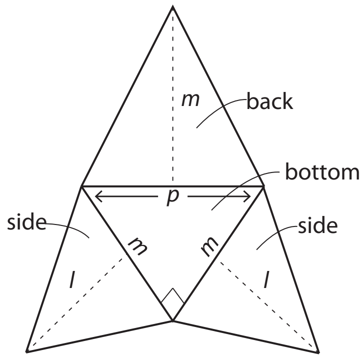
$$l = \sqrt{12^2 + (2.5)^2}$$
$$= 12.25$$

$$A = (5) \times (5) + 4\left[\frac{1}{2} \times 5 \times 12.25\right]$$
$$= 147.5$$
$$= 148 \text{ cm}^2$$

(iii) $V_p = \frac{1}{3} \cdot a^2 \cdot h$

If $a = m$, and $h = m$, and $V = \frac{1}{2}V_p$,

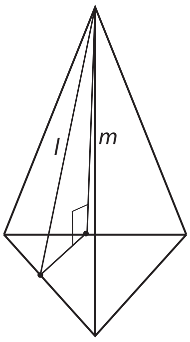
$$\Rightarrow \quad V = \frac{1}{2}\left[\frac{1}{3} \cdot m^2 \cdot m\right]$$
$$= \frac{m^3}{6}$$



Note: $l = \sqrt{m^2 + \left(\frac{m}{2}\right)^2}$

$$= \frac{\sqrt{5}}{2}m$$

$$p = \sqrt{m^2 + m^2}$$
$$= \sqrt{2}m$$



Area = $A_{back} + A_{bottom} + 2A_{sides}$

$$= \frac{1}{2}(\sqrt{2}m)m + \frac{1}{2}(m)^2 + 2\left(\frac{1}{2}m \cdot \frac{\sqrt{5}}{2}m\right)$$
$$= \frac{\sqrt{2}}{2}m^2 + \frac{1}{2}m^2 + \frac{\sqrt{5}}{2}m^2$$
$$= \frac{m^2}{2}[1 + \sqrt{2} + \sqrt{5}]$$

14. (i) (a) $V_{box} = (14) \times (14) \times (14)$

$$= 2744 \text{ cm}^3$$

(b) $V_{sphere} = \frac{4}{3}\pi \cdot (7)^3$

$$= 1437\tfrac{1}{3} \text{ cm}^3$$

(c) $V_{unoccupied} = \left(2744 - 1437\tfrac{1}{3}\right) \text{ cm}^3$

$$= 1306\tfrac{2}{3} \text{ cm}^3$$

$$\% \text{ unoccupied} = \frac{1306\frac{2}{3}}{2744}$$
$$= 47.62\%$$
$$= 48\%$$

(ii) $V_{\text{cylinder}} = \pi\,.\,(7)^2(14)$

$= 215\text{ cm}^3$

$V_{\text{unoccupied}} = 2155.13 - 1437.3\dot{3}$

$= 717.8\text{ cm}^3$

$\%\text{ unoccupied} = \dfrac{717.8}{2155.13}$

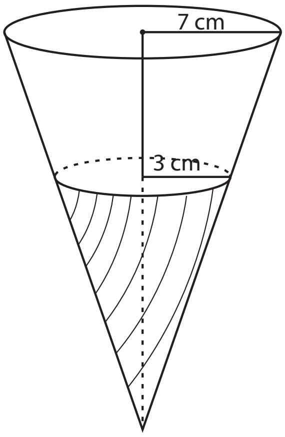
$= 33.3\%$

∴ less space is unoccupied in cylinder.

15. $\Rightarrow \dfrac{h-6}{3} = \dfrac{h}{7}$

$7h - 42 = 3h$

$h = 10.5\text{ cm}$



$\Rightarrow\quad V_{\text{large cone}} = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi \cdot 7^2 \cdot (10.5)$

$= 171.5\,\pi\text{ cm}^3$

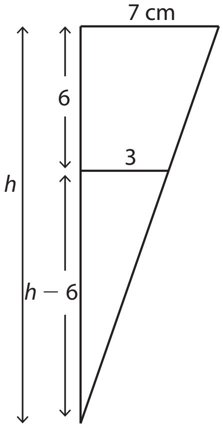
$V_{\text{small cone}} = \dfrac{1}{3}\pi \cdot 3^2 \cdot (4.5)$

$= 13.5\pi$

$V_{\text{stopper}} = 171.5\pi - 13.5\pi$

$= 158\pi$

$= 496.4\text{ cm}^3$



Exercise 6.4

1. Area $= \dfrac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots\, y_{n-1})]$

$= \dfrac{10}{2}[0 + 20 + 2(23 + 28 + 22 + 23 + 28)]$

$= 1340\text{ mm}^2$

Scale $= 1000 : 1$

$\Rightarrow 1000\text{ mm} : 1\text{ mm}$

$\Rightarrow (1000)(1000)\text{ mm}^2 : (1 \times 1)\text{ mm}^2$

$1 \times 10^6\text{ mm}^2 : 1\text{ mm}^2$

$\Rightarrow \text{Area}_{(\text{True})} = 1340 \times 10^6\,(\text{mm})^2$

$= 1340\text{ m}^2 \qquad [1\text{ m}^2 = (1000)^2\,(\text{mm})^2]$

$= 0.134$ hectares

2. Area $= \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots. y_{n-1})]$

$= \frac{1}{2}[0 + 3.66 + 2(2 + 5.42 + 3 + 1 + 4.36)]$

$= 17.61 \text{ cm}^2$

(i) % error $= \frac{17.62 - 17.23}{17.23} \times \frac{100}{1} = 2.26\%$

(ii) Area $= \frac{1}{4}[0 + 3.66 + 2(0.9 + 2 + 3.9 + 5.42 + 4.7 + 3 + 1.9 + 1 + 2 + 4.36 + 4)]$

$= 17.505 \text{ cm}^2$

3. (i) Area $= \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots. y_{n-1})]$

$= \frac{1}{2}[0 + 0 + 2(2.5 + 4 + 4.5 + 4 + 2.5)]$

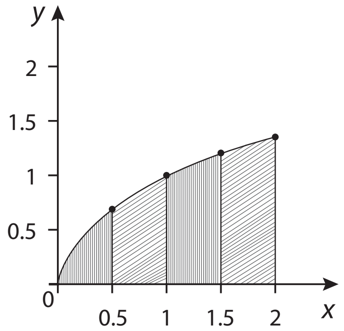
$= \frac{1}{2}[2(17.5)]$

$= 17.5$ sq. units

(ii) Area $= \frac{1}{4}[0 + 0 + 2(1.4 + 2.5 + 3.3 + 4 + 4.3 + 4.5 + 4.3 + 4 + 3.3 + 2.5 + 1.4)]$

$= 17.75$ sq. units

4.



$y = \sqrt{x}$;
at $x = 0.5$, $y = \sqrt{0.5} = 0.71$
at $x = 1$, $y = \sqrt{1} = 1$
at $x = 1.5$, $y = \sqrt{1.5} = 1.22$
at $x = 2$, $y = \sqrt{2} = 1.41$

$A = \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots. y_{n-1})]$

$= \frac{0.5}{2}[0 + 1.41 + 2(0.71 + 1 + 1.22)]$

$= 1.82$ sq. units

5. $A_1 = \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots. y_{n-1})]$

$= \frac{0.25}{2}[0.02 + 1.1 + 2(0.03 + 0.05 + 0.09 + 0.15 + 0.24 + 0.4 + 0.66)]$

$= 0.545$ sq. units

$A_2 = \frac{0.25}{2}[1.1 + 0.52 + 2(0.97 + 0.86 + 0.76 + 0.67 + 0.59)]$

$= 1.165$ sq. units

$A_1 : A_2 = 0.545 : 1.165$
$= 1 : 2.14$

Both areas equal $\Rightarrow A_2 = 0.545$
Let $x_{\text{maximum}} = 2.5$

$= 0.545 = \frac{0.25}{2}[1.1 + 0.86 + 2\,(0.97)]$

$= 0.487$ sq. units

Let $x_{\text{maximum}} = 2.75$

$$0.545 = \frac{0.25}{2}[1.1 + 0.76 + 2(0.97 + 0.86)]$$

$$= 0.69 \text{ sq. units}$$

For equal areas: $2.5 < x < 2.75$

6. Area $= \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots.)]$

$$= \frac{3}{2}[0 + 1 + 2(6 + 7.3 + 9.2 + 14.6 + 12 + 8.2 + 9.8)]$$

$$= 1.5\,[135.2]$$

$$= 202.8 \text{ cm}^2$$

1 cm = 20 km

$1 \text{ cm}^2 = 400 \text{ km}^2$

Area $= 202.8 \times 400$

$= 81120 \text{ km}^2$

Revision Exercise 6 (Core)

1. Area (A + C) = Area B

(same base and same perpendicular height)

Area (A + C) = Area (P + C)

(same base and same perpendicular height)

$\Rightarrow$ Area P = Area A

Area B = 3 × Area C

Area P = 2 × Area C

$\Rightarrow$ P : Q

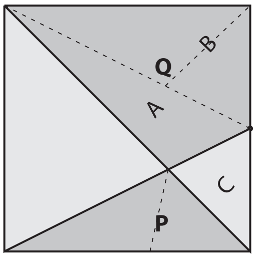
= 2C : A + B

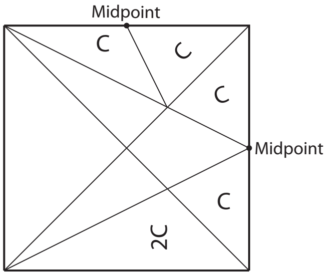
2C : P + B

2C : 2C + 3C

2C : 5C

2 : 5





2. (a) 8 – surfaces

$$\text{Area} = 2(20 \times 15) + 2 \times (10 \times 20) + (10 \times 15) + 2\left(\frac{1}{2}\pi(5)^2\right) + \frac{1}{2}(2\pi \times 5 \times 15)$$

$$= 1464.16$$

$$= 1464 \text{ cm}^2$$

$$\text{Volume} = (10 \times 15 \times 20) + \frac{1}{2}(\pi\,.\,5^2\,.\,15)$$

$$= 3589.05$$

$$= 3589 \text{ cm}^2$$

(b) 4 – surfaces

$$\text{Area} = (\pi \times (1.5)^2) + (2 \times \pi \times 1.5 \times 10) + \frac{1}{2}(4 \times \pi \times 6^2) + [\pi\,.\,(6)^2 - \pi\,.\,(1.5)^2]$$

$$= 433.54$$

$$= 434 \text{ mm}^2$$

$$\text{Volume} = (\pi \times (1.5)^2 \times 10) + \frac{1}{2}\left(\frac{4}{3} \times \pi \times (6)^3\right)$$

$$= 523.075$$

$$= 523 \text{ mm}^3$$

$$\begin{aligned}
\text{(c) Area of back} &= 75 \times 60 && = 4500\\
\text{Area of bottom} &= 75 \times 60 && = 4500\\
\text{Area of treads} &= 75 \times 60 && = 4500\\
\text{Area of rises} &= 75 \times 60 && = 4500\\
\text{Area of sides} &= 2 \times \frac{2}{3} \times 75 \times 75 && \underline{+7500}\\
& && = 25\,500 \text{ cm}^2
\end{aligned}$$

$$\begin{aligned}
\text{Volume} &= 6 \times (25 \times 25 \times 60)\\
&= 225\,000 \text{ cm}^3
\end{aligned}$$

3. (i) Let $|\angle COB| = \theta$

$$\begin{aligned}
\Rightarrow l &= r\theta && l = \text{arc length}\\
6.4 &= 5\theta && r = \text{radius}\\
\Rightarrow \theta &= 1.28 \text{ radians}
\end{aligned}$$

(ii)
$$\begin{aligned}
\text{Area} &= \frac{1}{2} \,.\, r^2 \,.\, \theta\\
&= \frac{1}{2}(5)^2 \,.\, (1.28)\\
&= 16 \text{ cm}^2
\end{aligned}$$

(iii)
$$\begin{aligned}
\text{Area of circle} = \pi \times (5)^2 \quad &= 78.539\\
\text{Area of minor sector} \quad &= \underline{16.000} \text{ (subtracting)}\\
\text{Area of major sector} \quad &= 62.539\\
\text{Area of minor: Area of major} \quad &= 16.00 : 62.539\\
&= 1 : 3.91
\end{aligned}$$

4.
$$\begin{aligned}
\text{Volume} &= \text{Area of trapezium} \times \text{width}\\
&= \left(\frac{3 + 1.5}{2}\right) \times 25 \times 8\\
&= 450 \text{ m}^3
\end{aligned}$$

$$\begin{aligned}
1 \text{ m}^3 &= (100 \text{ cm})^3\\
&= 1\,000\,000 \text{ cm}^3\\
&= 1000 \text{ litres} \qquad (1000 \text{ cm}^3 = 1 \text{ litre})
\end{aligned}$$

$$\begin{aligned}
\therefore \text{Capacity} &= 450 \times 1000\, l\\
&= 450\,000\, l\\
\text{Time} &= \frac{450\,000}{10}\\
&= 45\,000 \text{ minutes}\\
&= \frac{450\,00\cancel{0}}{6\cancel{0}} \text{ hours}\\
&= 750 \text{ hours}
\end{aligned}$$

5. (i)
$$\begin{aligned}
A &= \frac{1}{2} r^2\theta\\
12 &= \frac{1}{2} \,.\, x^2 \,.\, (1.2)\\
\frac{24}{1.2} &= x^2\\
\sqrt{20} &= x\\
x &= 2\sqrt{5} \text{ cm}
\end{aligned}$$

(ii)
$$\begin{aligned}
A &= \frac{1}{2} r^2\theta\\
15\cancel{\pi} &= \frac{1}{2} \,.\, x^2 \,.\, \frac{23\cancel{\pi}}{12} && \theta = \left(\frac{24\pi}{12} - \frac{\pi}{12}\right)\\
\frac{2 \times 15 \times 12}{23} &= x^2 && \quad = \frac{23\pi}{12}
\end{aligned}$$

$$\sqrt{\frac{360}{23}} = x$$

$$x = 3.96 \text{ cm}$$

(iii) $$A = \frac{1}{2}r^2\theta$$

$$20 = \frac{1}{2}(4.5)^2 \,.\, x$$

$$\frac{20}{(4.5)^2} = x$$

$$x = 0.99 \text{ radians}$$

6. Curved surface area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}l^2 \,.\, \theta$

But $s = l\theta$

and $s = 2\pi r$

$\Rightarrow l\theta = 2\pi r$

$\theta = \frac{2\pi r}{l}$

$\therefore$ Curved surface area $= \frac{1}{\cancel{2}}l^{\cancel{2}} \,.\, \frac{\cancel{2}\pi r}{\cancel{l}}$

$= \pi r l$

7. No.

The minimum possible size of the envelope is $12 - 0.5 = 11.5$ cm, while the maximum possible size of the card is $11 + 0.05 = 11.55$ cm.

8. (i) The tolerance of each dimension is ±0.5 cm.

(ii) Maximum possible volume $= 12.5 \times 7.5 \times 5.5$

$= 515.625 \text{ cm}^3$

Minimum possible volume $= 11.5 \times 6.5 \times 4.5$

$= 336.375 \text{ cm}^3$

Difference $= 515.625 - 336.375$

$= 179 \text{ cm}^3$

(iii) Error $= 515.625 - 450$

$= 65.625$

% Error $= \frac{65.625}{450} \times 100\%$

$= 14.6\%$

9. Maximum possible 3rd angle $= 180° - (60.5° + 45.5°) = 74°$

Minimum possible 3rd angle $= 180° - (61.5° + 46.5°) = 72°$

10. (i) The tolerance for each measurement is ± 0.5 mm $= \pm 0.05$ cm.

Thus the tolerance interval for the diameter is (9.8 ± 0.05) cm, and the tolerance interval for the height is (8.5 ± 0.05) cm.

(ii) For the cone, $r = 4.9$ cm and $h = 8.5$ cm.

Then the volume of the cone is

$\frac{1}{3}\pi r^2 h = \frac{\pi}{3}(4.9)^2\,(8.5) = 213.72 \text{ cm}^2$

(iii) The tolerance for the radius is half the tolerance for the diameter, i.e. the radius is (4.9 ± 0.025) cm.

Maximum possible volume $= \frac{\pi}{3}(4.925)^2(8.55) = 217.174 \text{ cm}^3$

Minimum possible volume $= \frac{\pi}{3}(4.875)^2(8.45) = 210.298 \text{ cm}^2$

$\text{Max error} = 217.174 - 213.72 = 3.454 \text{ cm}^3$

$\% \text{ Error} = \dfrac{3.454}{217.174} \times 100\% = 1.59\%$

11. Least possible breaking load of the crane is 1450 kg.
Greatest possible weight of each box is 50.5 kg.

$\text{Number of boxes} = \dfrac{1450}{50.5} = 28.71.$

To be sure that the cable does not break, greatest number of boxes is 28.

12. (i) $\text{Area of rectangle} = 5 \times 3.24$

$= 16.2 \text{ sq. units}$

$\text{Area of triangle} = \dfrac{1}{2}(5 \times 2.24)$

$= 5.6 \text{ sq. units}$

$\Rightarrow \text{ Area under graph} = 16.2 - 5.6$

$= 10.6 \text{ sq. units}$

(ii) $\text{Area} = \dfrac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots. y_{n-1})]$

$= \dfrac{1}{2}[1 + 3.24 + 2(1.39 + 1.95 + 2.42 + 2.85)]$

$= 10.73 \text{ sq. units}$

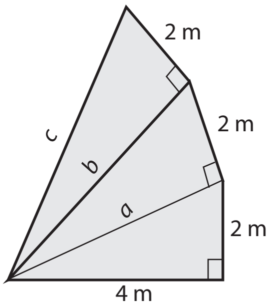
13. (i) $\text{Perimeter} = \sqrt{28} + 4 + 6$

$= (10 + 2\sqrt{7}) \text{ m}$

(ii) $\text{Area} = \dfrac{1}{2}(4 \times 2) + \dfrac{1}{2}(\sqrt{20} \times 2) + \dfrac{1}{2}(\sqrt{24} \times 2)$

$= 4 + \sqrt{20} + \sqrt{24}$

$= (4 + 2\sqrt{5} + 2\sqrt{6}) \text{ m}$



$a = \sqrt{4^2 + 2^2} = \sqrt{20} \text{ m}$

$b = \sqrt{(\sqrt{20})^2 + 2^2} = \sqrt{24} \text{ m}$

$c = \sqrt{(24)^2 + 2^2} = \sqrt{28} \text{ m}$

14. Large radius = 8 cm
Small radius = 4 cm

(i) $\text{Perimeter} = \dfrac{1}{2}\text{perimeter of large circle}$

$+ \dfrac{1}{2}\text{perimeter of small circle}$

$+ \dfrac{1}{2}\text{perimeter of small circle}$

$= \dfrac{1}{2}(2 \times \pi \times 8) + 2\left[\dfrac{1}{2}(2 \times \pi \times 4)\right]$

$= 8\pi + 8\pi$

$= 16\pi$

$= 50.27 \text{ cm}$

(ii) $\text{Area} = \dfrac{1}{2}\text{area of large circle} - \dfrac{1}{2}\text{area of small circle} + \dfrac{1}{2}\text{area of small circle}$

$= \dfrac{1}{2}\text{area of large circle}$

$= \dfrac{1}{2}\pi . 8^2$

$= 32\pi$

$= 100.53 \text{ cm}^2$

15. Cost = €(5200 + 35A)

$$\begin{aligned}\text{A} = \text{Surface area} &= \frac{1}{2}(4\pi r^2) + \pi r^2\\ &= 3\pi r^2\\ &= 3\pi 10^2\\ &= 300\pi\\ &= 942.47\end{aligned}$$

$$\begin{aligned}\text{Cost} &= €(5200 + 35(942.47))\\ &= €38\,186.72\end{aligned}$$

16. Volume = Volume of cube $= 2^3 = 8$

+ Volume of cylinder $= \pi \times 4^2 \times 2 = 32\pi$

+ Volume of cone $= \frac{1}{3}\pi \times 1^2 \times 4 = \frac{4\pi}{3}$

$$\begin{aligned}&= 8 + 32\pi + \frac{4\pi}{3}\\ &= 112.72 \text{ cm}^3\\ &= 112.7 \text{ cm}^3\end{aligned}$$

Revision Exercise 6 (Advanced)

1. (i) Arc length $= r\theta$

Perimeter $= r\theta + 2r$

$\Rightarrow\ 100 = r\theta + 2r$

$r\theta = 100 - 2r$

$\theta = \frac{100 - 2r}{r}$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} . r^{\cancel{2}} . \left(\frac{100 - 2r}{\cancel{r}}\right)$

$= 50r - r^2$

(ii) $A = -r^2 + 50r$

$= -[r^2 - 50r]$

$= -[r^2 - 50r + 25^2 - 25^2]$

$\text{A} = -[(r - 25)^2 - 625]$

$\text{A} = 625 - (r - 25)^2$

The maximum value of A occurs when $r - 25 = 0$,

i.e. $r = 25$.

(iii) $|\angle \text{CEB}| = \theta$

$\therefore\ \frac{1}{2}r^2\theta = 625$

$\theta = \frac{1250}{r^2} = \frac{1250}{625}$

$= 2$ radians

2. Hole of radius 1 cm

⇒ diameter = 2 cm

⇒ sheet of width and length 1 m (= 100 cm) can have 50 holes cut along length and along width

⇒ Method A: number of holes = 50 × 50 = 2500 holes

Area of sheet = (100 × 100) cm^2

= 10 000 cm^2

Area of hole $= \pi r^2 = \pi . 1^2$

$= \pi$ cm^2

Area of 2500 holes $= \pi \times 2500$

= 7853.98 cm^2

$$\Rightarrow \quad \text{Waste} = 10\,000 - 7{,}853.98 = 2146.02 \text{ cm}^2$$

$$\%\text{ waste} = \left(\frac{2146.02}{10\,000} \times \frac{100}{1}\right)\%$$

$$= 21.46\%$$

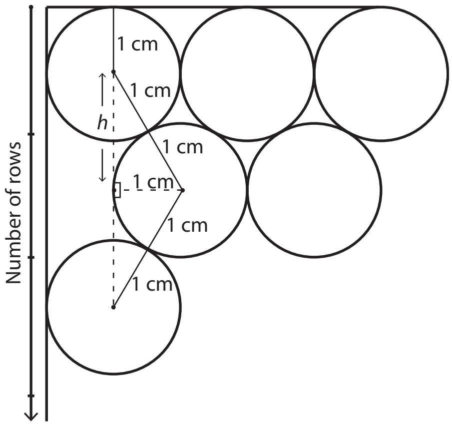
Method B

1^{st} row has 50 holes

2^{nd} row has 49 holes

3^{rd} row has 50 holes, etc.

$h = \sqrt{2^2 - 1^2} = \sqrt{3}$ cm



1 row needs 2 cm of metal

2 rows need $2 + \sqrt{3}$ cm of metal

3 rows need $2 + 2\sqrt{3}$ cm of metal

n rows need $2 + (n-1)\sqrt{3}$ cm of metal

$= 2 + (n-1)\sqrt{3} = 100$ cm, where n = number of rows

$$\Rightarrow \quad (n-1)\sqrt{3} = 98$$

$$n = \frac{98}{\sqrt{3}} + 1$$

$$= 57.58 \text{ rows}$$

⇒ 57 complete rows

⇒ 29 rows of 50 holes = 1450

28 rows of 49 holes = 1372

2822 holes

$$\text{Area of 2822 holes} = \pi \times 2822$$

$$= 8865.57 \text{ cm}^2$$

$$\Rightarrow \quad \text{Waste} = 10\,000 - 8865.57$$

$$= 1134.43 \text{ cm}^2$$

$$\%\text{ waste} = \frac{1134.43}{10\,000} \times \frac{100}{1}$$

$$= 11.34\%$$

3. (i) Area of $\Delta PQO = \frac{1}{2}(r)(r)\sin\theta$

$$= \frac{1}{2}r^2 \sin\theta$$

(ii) Area of segment = Area of sector − Area of triangle

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

(iii) Area of $\Delta PQN = 3 \times \frac{r^2}{2}(\theta - \sin\theta)$

Also, area of $\Delta PQN = \frac{1}{2}(r)(r)\sin(\pi - \theta)$

$$\therefore \ \frac{\cancel{1}}{\cancel{2}}\cancel{r^2}\sin(\pi - \theta) = 3\frac{\cancel{r^2}}{\cancel{2}}(\theta - \sin\theta)$$

$$\Rightarrow \sin\pi\cos\theta - \cos\pi\sin\theta = 3\theta - 3\sin\theta$$

$$\Rightarrow \quad (0).\cos\theta - (-1)\sin\theta = 3\theta - 3\sin\theta$$

$$\Rightarrow \quad \sin\theta = 3\theta - 3\sin\theta$$

$$4\sin\theta = 3\theta$$

$$\Rightarrow \quad 3\theta - 4\sin\theta = 0$$

4. (a) Surface area → 4 surfaces

Outer curved $= 2\pi rh = 2\pi(45)(100)$

$= 9000\pi$

Inner curved $= 2\pi rh = 2\pi(40)1000$

$= 8000\pi$

$2 \times$ end rings $= 2[\pi R^2 - \pi r^2]$

$= 2[\pi(45)^2 - \pi(40)^2]$

$= 850\pi$

Total area $= (9000 + 8000 + 850)\pi$

$= 56077.43 \text{ cm}^2$

$= 56077 \text{ cm}^2$

Volume $= \pi R^2 h - \pi r^2 h$

$= \pi h(R^2 - r^2)$

$= \pi \,.\, 100(45^2 - 40^2)$

$= 133517.69 \text{ cm}^3$

$= 133518 \text{ cm}^3$

(b) Surface area → 6 surfaces

Outer curved $= \frac{1}{2}(2\pi Rh) = \pi \,.\, (10) \,.\, 15$

$= 150\pi \text{ mm}^2$

Inner curved $= \frac{1}{2}(2\pi rh) = \pi \,.\, (5) \,.\, 15$

$= 75\pi \text{ mm}^2$

$2 \times$ end rings $= 2\left(\frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2\right)$

$= \pi(R^2 - r^2)$

$= \pi(10^2 - 5^2)$

$= 75\pi \text{ mm}^2$

$2 \times$ flat areas $= 2 \times (5 \times 15)$

$= 150 \text{ mm}^2$

Total area $= 150 + (75 + 75 + 150)\pi$

$= 1092 \text{ mm}^2$

Volume $= \frac{1}{2}(\pi R^2 h - (\pi r^2 h)$

$= \frac{\pi h}{2}(R^2 - r^2)$

$= \frac{\pi \,.\, 15}{2}(10^2 - 5^2)$

$= 1767.145 \text{ mm}^3$

$= 1767 \text{ mm}^3$

(c) Surface area → 2 surfaces

Hemisphere $= \frac{1}{2}(4\pi r^2)$

$= \frac{1}{2}(4 \,.\, \pi \,.\, 4^2)$

$= 32\pi$

$= \pi rl$

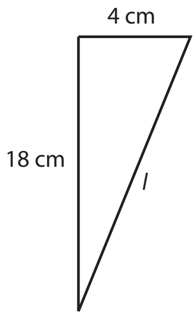
cone $= \pi \,.\, 4 \,.\, (18.44)$

$= 73.76\pi$

Total area $= (32 + 73.76)\pi$

$= 332.25 \text{ cm}^2$

$= 332 \text{ cm}^2$



$l = \sqrt{18^2 + 4^2}$

$= 18.44 \text{ cm}$

$$\begin{aligned}\text{Volume} &= \frac{1}{2}\left(\frac{4}{3}\pi\,.\,4^3\right) + \frac{1}{3}\pi(4)^2\,.\,(18)\\ &= 435.63\\ &= 436\text{ cm}^3\end{aligned}$$

5. (i) $\text{Area of sector} = \frac{1}{2}r^2\theta$

$$\begin{aligned}&= \frac{1}{2}\,.\,2^2\left(\frac{2\pi}{3}\right)\\ &= \frac{4\pi}{3} = 4.189\text{ cm}^2\end{aligned}$$

(ii) Area of segment = area of sector − area of triangle

$$\begin{aligned}&= 4.189 - \frac{1}{2}(2)(2)\sin\left(\frac{2\pi}{3}\right)\\ &= 2.457\text{ cm}^2\end{aligned}$$

Area bisected ⇒ area of triangle $= \frac{1}{2}(2)^2\,.\,\sin(1.895)$

$$= 1.895$$

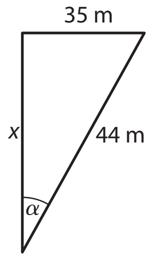
Area of segment $= \frac{1}{2}(2)^2(1.895) - \frac{1}{2}(2^2)\sin(1.895)$

$$= 1.895$$

6. (i) $\sin a = \frac{35}{44}$

$$\begin{aligned}a &= \sin^{-1}\left(\frac{35}{44}\right)\\ &= 0.91975\text{ radians}\end{aligned}$$

$\Rightarrow 2a = 1.84$ radians



$$\begin{aligned}x &= \sqrt{44^2 - 35^2}\\ &= 26.66\text{ m}\end{aligned}$$

(ii) $l = r\theta$

$$\begin{aligned}&= 44\,(1.84)\\ &= 80.96\text{ m}\end{aligned}$$

(iii) Shortest distance $= x = \sqrt{44^2 - 35^2}$

$$= 26.66\text{ m}$$

(iv) $\text{Area}_{\text{sector}} = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}(44)^2\,.\,(1.84) = 1781.12$$

$\text{Area}_{\text{triangle}} = \frac{1}{2}r^2\sin\theta$

$$= \frac{1}{2}(44)^2\,.\,\sin(1.84) = 933.14$$

$\text{Area}_{\text{segment}} = 1781.12 - 933.14$

$$= 848\text{ m}^2$$

7. Area of sector $= \frac{1}{2}r^2\theta$

Area of triangle $= \frac{1}{2}r^2\sin\theta$

⇒ area of segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$$= \frac{r^2}{2}(\theta - \sin\theta)$$

(i) Area of major segment $= \pi r^2$ − area of minor segment

$$\begin{aligned}&= \pi r^2 - \frac{r^2}{2}(\theta - \sin\theta)\\ 23.32 &= r^2\left[\pi - \frac{\theta}{2} + \frac{\sin\theta}{2}\right]\end{aligned}$$

When $\theta = 2$, $\quad 23.32 = r^2\left[\pi - 1 + \frac{\sin 2}{2}\right]$ (θ in radians)

$$= 2.596r^2$$

$$\Rightarrow \qquad r = \sqrt{\frac{23.32}{2.59}} = 3.0$$

(ii) $\sin \theta = \frac{r}{3}$

$r = 3 \sin \theta$

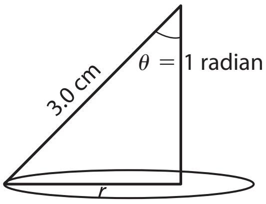
$= 3 \sin 1$

$= 2.52$ cm

Surface area $= \pi r^2$

$= \pi(2.52)^2$

$= 19.95 \text{ cm}^2$



8. (i)

A → B	accelerating (nearly uniform/constant)
B → C	accelerating (not uniform)
C → D	beginning to slow down
E → F	having stopped, begins to move off again.

(ii) Speed × time = distance

(iii) Area [distance] $= \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots.y_{n-1})]$

$= \frac{1}{2}[0 + 0 + 2(32 + 50 + 58 + 57 + 50 + 38 + 25 + 13 + 4)]$

$= 327 \text{ (kmh}^{-1} \times \text{min)}$

$= \frac{327}{60} \text{ (km min}^{-1} \times \text{min) } (= \text{km})$

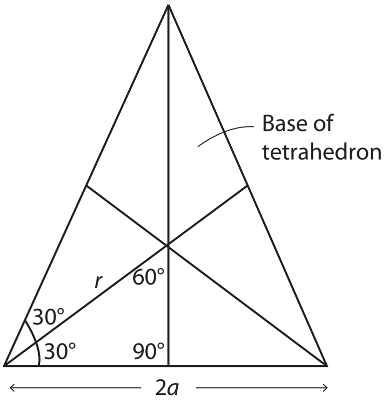
$= 5.5$ km

9. $\cos 30° = \frac{a}{r}$

$\Rightarrow r = \frac{a}{\cos 30°} = \frac{2a}{\sqrt{3}}$

$h = \sqrt{4a^2 - \left(\frac{2a}{\sqrt{3}}\right)^2}$

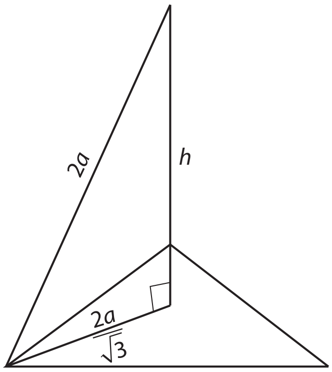
$= \sqrt{\frac{8a^2}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}a$



Volume of cylinder $= \pi r^2 h$

$= \pi \cdot \left(\frac{2a}{\sqrt{3}}\right)^2 \cdot \frac{2\sqrt{2}}{\sqrt{3}} \cdot a$

$= \frac{8\sqrt{6}}{9}\pi a^3$



10. $\frac{104}{78} = \frac{10 + x}{x}$

$104x = 780 + 78x$

$\Rightarrow 26x = 780$

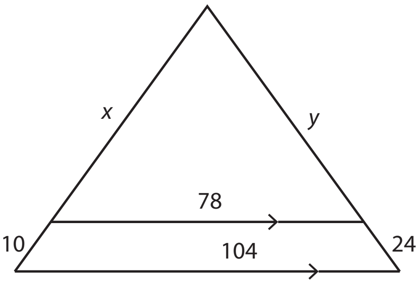
$x = 30$

$\frac{104}{78} = \frac{24 + y}{y}$

$104y = 24(78) + 78y$

$26y = 1872$

$y = 72$



$\cos a = \frac{104^2 + 40^2 - 96^2}{2(40)(104)}$

$= 0.3846$

$a = 67.38°$

$\Rightarrow$ Area of large triangle $= \frac{1}{2}(40)(104)\sin(67.38°)$

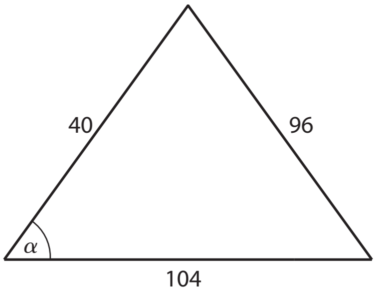
$= 1919.9 = 1920$

Area of small triangle $= \frac{1}{2}(30)(78)\sin(67.38°)$

$= 1079.9 = 1080$

Area of trapezium $= 1920 - 1080$

$= 840 \text{ mm}^2$



Revision Exercise 6 (Extended–Response Questions)

1. (i) (a) $l = r\theta$

(b) $A = \frac{1}{2}r^2\theta$

(ii) Length of wire $= 4 = r + r + r\theta$

$\Rightarrow 4 = 2r + r\theta$

$\theta = \frac{4 - 2r}{r}$

$\Rightarrow$ Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2\left[\frac{4 - 2r}{r}\right]$

$= 2r - r^2$

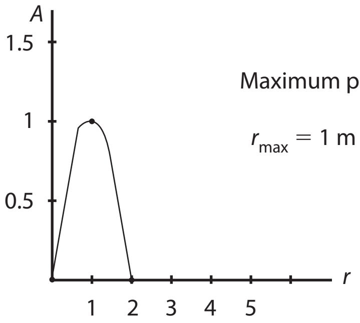
(iii) Area $= -[r^2 - 2r]$

$= -[r^2 - 2r + 1 - 1]$

$= -[(r - 1)^2 - 1]$

$= 1 - (r - 1)^2 \quad \Rightarrow q = 1, \quad p = 1$

(iv)



Maximum point = (1, 1)

$r_{\max} = 1$ m

(v) $A = \frac{1}{2}r^2\theta$

$1 = \frac{1}{2}(1)^2\theta$

$\Rightarrow \theta = 2$ radians

2. (i) **Group one** areas are the same.

Area of semi-circle

$$= \frac{1}{2}(\pi r^2)$$

$$= \frac{1}{2}\left(\pi\left(\frac{x}{2}\right)^2\right)$$

$$= \frac{\pi x^2}{8} = 0.393x^2$$

Area of equilateral triangle ($\theta = 60°$)

$$A = \frac{1}{2}(x)(x)\sin 60°$$

$$= \frac{x^2}{2}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4}x^2 = 0.433x^2$$

$\therefore$ Their claim is false.

Group two

$$\text{Difference in areas} = (0.4330 - 0.393)x^2$$

$$= 0.040x^2$$

$$\% \text{ difference} = \frac{0.040x^2}{0.393x^2} \times \frac{100}{1}$$

$$= 10.18\%$$

$\therefore$ Their claim is true.

Group three

$$\% \text{ difference} = \frac{0.04x^2}{0.433x^2} \times \frac{100}{1}$$

$$= 9.24\%$$

$\therefore$ Their claim is also true.

(ii) Area of semi-circle $= \frac{1}{2}(\pi r^2)$

$$= \frac{\pi x^2}{8}$$

Area of isosceles triangle

$$= \frac{1}{2}(x)h$$

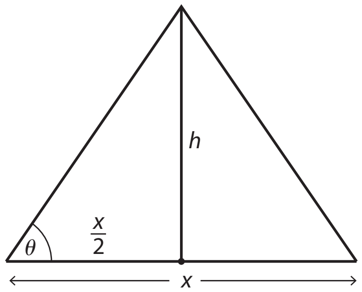
$$= \frac{1}{2}(x)(x)\frac{\tan\theta}{2}$$

$$= \frac{x^2}{4}\tan\theta$$

$$\Rightarrow \frac{\cancel{x^2}}{4}\tan\theta = \frac{\pi \cancel{x^2}}{8}$$

$$\tan\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = 57.52°$$



$$\tan\theta = \frac{h}{\frac{x}{2}} = \frac{2h}{x}$$

$$\Rightarrow h = \frac{x\tan\theta}{2}$$

(iii) Volume of hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$$= \frac{2}{3}\pi x^3 = 2.09x^3$$

Volume of cone $= \frac{1}{3}\pi x^2 h$

$$= \frac{1}{3}\pi \,.\, x^2 \,.\left(\frac{x\tan(57.52°)}{2}\right)$$

$$= 0.822x^3$$

$\therefore$ The volumes are not the same.

3. (i) Area of base $= (20 - 2h)(20 - 2h)$

$= (4h^2 - 80h + 400)\text{ cm}^2$

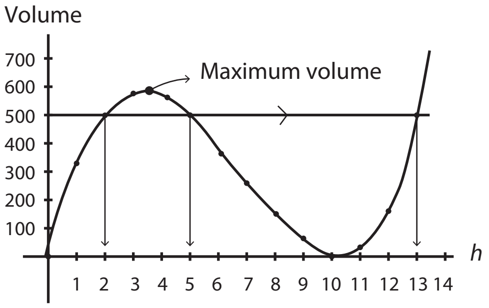
(ii) Cube $\Rightarrow h = 20 - 2h$

$\Rightarrow h = \dfrac{20}{3}\text{ cm}$

(iii) Volume $= (20 - 2h)^2(h)$

$= 4h^3 - 80h^2 + 400h$

(iv)

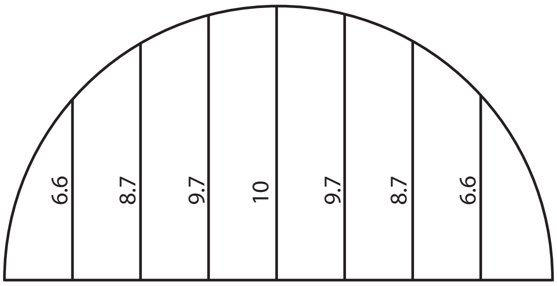


(v) $h = 2, 5, 13$

(vi) $h = 13$ is not possible

since the base is $(20 - 2h) \Rightarrow (20 - 26) = -6$

4. (i)



Area $= \dfrac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + \ldots.\ y_{n-1})]$

$= \dfrac{2.5}{2}[0 + 0 + 2(6.6 + 8.7 + 9.7 + 10 + 9.7 + 8.7 + 6.6)]$

$= 1.25\,[120]$

$= 150\text{ cm}^2$

(ii) $A = \dfrac{\pi r^2}{2} = \dfrac{\pi\,.\,10^2}{2}$

$= 157.08$

% error $= \dfrac{157.08 - 150}{157.08} \times \dfrac{100}{1}$

$= 4.5\%$

(iii) $ar = 6.6$ $\quad br = 8.7$ $\quad cr = 9.6$

$a = \dfrac{6.6}{10}$ $\quad b = \dfrac{8.7}{10}$ $\quad c = \dfrac{9.6}{10}$

$a = 0.66$ $\quad b = 0.87$ $\quad c = 0.96$

(iv) Area $= \dfrac{\frac{r}{4}}{2}[0 + 0 + 2(ar + br + cr + r + cr + br + ar)]$

$= \dfrac{r}{8}[2r(2a + 2b + 2c + 1)]$

$= \dfrac{r^2}{4}(2a + 2b + 2c + 1)$

(v) Area $= \dfrac{r^2}{4}(2(0.66) + 2(0.87) + 2(0.96) + 1)$

$= \dfrac{r^2}{4}(5.98)$

$= 1.495r^2$

(vi) $r = 5$ cm $\Rightarrow \quad A = 1.495(5)^2$
$= 37.38 \text{ cm}^2$

$r = 10$ cm $\Rightarrow \quad A = 1.495(10)^2$
$= 149.5 \text{ cm}^2$

$r = 15$ cm $\Rightarrow \quad A = 1.495(15)^2$
$= 336.38 \text{ cm}^2$

(vii) Using $A = \pi r^2$:

at $r = 5$, $\quad A = \dfrac{\pi(5)^2}{2} = 39.27 \text{ cm}^2$

at $r = 10$, $\quad A = \dfrac{\pi(10)^2}{2} = 157.08 \text{ cm}^2$

at $r = 15$, $\quad A = \dfrac{\pi(15)^2}{2} = 353.43 \text{ cm}^2$

$\therefore$ % error $\quad = \dfrac{39.27 - 37.38}{39.27} \times \dfrac{100}{1} = 4.8\%$

$= \dfrac{157.08 - 149.5}{157.08} \times \dfrac{100}{1} = 4.8\%$

$\dfrac{353.43 - 336.38}{353.43} \times \dfrac{100}{1} = 4.8\%$

As can be seen, there is a consistent error of 4.8% with this formula.
We also note that the formula reading is always 4.8% less than the true reading as would seem reasonable as the trapezia all lie below the real curve.

5. (i) (a) Length:
$31 = 1 + l + h + l + h$
$\Rightarrow 2l = 30 - 2h$
$l = 15 - h$

(b) Width:
$22 = 1 + h + w + h + 1$
$w = 20 - 2h$

(c) Height:
Height $= h$

(ii) Capacity :
Volume $= l \times w \times h$
$= (15 - h)(20 - 2h)(h)$
$= 2h^3 - 50h^2 + 300h$

(iii) Square base :
$\Rightarrow \quad l = w$
$\Rightarrow 15 - h = 20 - 2h$
$\Rightarrow \quad h = 5$ cm

(iv) $V = 2h^3 - 50h^2 + 300h$
$= 2(5)^3 - 50(h)^2 + 300(5)$
$= 500 \text{ cm}^3$

(v) $2h^3 - 50h^2 + 300h = 500$
$\Rightarrow 2h^3 - 50h^2 + 300h - 500 = 0$
$h = 5$ is a root (solution)
$\Rightarrow h - 5 = 0$ is a factor.

$$\begin{array}{r|l} & 2h^2 - 40h + 100 \\ \hline h - 5 & 2h^3 - 50h^2 + 300h - 500 \\ & \underline{2h^3 - 10h^2} \\ & \quad -40h^2 + 300h - 500 \\ & \quad \underline{-40h^2 + 200h} \\ & \qquad\qquad 100h - 500 \\ & \qquad\qquad \underline{100h - 500} \end{array}$$

$\Rightarrow 2h^2 - 40h + 100 = 0$ is the second factor

$\Rightarrow h^2 - 20h + 50 = 0$

$\Rightarrow a = 1, b = -20, c = 50$

$$\Rightarrow \; h = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(50)}}{2(1)}$$

$$= \frac{20 \pm \sqrt{200}}{2}$$

$$= \frac{20 \pm 10\sqrt{2}}{2}$$

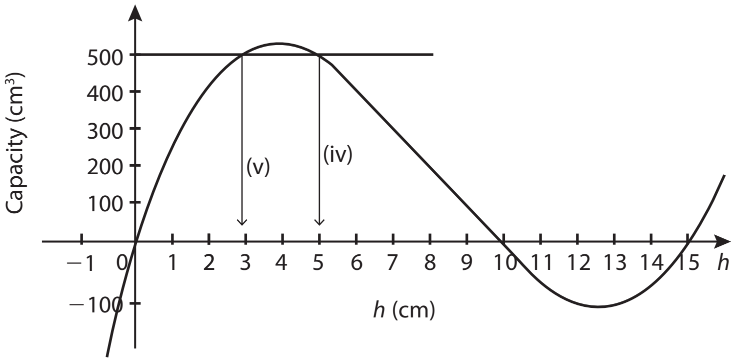
$$= 10 \pm 5\sqrt{2}$$

$$= 17.07 \text{ or } 2.929$$

$h \neq 17.07$, as $2h = 34 > 31$, the length of cardboard.

$\therefore h = 2.9$ cm would also give a capacity of 500 cm^3.

(vi)



(vii) Capacity increased by 10% $\Rightarrow$ Volume $= 500 + 50$

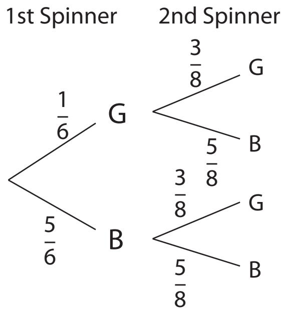
$= 550$ cm^3

This is not possible since no value of h could produce a value of 550 cm^3.

Chapter 7

Exercise 7.1

1. (i)



$\text{GG} = \frac{1}{6} \times \frac{3}{8}$

GB

BG

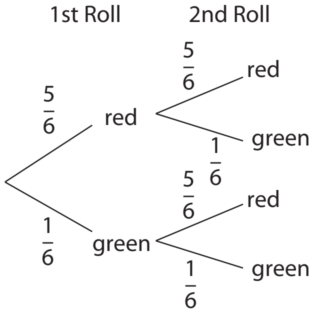
$\text{BB} = \frac{5}{6} \times \frac{5}{8}$

(ii) P(the two spinners show the same colour)

$$= \left(\frac{1}{6} \times \frac{3}{8}\right) + \left(\frac{5}{6} \times \frac{5}{8}\right)$$

$$= \frac{28}{48} = \frac{7}{12}$$

2. (i)



$\text{RR} = \frac{5}{6} \times \frac{5}{6}$

RG

GR

$\text{GG} = \frac{1}{6} \times \frac{1}{6}$

(ii) $P(\text{RR}) = \frac{25}{36}$

$P(\text{GG}) = \frac{1}{36}$

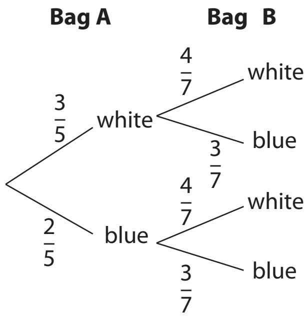
P(same colour) $=$ P(both red) **or** P(both green)

$$= \frac{25}{36} + \frac{1}{36}$$

$$= \frac{26}{36} = \frac{13}{18}$$

(iii) $P(\text{G and R}) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

3.



$\text{WW} = \frac{3}{5} \times \frac{4}{7}$

$\text{WB} = \frac{3}{5} \times \frac{3}{7}$

$\text{BW} = \frac{2}{5} \times \frac{4}{7}$

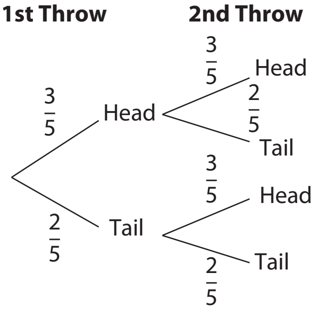
$\text{BB} = \frac{2}{5} \times \frac{3}{7}$

(i) P(both counters white) $= \frac{3}{5} \times \frac{4}{7} = \frac{12}{35}$

(ii) P(both blue) $= \frac{2}{5} \times \frac{3}{7} = \frac{6}{35}$

(iii) $P(\text{both white})$ or $P(\text{both blue}) = \frac{12}{35} + \frac{6}{35} = \frac{18}{35}$

4. (i)



$HH = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$

$HT = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

$TH = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$

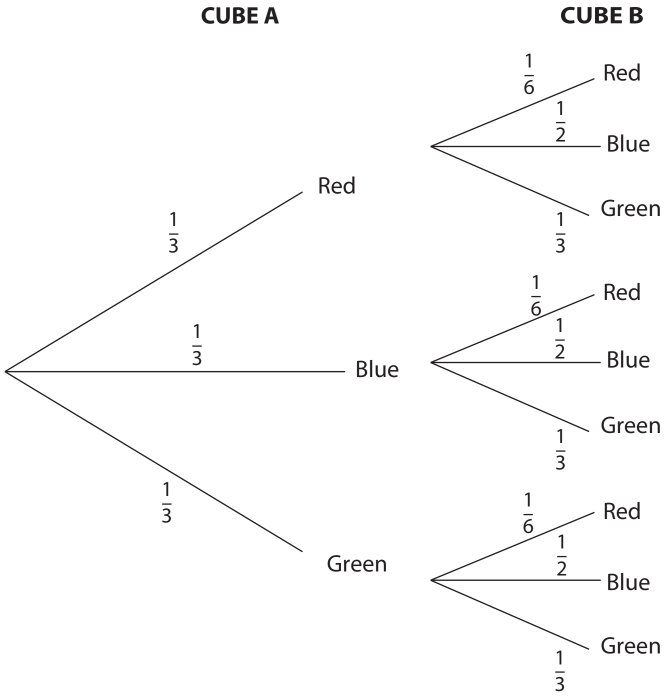
$TT = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

(ii) $P(\text{two heads}) = P(H, H) = \frac{9}{25}$

(iii) $P(H, T)$ or $P(T, H) = \frac{6}{25} + \frac{6}{25}$

$= \frac{12}{25}$

5. (i)



$RR = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$RB = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$RG = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$BR = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$BB = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$BG = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$GR = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$GB = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$GG = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

(ii) $P(RR)$ or $P(BB)$ or $P(GG)$

$= \frac{1}{18} + \frac{1}{6} + \frac{1}{9}$

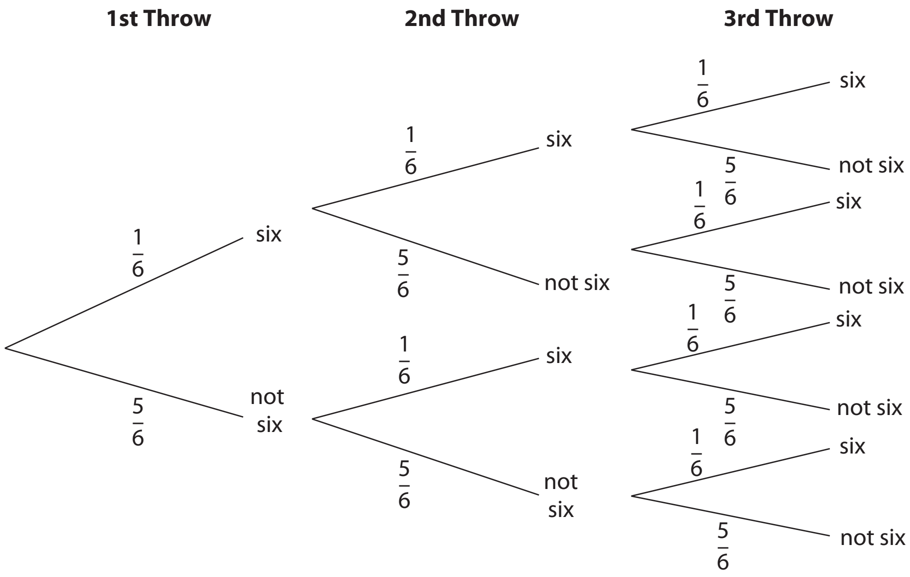
$= \frac{6}{18} = \frac{1}{3}$

(iii) $P(BG)$ or $P(GB)$

$= \frac{1}{9} + \frac{1}{6}$

$= \frac{5}{18}$

6. (i)



$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$

$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$

$= \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$

$= \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$

(ii) P(two sixes) or P(three sixes)

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{2}{27}$$

7. (i) P(1st Black) and P(2nd Black) **or** P(1st White) and P(2nd White)

$\therefore$ P(1st Black) $= \frac{1}{3}$ P(2nd Black) $= \frac{1}{5}$

$\therefore$ P(1st White) $= \frac{2}{3}$ P(2nd White) $= \frac{3}{5}$

$\therefore$ P(same colour) $= \left(\frac{1}{3} \times \frac{1}{5}\right) + \left(\frac{2}{3} \times \frac{3}{5}\right)$

$\therefore$ $= \frac{1}{15} + \frac{6}{15}$

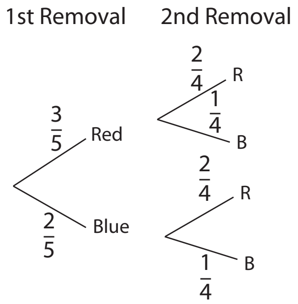
$= \frac{7}{15}$

(ii) P(different colours) $= 1 - P$(same colour)

$= 1 - \frac{7}{15}$

$= \frac{8}{15}$

8. (i)



$RR = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$

$RB = \frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$

$BR = \frac{2}{5} \times \frac{2}{4} = \frac{4}{20}$

$BB = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$

(ii) P(both cubes same colour)

$= P(RR)$ OR $P(BB)$

$= \left(\frac{3}{5} \times \frac{2}{4}\right) + \left(\frac{2}{5} \times \frac{1}{4}\right)$

$= \frac{6}{20} + \frac{2}{20}$

$= \frac{8}{20} = \frac{2}{5}$

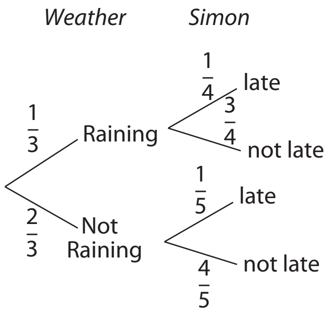
(iii) P(cubes are different colours)

$= 1 - P$(both same colour)

$= 1 - \frac{2}{5}$

$= \frac{3}{5}$

9. (i)



R/Late $= \frac{1}{3} \times \frac{1}{4}$

R/not late $= \frac{1}{3} \times \frac{3}{4}$

not Rain/Late $= \frac{2}{3} \times \frac{1}{5}$

not Rain/not Late $= \frac{2}{3} \times \frac{4}{5}$

(ii) P(simon late) $= P$(raining and late)

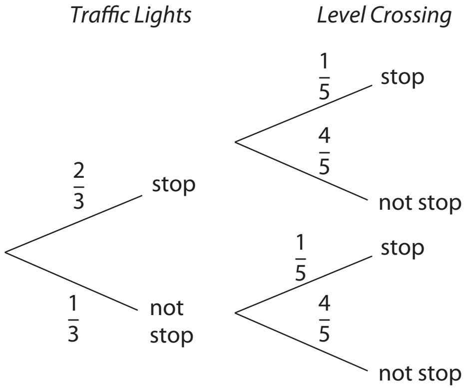
or P(not raining and late)

$$\therefore \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{2}{3} \times \frac{1}{5}\right)$$

$$\therefore \frac{1}{12} + \frac{2}{15}$$

$$= \frac{13}{60}$$

10. (i)



SS $= \frac{2}{3} \times \frac{1}{5}$

SN $= \frac{2}{3} \times \frac{4}{5}$

NS $= \frac{1}{3} \times \frac{1}{5}$

NN $= \frac{1}{3} \times \frac{4}{5}$

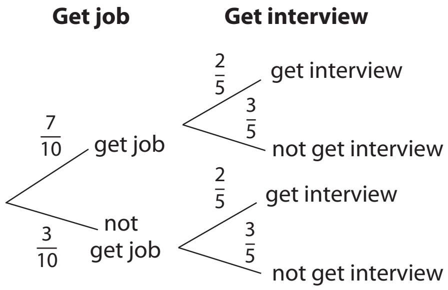
(ii) P(not have to stop at lights or crossing)

$= P$(not stop lights) **and** P(not stop crossing)

$$= \frac{1}{3} \times \frac{4}{5}$$

$$= \frac{4}{15}$$

11.



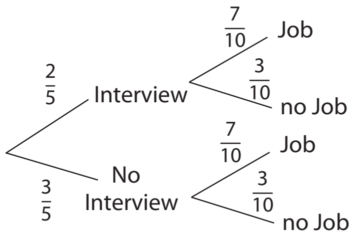
J, I

J, Not I

Not J, I

Not J, not I

or



(i) $P(\text{interview with no job}) = \frac{2}{5} \times \frac{3}{10} = \frac{6}{50} = \frac{3}{25}$

P(interview with no job) and P(no interview, no job)

$$\left(\frac{2}{5} \times \frac{3}{10}\right) + \left(\frac{3}{5} \times \frac{3}{10}\right)$$

$$= \frac{6}{50} + \frac{9}{50}$$

$$= \frac{15}{50} = 0.3$$

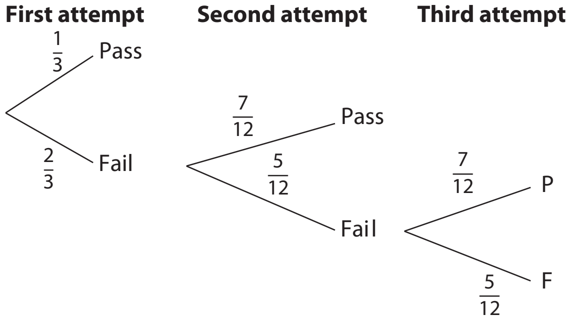
Or probability Karen does not get the job = 30%

(ii) $P(\text{Karen not get job}) = 1 - P(\text{Karen get interview and get job})$

$$= 1 - \left(\frac{7}{10} \times \frac{2}{5}\right)$$

$$= 1 - \frac{14}{50} = \frac{36}{50} = \frac{18}{25}$$

12.



$P = \frac{1}{3}$

$FP = \frac{2}{3} \times \frac{7}{12}$

$FFP = \frac{2}{3} \times \frac{5}{12} \times \frac{7}{12}$

$FFP = \frac{2}{3} \times \frac{5}{12} \times \frac{5}{12}$

P(pass at 3rd attempt) = FFP

$$= \frac{2}{3} \times \frac{5}{12} \times \frac{7}{12} = \frac{70}{432} = \frac{35}{216}$$

Exercise 7.2

1.

Outcome (x)	**Probability (P)**	**$x \times P$**
10	$\frac{1}{4}$	$2\frac{1}{2}$
12	$\frac{1}{2}$	6
6	$\frac{1}{4}$	$1\frac{1}{2}$

$$\therefore \sum x \,.\, P(x) = 2.5 + 6 + 1.5$$
$$= 10$$

2.

Outcome (x)	2	6	8	9	12
Probability (P)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$x \,.\, P(x)$	$\frac{2}{3}$	1	$\frac{8}{6}$	$\frac{9}{6}$	2

$$\therefore \sum x \,.\, P(x) = \frac{2}{3} + 1 + 1\frac{1}{3} + 1\frac{1}{2} + 2$$
$$= 6\frac{1}{2} = 6.5$$

3.

Outcome (*x*)	2	10	15	20
Probability (*P*)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
***x* . *P*(*x*)**	$\frac{1}{4}$	$3\frac{3}{4}$	$3\frac{3}{4}$	5

$$\begin{aligned}\sum x \,.\, P(x) &= \frac{1}{4} + 3\frac{3}{4} + 3\frac{3}{4} + 5\\ &= €12.75\end{aligned}$$

4. $$\begin{aligned}\sum x \,.\, P(x) &= 0.1 + 0.1 + 0.75 + 0.6 + 1.25 + 1.2\\ &= 4\end{aligned}$$

5. Expected value of x

i.e. $$\begin{aligned}\sum x \,.\, P(x) &= -0.6 - \cancel{0.1} + 0 + 0.4 + \cancel{0.1}\\ &= -0.6 + 0.4\\ &= -0.2\end{aligned}$$

6.

Outcome (*x*)	0	1	2	3	4	5
Probability (*P*)	0.21	0.37	0.25	0.13	0.03	0.01
***x* . *P*(*x*)**	0	0.37	0.50	0.39	0.12	0.05

$$\begin{aligned}\sum x \,.\, P(x) &= 0.37 + 0.5 + 0.39 + 0.12 + 0.05\\ &= 1.43\end{aligned}$$

7.

Outcome (*x*)	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
Probability (*P*)	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{0}{8}$

$$\begin{aligned}\sum x \,.\, P(x) &= \frac{3}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\\ &= \frac{12}{8} = 1.5\end{aligned}$$

8.

Outcome (*x*)	5	10	20
Probability (*P*)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
***x* . *P*(*x*)**	$\frac{5}{3}$	$\frac{10}{6}$	10

Costs €10 to play the game.

$\therefore$ Since $\sum x \,.\, P(x) = \frac{5}{3} + \frac{10}{6} + 10 = €13\frac{1}{3}$,

you expect to win $13\frac{1}{3} - 10$

$= €3\frac{1}{3}$

The game is not fair as mathematical expectation $\neq 0$.

9.

Outcome (x)	1	2	3	4	5	6
Probability (P)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$x \,.\, P(x)$	$+\frac{10}{6}$	$+\frac{10}{6}$	$-\frac{5}{6}$	$-\frac{5}{6}$	$-\frac{5}{6}$	$-\frac{5}{6}$

$$\sum x \,.\, P(x) = \frac{20}{6} - \frac{20}{6} = 0$$

Yes, the game is fair since the expected amount is 0 (zero).

10. (i) $\sum x \,.\, P(x) = 3.52 + 4.76 + 4.62 + 4.8 + 4.8$
$= €22.50$

(ii) Granded will have a **loss**, since his bet on the 5 horses was €25.

11. $P(\text{dying}) = \frac{1}{1000} = 0.001$

$P(\text{disability}) = \frac{3}{1000} = 0.003$

$$\sum x \,.\, P(x) = 50\,000(0.001) + 20\,000(0.003)$$
$$= 50 + 60$$
$$= €110$$

Profit $= €300 - €110 = €190$

12. (i) $y = 1 - (0.1 + 0.3 + 0.2 + 0.1)$
$= 1 - 0.7$
$= 0.3$

(ii) $\sum x \,.\, P(x) = 1(0.1) + 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1)$
$= 0.1 + 0.6 + 0.9 + 0.8 + 0.5$
$= 2.9$

13.

Outcome (x)	1	2	3	4	5	6
Probability (P)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$x \,.\, P(x)$	$-\frac{15}{6}$	$\frac{20}{6}$	0	0	0	$\frac{20}{6}$

$$\sum x \,.\, P(x) = \frac{25}{6} = €4.17$$

Costs €5 to play, $\therefore$ lose $5 - 4.17 = 0.83$
In 20 games $\therefore$ $20 \times 0.83 = €16.67$

14. (i) $E(x) = 3$

$\therefore\ 0.1 + 2p + 0.9 + 4q + 1 = 3$
$\therefore\ 2p + 4q = 3 - 2$
$\therefore\ 2p + 4q = 1 \qquad \ldots\ldots.\ (1)$
Since $P(x) = 1$,
$\therefore\ 0.1 + p + 0.3 + q + 0.2 = 1$
$\therefore\ p + q = -0.6 + 1$
$= 0.4 \qquad \ldots\ldots.\ (2)$

(ii) Solve

$$\begin{aligned} 2p + 4q &= 1 & \ldots\ldots\ (1)\\ p + q &= 0.4 & \ldots\ldots\ (2) \end{aligned}$$

$$\begin{aligned} \cancel{2p} + 4q &= 1 & \ldots\ldots\ (1)\\ -\cancel{2p} - 2q &= -0.8 & \ldots\ldots\ (2) \times -2\\ 2q &= 0.2\\ q &= 0.1\\ p &= 0.4 - q\\ &= 0.4 - 0.1\\ &= 0.3 \end{aligned}$$

$\therefore\ p = 0.3, \quad q = 0.1$

15. (i) $P(\text{rural claim}) = \dfrac{210}{4600} = 0.0456$

(ii) Expected value of cost

$$\begin{aligned} &= 0.0456 \times €1705\\ &= 77.836\\ &= €77.84 \end{aligned}$$

(iii) No. of households $= 6250$; Premium $= €580$

$6250 \times 580 = €3\,625\,000$	payments
$480 \times 2840 = €1\,363\,200$	claims
$€2\,261\,800$	profit

Profit per household

$$\begin{aligned} &= \frac{2\,261\,800}{6250}\\ &= 361.888\\ &= €361.89 \end{aligned}$$

(iv) $P(\text{rural claim}) = 0.05$

$\therefore\ 1550 \times 0.05 = €77.5$

Profit $= €350$

$\therefore$ annual premium

$$\begin{aligned} &= €350 + €77.5\\ &= €427.50 \end{aligned}$$

16. Section 1

$P(\text{A}), P(\text{B}), P(\text{C}), P(\text{D})$

$= \frac{1}{4} \quad = \frac{1}{4} \quad = \frac{1}{4} \quad = \frac{1}{4}$

$\therefore$ 20 questions; expected number of correct answers

$$\begin{aligned} &= 20 \times \frac{1}{4}\\ &= 5 \end{aligned}$$

Section 2

$P(\text{T}) = \frac{1}{2} \quad P(\text{F}) = \frac{1}{2}$

$\therefore$ with 10 questions, expected number of correct answers

$$\begin{aligned} &= 10 \times \frac{1}{2}\\ &= 5 \end{aligned}$$

Section 3

$P(\text{A}) = \frac{1}{3}, \quad P(\text{B}) = \frac{1}{3}, \quad P(\text{C}) = \frac{1}{3}$

$\therefore$ 10 questions give $\quad 10 \times \frac{1}{3}$

Expected no. of correct answers $= 3\frac{1}{3}$

$\therefore$ Total correct answers expected

$= 5 + 5 + 3\frac{1}{3}$

$= 13\frac{1}{3}$

17. Table 1

Deck of cards = 52 cards

$P(\text{pick one card}) = \frac{1}{52}$

Outcome (*x*)	Heart	Other suit
Probability (*P*)	$\frac{13}{52}$	$\frac{39}{52}$
***x* . *P*(*x*)**	$\frac{13}{52} \times 30$	$\frac{39}{52} \times -5$

$\therefore$ Expected payout $= (0.25 \times 30) + (0.75 \times (-5))$

$= €7.5 - €3.75$

$= €3.75$

Costs €10 to play the table

$\therefore$ €10 − 3.75

= expected loss of €6.25

Table 2

Throw 2 dice

Outcome (*x*)	Dice total 10	Dice total 11	Dice total 12
Probability (*P*)	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$P(\text{sum 10, sum 11, sum 12}) = \frac{6}{36} = \frac{1}{6}$

$\therefore P(\text{any other sum total}) = 1 - \frac{1}{6} = \frac{5}{6}$

$\therefore$ Expected value

$= \left(\frac{1}{6} \times 50\right) + \frac{5}{6}(-2)$

$= \frac{50}{6} - \frac{10}{6} = \frac{40}{6} = €6\frac{2}{3}$

Costs €10 to play the table

$\therefore$ €10 − $6\frac{2}{3}$

= €3.33 expected loss

Hence, to get the better expected return, play the dice table since with cards we lose 6.25 and with dice we lose 3.33.

The difference between the two expected returns is:

€6.25 − €3.33

= €2.92

Exercise 7.3

1. $p = \frac{1}{6}, \quad q = \frac{5}{6}$

$P(\text{1st success in 4th trial}) = q^3 \,.\, p$

$= \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) = \frac{125}{1296}$

2. Single trial: $n = 36$

success $= (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$

$r = 6$

$p = P(\text{success}) = \frac{6}{36} = \frac{1}{6}$ and $q = \frac{5}{6}$

$P(\text{1st success in 3rd trial}) = q^2 \,.\, p$

$= \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) = \frac{25}{216}$

3. $p = \frac{2}{5}, \quad q = \frac{3}{5}$

$P(\text{1st win in 5th trial}) = q^4 \,.\, p$

$= \left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) = \frac{162}{3125}$

4. $p = 0.1, \quad q = 0.9$

$P(\text{1st success in 6th trial}) = q^5 \,.\, p$

$= (0.9)^5(0.1)$

$= 0.059$

5. (i) $p = \frac{12}{52} = \frac{3}{13}, \quad q = \frac{10}{13}$

$P(\text{1st success in 3rd trial}) = q^2 \,.\, p$

$= \left(\frac{10}{13}\right)^2\left(\frac{3}{13}\right) = \frac{300}{2197}$

(ii) $p = \frac{13}{52} = \frac{1}{4}, \quad q = \frac{3}{4}$

$P(\text{1st success in 4th trial}) = q^3 \,.\, p$

$= \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \frac{27}{256}$

(iii) $p = \frac{26}{52} = \frac{1}{2}, \quad q = \frac{1}{2}$

$P(\text{1st success in 2nd trial}) = q \,.\, p$

$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

6. (i) There is a fixed number of independent trials, with two outcomes that have constant probabilities.

(ii) $p = \frac{1}{2}, \quad q = \frac{1}{2}, \quad n = 8$

7. (i) $\binom{5}{1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = 5 \cdot \frac{1}{16} \cdot \frac{1}{2} = \frac{5}{32}$

(ii) $\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32}$

$= \frac{5}{16}$

8. (i) $P(\text{success}) = \frac{1}{6} \qquad P(\text{failure}) = \frac{5}{6}$

$\therefore\ P(\text{a three}) = \frac{1}{6} \qquad P(\text{not a three}) = \frac{5}{6}$

$\binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = \frac{3125}{7776}$

(ii) $\binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = \frac{3125}{7776}$

(iii) $\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 10 \cdot \frac{1}{36} \cdot \frac{125}{216} = \frac{625}{3888}$

9. $P(\text{success}) = \frac{1}{3}$ $\quad P(\text{failure}) = \frac{2}{3}$

$$\binom{7}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^4 = 35 \cdot \frac{1}{27} \cdot \frac{16}{81}$$

$$= \frac{560}{2187}$$

10. $P(\text{boy}) = \frac{1}{2}$, $\quad P(\text{girl}) = \frac{1}{2}$

$$P(3\text{ boys}) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32}$$

$$\therefore \frac{10}{32} = \frac{5}{16}$$

$$P(2\text{ girls}) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 10 \cdot \frac{1}{4} \cdot \frac{1}{8} = \frac{10}{32}$$

$$\therefore \frac{10}{32} = \frac{5}{16}$$

11. (i) $P(\text{success}) = 0.7$ $\quad P(\text{failure}) = 0.3$

Here "Success" equals walking to school and not walking equal "failure".

$P(\text{walks to school once})$

$$= \binom{5}{1}(0.7)^1(0.3)^4 = 5(0.7)(0.0081)$$

$$= 0.028$$

(ii) $P(\text{walks to school 3 times})$

$$= \binom{5}{3}(0.7)^3(0.3)^2 = 10(0.343)(0.09)$$

$$= 0.3087$$

$$= 0.31$$

12. $P(\text{success}) = P(\text{vote } X) = \frac{3}{5}$

$P(\text{failure}) = P(\text{not vote for } X) = \frac{2}{5}$

$P(3\text{ people vote for party } X)$

$$= \binom{8}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^5 = 56 \cdot \frac{27}{125} \cdot \frac{32}{305} = \frac{48\,384}{390\,625}$$

$$= 0.1238$$

$$= 0.124$$

13. $P(\text{success}) = \frac{1}{3}$ $\quad P(\text{failure}) = \frac{2}{3}$

$= p \qquad\qquad = q$

$P(3\text{ students completing 4 yrs})$

$$= \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 = 4\left(\frac{1}{27}\right)\left(\frac{2}{3}\right)$$

$$= \frac{8}{81}$$

$P(4\text{ students completing 4 yrs})$

$$= \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = 1\left(\frac{1}{81}\right)(1)$$

$$= \frac{1}{81}$$

$\therefore$ $P(3\text{ students at least completing 4 yrs study})$

$$= \frac{8}{81} + \frac{1}{81} = \frac{9}{81} = \frac{1}{9}$$

14. (i) 20% defective $= \frac{20}{100} = \frac{1}{5}$

$P(\text{defective}) = \frac{1}{5}$

$P(\text{not defective}) = \frac{4}{5}$

$P(\text{two bolts defective}) = \binom{4}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2 = 6\left(\frac{1}{25}\right)\left(\frac{16}{25}\right)$

$= \frac{96}{625}$

(ii) P(not more than 2 defective)

$=$ P(none defective) **or** P(one defective)

or P(two defective)

$P(1\text{ defective}) = \binom{4}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^3 = 4\left(\frac{1}{5}\right)\left(\frac{64}{125}\right)$

$= \frac{256}{625}$

$P(0\text{ defective}) = \binom{4}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^4 = \frac{256}{625}$

$\therefore P(\text{not more than 2 defective}) = \frac{256}{625} + \frac{256}{625} + \frac{96}{625}$

$= \frac{608}{625}$

15. $P(\text{success}) = \frac{2}{5} = p$

$P(\text{failure}) = \frac{3}{5} = q$

(i) $P(\text{none travel by bus}) = \binom{4}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^4$

$= 1\left(\frac{81}{625}\right) = \frac{81}{625}$

(ii) $P(\text{three travel by bus}) = \binom{4}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^1$

$= 4\left(\frac{8}{125}\right)\left(\frac{3}{5}\right)$

$= \frac{96}{625}$

(iii) P(at least one of the children travel by bus)

$= 1 - P$(none travel by bus)

$\therefore \quad 1 - \frac{81}{625}$

$= \frac{544}{625}$

16. $P(\text{sink a putt}) = \frac{7}{10} \qquad = p$

$P(\text{not sink a putt}) = \frac{3}{10} \qquad = q$

(i) $n = 3$

P(sink 2 putts in 3 attempts)

$= \binom{3}{2}\left(\frac{7}{10}\right)^2\left(\frac{3}{10}\right)^1 = 3\left(\frac{49}{100}\right)\left(\frac{3}{10}\right) = \frac{441}{1000}$

(ii) P(miss 3 putts in 4 attempts) $\qquad n = 4$

$= \binom{4}{3}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^1 = 4\left(\frac{343}{1000}\right)\left(\frac{3}{10}\right) = \frac{1029}{2500}$

17. $P(A \text{ will win race}) = \frac{2}{5} = p$

$P(A \text{ not win race}) = \frac{3}{5} = q$

(i) $n = 5$

$$\binom{5}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^1 = 10\left(\frac{8}{125}\right)\left(\frac{9}{25}\right) = \frac{144}{625}$$

$= P(\text{winning exactly 3 races})$

(ii) $P(A \text{ win 1st, 3rd, 5th races}) \qquad n = 5$

$$P(A \text{ win 1st race}) = \binom{5}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^4 = \cancel{5}\left(\frac{2}{\cancel{5}}\right)\left(\frac{81}{625}\right)$$
$$= \frac{162}{625}$$

$$P(A \text{ win 3rd race}) = \binom{5}{3} = \frac{144}{625}$$

$$P(A \text{ win 5th race}) = \binom{5}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^0 = \frac{32}{3125}$$

$$P(A \text{ lose 2nd race}) = \binom{5}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 = 10\left(\frac{9}{25}\right)\left(\frac{8}{125}\right)$$
$$= \frac{720}{3125}$$

$$P(A \text{ lose 4th race}) = \binom{5}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1 = 5\left(\frac{81}{3125}\right)\left(\frac{2}{5}\right)$$
$$= \frac{162}{3125}$$

$\therefore$ $P(\text{win 1st, 3rd, 5th and lose 2nd \& 4th races})$

$$= \frac{144}{625} + \frac{162}{625} + \frac{32}{3125} - \left(\frac{720}{3125} + \frac{162}{3125}\right)$$
$$- \frac{882}{3125}$$

18. $P(\text{boy}) = \frac{1}{2} \qquad P(\text{girl}) = \frac{1}{2} \qquad n = 4$

(i) $P(2 \text{ boys}) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$

In 2000 families, those with 4 children (2 boys) are expected to number:

$$2000 \times \frac{3}{8} = 750 \text{ families}$$

(ii) $P(\text{no girls})$ i.e. 4 boys 0 girls

$$\binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 1\left(\frac{1}{16}\right)$$

With 2000 families expect:

$$\frac{1}{16} \times 2000 = 125 \text{ families}$$

(iii) $P(\text{at least one boy}) = 1 - P(\text{no boy})$

$$\therefore\ 1 - \binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = 1 - 1\left(\frac{1}{16}\right)$$
$$= \frac{15}{16}$$

In 2000 families: $\therefore \frac{15}{16} \times 2000$

$= 1875$ families

19. $P(\text{answer correct}) = \frac{1}{3}$

$P(\text{answer incorrect}) = \frac{2}{3}$

(i)
- Suitable because there is a fixed number of independent trials
- There are two outcomes (correct or incorrect)
- Outcomes have constant probabilities

(ii) $P(\text{all 4 answers correct}) = \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0$

$$= 1\left(\frac{1}{81}\right)1 = \frac{1}{81}$$

(iii) $P(\text{one answer correct}) = \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$

$$= 4 \cdot \frac{1}{3} \cdot \frac{8}{27} = \frac{32}{81}$$

Probability that Ray gets the first answer correct $= \frac{1}{3}$ since in the test there are 3 alternative answers of which exactly one is correct, and he is guessing.

20. When a coin is tossed there are only two outcomes:

(1) Getting Head (2) Getting Tail

$P(\text{success}) = P(\text{head}) = p$

$P(\text{failure}) = P(\text{tail}) = q$

21. (i) $P(\text{getting a 5 on a throw}) = \frac{1}{6} = p$

$P(\text{not getting a 5 on a throw}) = \frac{5}{6} = q$

$n = 10$

$$P(\text{two 5's}) = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$$

$$= 45\left(\frac{1}{36}\right)\left(\frac{5}{6}\right)^8$$

$$= 0.29071$$

(ii) $P(\text{getting 3rd five on 11th throw})$

$= P(\text{getting 2 fives in 10 throws}) \times P(5)$

$= 0.29071 \times \frac{1}{6}$

$= 0.048451$

$= 0.04845$

22. (i) $n = 52$ cards

$P(\text{card is picture card}) = \frac{12}{52} = \frac{3}{13}$

(ii) $P(\text{card not picture card}) = \frac{10}{13}$

$P(\text{3rd picture card on 13th attempt})$

i.e. 2 picture cards in 12 selections

$$\therefore \binom{12}{2}\left(\frac{3}{13}\right)^2\left(\frac{10}{13}\right)^{10} = 66 \times \frac{9}{169} \times \left(\frac{10}{13}\right)^{10}$$

$$= 0.2548$$

$P(\text{picture card on 13th selection}) = \frac{3}{13}$

Thus, $P(\text{3rd picture card on 13th selection})$

$= 0.2548 \times \frac{3}{13}$

$= 0.0588$

23. Probability (spinner stops on red) $= 0.3$

P(spinner stops on another colour) $= 0.7$

$\therefore\ p = 0.3 \qquad q = 0.7$

For 4th red on 10th spin,

$\therefore$ there must be 3 red on first 9 spins.

$$\therefore \binom{9}{3}(0.3)^3(0.7)^6 = 84(0.027)(0.117649)$$
$$= 0.2668$$

P(red on 10th spin) $= 0.3$

$\therefore\ P$(4th red on 10th spin) $= 0.2668 \times 0.3$
$= 0.08$

24. (i) P(red counter) $= 40\% = \frac{2}{5}$

P(yellow counter) $= 60\% = \frac{3}{5}$

$n = 8$

$$P(\text{3 red counters}) = \binom{8}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^5$$
$$= 56 \cdot \frac{8}{125} \cdot \frac{243}{3125}$$
$$= 0.27869$$

(ii) P(red counter on ninth draw) $= \frac{2}{5}$

$\therefore\ P$(4th red counter on 9th draw)

$$= 0.27869 \times \frac{2}{5} = 0.11148$$

25. (i) P(correct answer) $= \frac{1}{4}$ $\qquad P$(incorrect answer) $= \frac{3}{4}$

$p = \frac{1}{4} \qquad q = \frac{3}{4} \qquad n = 10$

$$P(\text{no correct answer out of 10}) = \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10}$$
$$= (1)(1)(0.75)^{10} = 0.0563$$

(ii) P(7 correct answers) $= \binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$

$$\therefore\ 120 \times \frac{1}{16\,384} \times \frac{27}{64}$$
$$= 0.003089$$
$$= 0.00309$$

$$P(\text{2 correct answers in 9 questions}) = \binom{9}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^7$$
$$= 36 \times \frac{1}{16} \times \frac{2187}{16\,384}$$
$$= 0.3003387$$

P(correct answer on 10th question) $= \frac{1}{4}$

$\therefore\ P$(3rd correct answer on 10th question)

$$= 0.3003387 \times \frac{1}{4}$$
$$= 0.07508$$

Exercise 7.4

1. (i) $P(\text{A}) = \frac{12}{30} = \frac{2}{5}$

(ii) $P(\text{B}) = \frac{10}{30} = \frac{1}{3}$

(iii) $P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B})$

$= \frac{2}{5} \times \frac{1}{3}$

$= \frac{2}{15}$

From diagram, $P(\text{A} \cap \text{B}) = \frac{4}{30} = \frac{2}{15}$

$\therefore$ since $P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B}) = \frac{2}{15}$

$\therefore$ A and B are independent events

2. (i) $P(\text{A}) = \frac{1}{3}$

(ii) $P(\text{B}) = \frac{1}{4}$

From diagram, $P(\text{A} \cap \text{B}) = \frac{1}{12}$

$P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B})$

$= \frac{1}{3} \cdot \frac{1}{4}$

$= \frac{1}{12}$

$\therefore P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B}) = \frac{1}{12}$

$\therefore$ A and B are independent

3. $P(\text{A}) = 0.8 \quad P(\text{B}) = 0.6$

$P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B})$

$0.48 = 0.8 \times 0.6$ (given)

$= 0.48$

$\therefore$ Yes, A and B are independent

since $P(\text{A}) \,.\, P(\text{B}) = P(\text{A} \cap \text{B})$

4. $P(\text{A}) = 0.4 \quad P(\text{B}) = 0.25$

$P(\text{A} \cap \text{B}) = P(\text{A}) \,.\, P(\text{B})$

$= 0.4 \times 0.25$

$= 0.1$

5. $P(\text{A}) = 0.4 \quad P(\text{A} \cup \text{B}) = 0.7$

$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

$0.7 = 0.4 + P(\text{B}) - [P(\text{A}) \,.\, P(\text{B})]$

$0.7 - 0.4 = P(\text{B}) - 0.4P(\text{B})$

$0.3 = 0.6P(\text{B})$

$\therefore P(\text{B}) = \frac{0.3}{0.6}$

$= 0.5$

6. (i) $P(\mathrm{A}) = 0.45 \quad P(\mathrm{B}) = 0.35$

$P(\mathrm{A} \cup \mathrm{B}) = 0.7$

$$P(\mathrm{A} \cup \mathrm{B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A} \cap \mathrm{B})$$

$$0.7 = 0.45 + 0.35 - P(\mathrm{A} \cap \mathrm{B})$$

$$0.7 - 0.45 - 0.35 = -P(\mathrm{A} \cap \mathrm{B})$$

$$0.7 - 0.8 = -P(\mathrm{A} \cap \mathrm{B})$$

$$\therefore -0.1 = -P(\mathrm{A} \cap \mathrm{B})$$

$$\therefore P(\mathrm{A} \cap \mathrm{B}) = 0.1$$

(ii) $P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}) \,.\, P(\mathrm{B})$

$$= 0.45 \times 0.35$$

$$= 0.1575$$

$\therefore P(\mathrm{A} \cap \mathrm{B}) \neq P(\mathrm{A}) \,.\, P(\mathrm{B})$

$\Rightarrow$ Events are not independent

(iii) $P(\mathrm{A} | \mathrm{B}) = \dfrac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})} = \dfrac{0.1}{0.35}$

$$= \frac{2}{7}$$

7. $P(\mathrm{A}) = 0.8 \quad P(\mathrm{B}) = 0.7$

$P(\mathrm{A} | \mathrm{B}) = 0.8$

(i) To find

$P(\mathrm{A} \cap \mathrm{B})$:

$$P(\mathrm{A} | \mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}$$

$$\therefore P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A} | \mathrm{B}) \times P(\mathrm{B})$$

$$= 0.8 \times 0.7$$

$$= 0.56$$

(ii) $P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}) \times P(\mathrm{B})$

$$= 0.8 \times 0.7$$

$$= 0.56$$

A and B are independent events

since $P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}) \times P(\mathrm{B}) = 0.56$

8. $P(\mathrm{A}) = \dfrac{2}{5} \quad P(\mathrm{B}) = \dfrac{1}{6}$

$P(\mathrm{A} \cup \mathrm{B}) = \dfrac{13}{30}$

(i) $P(\mathrm{A} \cup \mathrm{B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A} \cap \mathrm{B})$

$$\frac{13}{30} = \frac{2}{5} + \frac{1}{6} - P(\mathrm{A} \cap \mathrm{B})$$

$$\therefore \frac{13}{30} - \frac{2}{5} - \frac{1}{6} = -P(\mathrm{A} \cap \mathrm{B})$$

$$\frac{2}{15} = P(\mathrm{A} \cap \mathrm{B})$$

$$\therefore P(\mathrm{A} \cap \mathrm{B}) = \frac{2}{15}$$

(ii) $P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}) \,.\, P(\mathrm{B})$

$$= \frac{2}{5} \times \frac{1}{6}$$

$$= \frac{2}{30} = \frac{1}{15}$$

Since $P(\mathrm{A} \cap \mathrm{B}) = \dfrac{2}{15}$ and $P(\mathrm{A}) \times P(\mathrm{B}) = \dfrac{1}{15}$ they are not equal.

$\therefore$ Events A and B are not independent.

9. Given $P(C|D) = \frac{2}{3}$ and $P(C \cap D) = \frac{1}{3}$

(i) $P(C|D) = \frac{P(C \cap D)}{P(D)}$

$$\therefore \frac{2}{3} = \frac{\frac{1}{3}}{P(D)}$$

$$\therefore \frac{2}{3}P(D) = \frac{1}{3}$$

$$\therefore P(D) = \frac{1}{3} \div \frac{2}{3}$$

$$= \frac{1}{2}$$

(ii) Since events are independent

$$P(C \cap D) = P(C) \times P(D)$$

$$\therefore \frac{1}{3} = P(C) \times \frac{1}{2}$$

$$\therefore P(C) = \frac{1}{3} \div \frac{1}{2}$$

$$= \frac{2}{3}$$

10. Given $P(B) = 0.7,\quad P(C) = 0.6,\quad P(C|B) = 0.7$

To find $P(B \cap C)$:

$$P(C|B) = \frac{P(C \cap B)}{P(B)}$$

$$\therefore 0.7 = \frac{P(C \cap B)}{0.7}$$

$$\therefore \quad P(C \cap B) = 0.7 \times 0.7$$

$$= 0.49$$

Also, $P(C \cap B) = P(C) \times P(B)$

$$= 0.6 \times 0.7$$

$$= 0.42$$

B and C are not independent

since $0.49 \neq 0.42$

11. Given $P(A) = 0.2 \quad P(B) = 0.15$

(i) To find $P(A \cap B)$, we use

$P(A \cap B) = P(A) \times P(B)$ since events are independent.

$$\therefore P(A \cap B) = 0.2 \times 0.15$$

$$= 0.03$$

(ii) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$= \frac{0.03}{0.15}$$

$$= 0.2$$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.2 + 0.15 - 0.03$$

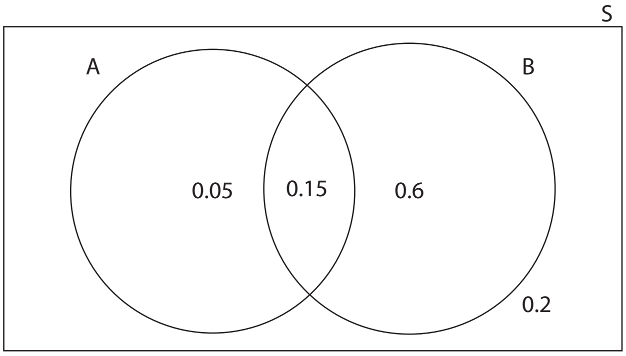
$$= 0.32$$

12. Given:

$P(A) = 0.2 \qquad P(A \cap B) = 0.15$

$P(A' \cap B) = 0.6$

(i)



(ii) $P(\text{neither A nor B}) = 1 - P(A \cup B)$

$= 1 - 0.8$

$= 0.2$

(iii) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{0.15}{0.75}$

$= 0.2$

(iv) $P(A \cap B) = P(A) \times P(B)$

$= 0.2 \times 0.75$

$= 0.15$

Yes, A and B are independent as $P(A \cap B) = P(A) \times P(B) = 0.15$.

13. Given $P(A) = \dfrac{8}{15} \quad P(B) = \dfrac{1}{3} \quad P(A \mid B) = \dfrac{1}{5}$

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$\therefore \dfrac{1}{5} = \dfrac{P(A \cap B)}{\frac{1}{3}}$

$\therefore P(A \cap B) = \dfrac{1}{5} \times \dfrac{1}{3}$

$= \dfrac{1}{15}$

$\therefore P(\text{both events occur}) = \dfrac{1}{15}$

(ii) $P(\text{only A or B occurs})$ i.e. $P(A) + P(B)$

$= \dfrac{8}{15} + \dfrac{1}{3}$

$= \dfrac{13}{15}$

14. (i) A and B are independent events whereby the outcome of A does not affect the outcome of B; e.g. B is the event obtaining a head when a coin is tossed.

(ii) If $P(\text{C or D}) = P(C) + P(D)$, then we can say that C and D are mutually exclusive events; value of $P(\text{C and D}) = 0$.

15. Given $P(A|B) = 0.4$
$P(B|A) = 0.25$
$P(A \cap B) = 0.12$

(i) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\therefore 0.4 = \frac{0.12}{P(B)}$

$\therefore 0.4P(B) = 0.12$

$\therefore P(B) = \frac{0.12}{0.4}$

$\therefore P(B) = 0.3$

$P(B|A) = \frac{P(B \cap A)}{P(A)}$

$\therefore 0.25 = \frac{0.12}{P(A)}$

$\therefore P(A) \times 0.25 = 0.12$

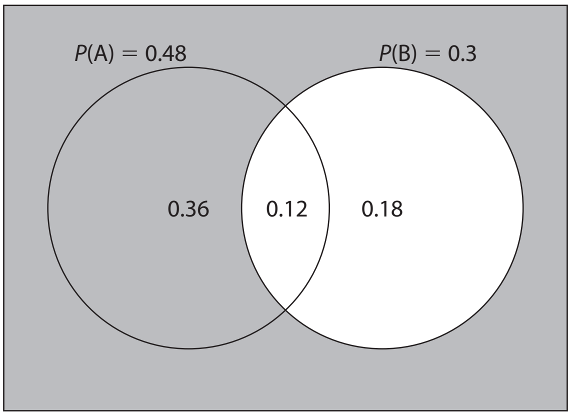
$\therefore P(A) = \frac{0.12}{0.25}$

$\therefore P(A) = 0.48$

(ii) A and B are not independent since $P(A \cap B) \neq P(A) \times P(B)$ as $0.144 \neq 0.12$.

(iii) $P(A \cap B')$
$P(A) = 0.48 \quad P(A \cap B) = 0.12$

$\therefore P(A \cap B') = P(A) - P(A \cap B)$
$= 0.48 - 0.12$
$= 0.36$



B′ = Shaded

16. Given $P(E) = \frac{2}{5}, P(F) = \frac{1}{6}, P(E \cup F) = \frac{13}{30}$

$P(E \cap F) = P(E) \times P(F)$

$= \frac{2}{5} \times \frac{1}{6} = \frac{2}{30} = \frac{1}{15}$

Also, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\therefore \frac{13}{30} = \frac{2}{5} + \frac{1}{6} - P(E \cap F)$

$\therefore -\frac{4}{30} = -P(E \cap F)$

$\therefore P(E \cap F) = \frac{2}{15}$

Hence, since $P(E \cap F) \neq P(E) \times P(F)$ $\left(\text{as } \frac{2}{15} \neq \frac{1}{15}\right)$ events E and F are *not* independent.

$P(E \cap F) \neq 0$, so it can be concluded that E and F are *not* mutually exclusive.

Exercise 7.5

1. (i) 4 cards can be selected from a pack of 52 in

$\binom{52}{4}$ ways $= 270\,725$

2 queens can be selected in $\binom{4}{2}$ ways

$\therefore P(\text{exactly 2 queens}) = \dfrac{6}{270\,725}$

(ii) 4 spades can be selected in $\binom{13}{4}$ ways

$\therefore P(\text{4 spades}) = \dfrac{\binom{13}{4}}{\binom{52}{4}} = \dfrac{715}{270\,725} = \dfrac{11}{4165}$

or 0.00264

(iii) 4 red cards can be selected in $\binom{26}{4}$ ways

$\therefore P(\text{4 red cards}) = \dfrac{\binom{26}{4}}{\binom{52}{4}} = \dfrac{14\,950}{270\,725} = \dfrac{46}{833}$

(iv) 4 cards of the same suit can be

4 spades *or* 4 clubs *or* 4 hearts *or* 4 diamonds

$\therefore P(\text{4 cards of the same suit})$

$= 4 \times P(\text{4 spades})$

$= 4 \times \dfrac{11}{4165} = \dfrac{44}{4165}$

2. (i) A team of 4 can be chosen

in $\binom{11}{4}$ ways $= 330$

Selecting 2 men & 2 women on team $= \binom{6}{2} \times \binom{5}{2}$ ways

$= 15 \times 10$

$= 150$

$\therefore P(\text{team of 2 men and 2 women}) = \dfrac{150}{330} = \dfrac{5}{11}$

(ii) 1 man and 3 women can be selected

in $\binom{6}{1} \times \binom{5}{3}$ ways $= 60$

$\therefore P(\text{team of 1 man and 3 women}) = \dfrac{60}{330} = \dfrac{2}{11}$

(iii) A team of all women can be selected

in $\binom{5}{4}$ ways $= 5$

$\therefore P(\text{team of all women}) = \dfrac{5}{330} = \dfrac{1}{66}$

3. Four discs are chosen from 16 in

$\binom{16}{4}$ ways $= 1820$

(i) P(four discs are blue) $= \dfrac{\binom{5}{4}}{\binom{16}{4}} = \dfrac{5}{1820}$

$= \dfrac{1}{364}$

(ii) 4 discs same colour means:

4 blue, 4 red

$\therefore$ P(4 discs blue) *or* P(4 discs red)

$= \dfrac{1}{364} + \dfrac{\binom{6}{4}}{1820}$

$= \dfrac{1}{364} + \dfrac{15}{1820} = \dfrac{1}{364} + \dfrac{3}{364}$

$= \dfrac{4}{364} = \dfrac{1}{91}$

$\therefore$ P(4 discs of same colour) $= \dfrac{1}{91}$

(iii) P(4 discs of different colours)

means P(red disc) *and* P(blue disc)

and P(yellow disc) *and* P(green disc)

$\therefore \dfrac{\binom{6}{1} \times \binom{5}{1} \times \binom{3}{1} \times \binom{2}{1}}{1820} = \dfrac{180}{1820} = \dfrac{9}{91}$

$\therefore$ P(4 discs of different colours) $= \dfrac{9}{91}$

(iv) P(2 blue and 2 not blue)

$= \dfrac{\binom{5}{2} \times \binom{11}{2}}{1820} = \dfrac{550}{1820} = \dfrac{55}{182}$

$\therefore$ P(2 blue discs and 2 not blue) $= \dfrac{55}{182}$

4. (i) Disc numbers are 2, 3, … 10

Prime numbers are 2, 3, 5, 7

P(1st number prime) $= \dfrac{4}{9}$

P(2nd number prime) $= \dfrac{4}{9}$

$\therefore$ P(both discs show prime numbers)

$= \dfrac{4}{9} \times \dfrac{4}{9} = \dfrac{16}{81}$

(ii) 3 discs can be picked in $\binom{9}{3}$ ways

$= 84$

Odd-numbered discs are 3, 5, 7, 9

Even-numbered discs are 2, 4, 6, 8, 10

$$P(\text{picking 3 odd-numbered discs}) = \frac{\binom{4}{3}}{\binom{9}{3}} = \frac{4}{84}$$

$$P(\text{picking 3 even-numbered discs}) = \frac{\binom{5}{3}}{\binom{9}{3}} = \frac{10}{84}$$

$\therefore P(\text{3 odd- or 3 even-numbered discs})$

$$= \frac{4}{84} + \frac{10}{84} = \frac{14}{84} = \frac{1}{6}$$

5. 3 cards drawn from 9 $= \binom{9}{3} = 84$

Drawing the card numbered 8 means there are only 8 numbers to draw 3 numbers from.

$\therefore \binom{8}{3} = 56$

(i) $P(\text{card number 8 not drawn}) = \frac{56}{84} = \frac{2}{3}$

(ii) Odd-numbered cards are 1, 3, 5, 7, 9

$$P(\text{all 3 cards have odd numbers}) = \frac{\binom{5}{3}}{\binom{9}{3}}$$

$$= \frac{10}{84} = \frac{5}{42}$$

6. Sample space $= \binom{24}{3} = 2024$

(i) $P(\text{3 boys celebrating birthday}) = \dfrac{\binom{14}{3}}{2024} = \dfrac{364}{2024}$

$P(\text{3 girls celebrating birthday}) = \dfrac{\binom{10}{3}}{2024} = \dfrac{120}{2024}$

$\therefore P(\text{students are 3 boys or 3 girls})$

$$= \frac{364}{2024} + \frac{120}{2024} = \frac{484}{2024} = \frac{11}{46}$$

(ii) *P*(a person has a birthday on a particular day in the week)

$= \frac{1}{7}$

P(a person does *not* have a birthday on a particular day in the week)

$= \frac{6}{7}$

Total probability of a birthday is $\frac{7}{7}$ (i.e. a certainty)

P(one of the 3 has a birthday on a particular day of the week)

$= \frac{7}{7}$ i.e. 1

P(the next of the 3 has a birthday on a different day from the first)

$= \frac{6}{7}$

P(the third has a birthday on a different day from the two above)

$= \frac{5}{7}$

Hence,

P(their birthdays fall on different days of the week)

$= 1 \times \frac{6}{7} \times \frac{5}{7} = \frac{30}{49}$

7. (i) $\binom{10}{7}$ ways $= 120$

(ii) Include Q_1, Q_2: $\therefore \binom{8}{5}$ ways $= 56$

(iii) P(choosing both Q_1 and Q_2) $= \frac{56}{120} = \frac{7}{15}$

(iv) P(choosing at least one of Q_1 and Q_2):

We can use $1 - P$(neither Q_1 nor Q_2 chosen)

Excluding Q_1 and Q_2 requires choice of selecting from 8 questions

$\therefore$ Selection is $\binom{8}{7} = 8$

$\therefore P$(neither Q_1 nor Q_2 chosen) $= \frac{8}{120}$

$\therefore 1 - P$(neither Q_1 nor Q_2)

$= 1 - \frac{8}{120}$

$= 1 - \frac{1}{15} = \frac{14}{15}$

8. 2 pupils to be chosen as prefects can be done in $\binom{16}{2}$ ways $= 120$

(i) P(one girl and one boy):

One girl can be selected in $\binom{10}{1}$ ways

One boy can be selected in $\binom{6}{1}$ ways

$\therefore P$(one boy and one girl)

$$= \frac{\binom{10}{1} \times \binom{6}{1}}{\binom{16}{2}} = \frac{10 \times 6}{120} = \frac{60}{120}$$

$$= \frac{1}{2}$$

(ii) To select left-handed girl is $\binom{3}{1}$

To select left-handed boy is $\binom{1}{1}$

$\therefore P$(one girl left-handed and one boy left-handed)

$$= \frac{\binom{3}{1} \times \binom{1}{1}}{\binom{16}{2}} = \frac{3 \times 1}{120} = \frac{3}{120}$$

$$= \frac{1}{40}$$

(iii) P(two left-handed pupils)

$$= \frac{\binom{4}{2}}{\binom{16}{2}} = \frac{6}{120} = \frac{1}{20}$$

(iv) P(at least one pupil who is left-handed)

$= P$(one pupil left-handed) **and** P(two pupils left-handed)

P(one pupil left-handed and one not left-handed)

$$= \frac{\binom{4}{1} \times \binom{12}{1}}{\binom{16}{2}} = \frac{4 \times 12}{120} = \frac{48}{120}$$

P(two left-handed pupils) $= \frac{1}{20}$ [see part (iii)]

$\therefore$ P(at least one pupil left-handed)

$$= \frac{48}{120} + \frac{1}{20} = \frac{48}{120} + \frac{6}{120}$$

$$= \frac{54}{120} = \frac{9}{20}$$

9. Given 1 fair dice and 2 biased dice. Bias assigns 6 as twice as likely as any other score.

$\therefore$ Scoring on bias dice $= P(6) = \frac{2}{7}$ and $P(\text{not } 6) = \frac{5}{7}$

P(rolling exactly two sixes) $=$

P(6 on 1st, 6 on second, not 6) *or*

P(6 on 1st, not 6 on second, 6) *or*

P(not six on 1st, 6 on second, 6)

$$\therefore \left(\frac{1}{6} \times \frac{2}{7} \times \frac{5}{7}\right) + \left(\frac{1}{6} \times \frac{5}{7} \times \frac{2}{7}\right) + \left(\frac{5}{6} \times \frac{2}{7} \times \frac{2}{7}\right)$$

$$= \frac{10}{294} + \frac{10}{294} + \frac{20}{294}$$

$$= \frac{40}{294} = \frac{20}{147}$$

10. Of the 8 letters, there are 2 A's, 3 P's and C, E, L.

(i) P(letters P, E, A drawn in that order)

$$= \frac{1}{\binom{8}{3}} = \frac{1}{56}$$

(ii) P(letters P, E, A are drawn in any order)

$$= \frac{\binom{3}{1} \times \binom{2}{1} \times \binom{1}{1}}{\binom{8}{3}} = \frac{3 \times 2 \times 1}{56}$$

$$= \frac{6}{56} = \frac{3}{28}$$

(iii) P(Excluding letters E and P)

$$= \frac{\binom{4}{3}}{\binom{8}{3}} = \frac{4}{56} = \frac{1}{14}$$

(iv) Consonants $= C, L, P$

Vowels $= A, E$

P(three letters all vowels)

$$= \frac{\binom{5}{3}}{\binom{8}{3}} = \frac{10}{56}$$

$$P(\text{3 letters all consonants}) = \frac{\binom{3}{3}}{\binom{8}{3}} = \frac{1}{56}$$

$\therefore$ P(3 letters are all consonants *or* all vowels)

$$= \frac{10}{56} + \frac{1}{56}$$

$$= \frac{11}{56}$$

Exercise 7.6

1. A possible generation can be carried out by generating random numbers 1–20 on a calculator. A simulation like the one above indicates that you need to buy 34 packets of crisps to get the full set. Repeat the simulation as many times as you like. The more times you repeat the experiment, the more confidence you can have in your results.

2. 3 food options = meat, fish, vegetarian

Allocate numbers 1–8, allowing No. 1 and 2 be fish (told probability is $\frac{2}{8}$)

Allocate No. 3 to vegetarian i.e. $\frac{1}{8}$

Allocate numbers 4, 5, 6, 7 and 8 to meat i.e. meat $= \frac{5}{8}$

3. A possible simulation would be to toss 4 coins where

H (head) stands for boy

T (tail) stands for girl

Outcomes of one such experiment

1.	HHTT	2B 2G
2.	HHHT	3B 1G
3.	TTTH	1B 3G
4.	HHTT	2B 2G
5.	HTTH	2B 2G
6.	HHTT	2B 2G
7.	HTTT	1B 3G
8.	HHHT	3B 1G
9.	HHTT	2B 2G
10.	TTTH	1B 3G
11.	HHTT	2B 2G
12.	TTHH	2B 2G
13.	TTHH	2B 2G
14.	TTTT	0B 4G
15.	HTTT	1B 3G
16.	HHTT	2B 2G

After 16 tosses:

(i) Probability that the girls outnumber the boys is $\frac{5}{16} = 0.3125$

(ii) Probability that all the 4 children are girls is $\frac{1}{16} = 0.0625$

4. You could generate random numbers; Allocate numbers 0 and 1 for cars turning right. Since 80% of cars turn left, allocate numbers 2, 3, 4, 5, 6, 7, 8, 9 for cars turning left. (The random numbers can be generated on a calculator or use a random number table.)

5. (i) $P(\text{win away}) = 0.4$

$P(\text{win at home}) = 0.7$

$\therefore$ In 12 home games $P(\text{winning})$

$= 12 \times 0.7$

$= 8.4$ games

$\therefore$ In 13 away games $P(\text{winning})$

$= 13 \times 0.4$

$= 5.2$ games

$\therefore$ The **Ringdogs** should win

$8.4 + 5.2 = 13.6$ games

$= 14$ games

(ii) The results of a simulation do approximately agree with the result above.

6. Possible simulations with discs, counters, calculators, computers, or even get your friends to buy the same breakfast cereal so they will have all 8 superhero figures.
Two possible simulations are presented by generating random number tables (numbers 1–8).

Simulation result:

1	4	3	7
5	6	8	1
6	7	2	5
8	2		

Based on this simulation, you would need to buy 14 packets.
Another simulation resulted in:

5	1	6	4	2
6	4	6	1	6
3	4	3	1	4
1	7	6	6	1
7	6	8		

In this case, 23 packets of *Chocopops* were purchased in order to collect the full set.
The more the experiment is repeated, the more confidence you have in the results.

7. $P(\text{at least one } 6) = 1 - P(\text{no six})$

$P(\text{no six in 4 rolls of a dice})$

$$= \binom{4}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4$$

$$= (1)(1)\,\frac{625}{1296}$$

$$= 0.48$$

Since $P(6) = \frac{1}{6}$ and $P(\text{not } 6) = \frac{5}{6}$

$\therefore$ $P(\text{at least one } 6)$

$= 1 - 0.48$

$= 0.52$

8. The likely size of a family that contains (at least) one child of each gender is 3.
A simulation could assume an equal chance of being a boy or a girl. You could toss coins, or roll dice, to simulate the gender of the children.
Generally, the probability of boys and girls in families are approximately $\frac{1}{2}$.

Revision Exercise 7 (Core)

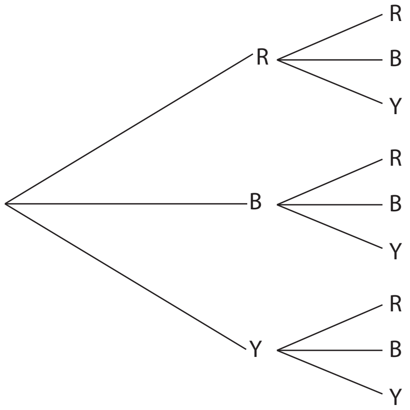
1. (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= (0.5) + (0.4) - (0.7)$$
$$= 0.2$$

(ii) $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{0.2}{0.4}$$
$$= 0.5$$

(iii) $$P(B|A) = \frac{P(A \cap B)}{P(A)}$$
$$= \frac{0.2}{0.5}$$
$$= 0.4$$

2. (ii) Event; selecting two counters from a bag of red, blue and yellow counters.



3. $P(\text{sink a 1 m putt}) = 0.7$

$P(\text{not sink 1 m putt}) = 0.3$

$\therefore P(\text{sink 3 in 4 attempts})$

$$= \binom{4}{3}(0.7)^3(0.3)^1$$
$$= 4 \times 0.343 \times 0.3$$
$$= 0.4116$$

4. (i) Children can be selected in $\binom{30}{5}$ ways

$$= 142\,506$$

(ii) No. of selections with 2 boys and 3 girls

$$= \binom{10}{2} \times \binom{20}{3}$$
$$= 45 \times 1140$$
$$= 51\,300$$

(iii) $P(\text{exactly 2 boys selected})$

$$= \frac{51\,300}{142\,506}$$
$$= \frac{950}{2639}$$
$$= 0.0359$$
$$= 0.36$$

5. (i) $p = 0.8, \quad q = 0.2$

$$P(\text{1st success in 4th trial}) = q^3 \, . \, p$$

$$= (0.2)^3(0.8) = \frac{4}{625}$$

(i) $p = 0.2, \quad q = 0.8$

$$P(\text{1st success in 4th trial}) = q^3 \, . \, p$$

$$= (0.8)^3(0.2) = \frac{64}{625}$$

6. $P(\text{success} - \text{defective}) = \frac{1}{5}$

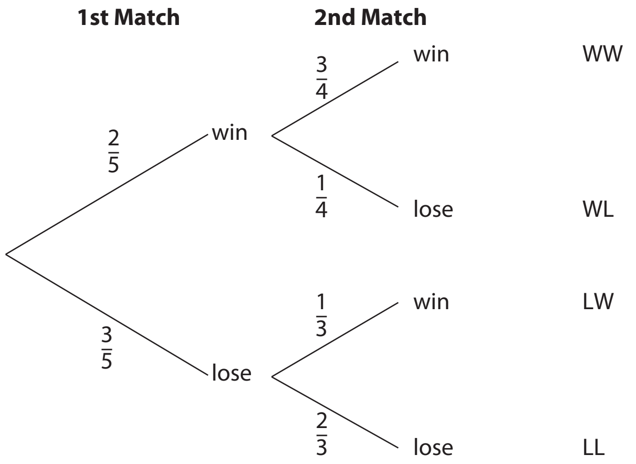
$P(\text{failure} - \text{not defective}) = \frac{4}{5}$

$$P(\text{no item defective}) = \binom{4}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^4$$

$$= (1)(1)\left(\frac{256}{625}\right)$$

$$= \frac{256}{625}$$

7.



(i) $P(\text{loses both matches}) = \frac{3}{5} \times \frac{2}{3}$

$$= \frac{6}{15} = \frac{2}{5}$$

(ii) $P(\text{wins only one match}) = P(\text{wins 1st, loses 2nd}) \quad \text{or} \quad P(\text{loses 1st, wins 2nd})$

$$= \left(\frac{2}{5} \times \frac{1}{4}\right) + \left(\frac{3}{5} \times \frac{1}{3}\right)$$

$$= \frac{2}{20} + \frac{3}{15}$$

$$= \frac{3}{10}$$

8. (i) $P(E) = 0.5$

(ii) $P(F) = 0.8$

(iii) $P(E \cup F) = 0.9$

If E and F are independent, then from diagram, $P(E \cap F) = 0.4$

Also, $P(E \cap F) = P(E) \times P(F)$

$$= 0.5 \times 0.8$$

$$= 0.4$$

$\therefore$ Since $P(E \cap F) = P(E) \times P(F) = 0.4$, events E and F are independent.

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$
$$= \frac{0.4}{0.8} = 0.5$$
$$\therefore P(E \mid F) = 0.5$$

9. $P(\text{ace}) = \frac{4}{52} = \frac{1}{13}$

$P(\text{not ace}) = \frac{12}{13}$

Drawing an ace wins €10,

so $\therefore$ Net win = 10 − 1 (entry cost)

= €9

Customer spends €12 on the other turns of not getting an ace.

$\therefore$ Expected profit $= \frac{12-9}{13} = \frac{3}{13}$

= 0.23 cents

10. The first number can be taken in 4 ways.

The second number can be taken in 3 ways.

$\therefore$ the two cards can be picked in 4 × 3 (i.e. 12) ways.

If 1 is picked, then 2, 3, 4 are higher ⇒ 3 ways
If 2 is picked, then 3, 4 are higher ⇒ 2 ways
If 3 is picked, then 4 only is higher ⇒ 1 way
[Note: Obviously if 4 is picked then the 2nd card cannot be higher, i.e. "0 ways"]

$$\therefore P(\text{2nd number is higher than first number}) = \frac{3+2+1}{12}$$
$$= \frac{6}{12} = \frac{1}{2}$$

Revision Exercise 7 (Advanced)

1. A tennis match has 2 or 3 sets.

$P(A \text{ wins a set}) = \frac{2}{3}$; $P(B \text{ wins a set}) = \frac{1}{3}$

To find $P(A$ wins the match in two or three sets) is made up of these three probabilities:

(i) $P(A \text{ wins}, A \text{ wins}) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

or (ii) $P(A \text{ wins}, A \text{ loses}, A \text{ wins}) = \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{27}$

or (iii) $P(A \text{ loses}, A \text{ wins}, A \text{ wins}) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$

$$\therefore \quad P(A \text{ wins the match}) = \frac{4}{9} + \frac{4}{27} + \frac{4}{27}$$
$$= \frac{12}{27} + \frac{4}{27} + \frac{4}{27} = \frac{20}{27}$$

2. $P(\text{team fully fit and win game}) = \frac{7}{10} \times \frac{9}{10} = \frac{63}{100}$

$P(\text{team not fully fit and win}) = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100}$

$$\therefore P(\text{team wins next home game}) = \frac{63}{100} + \frac{12}{100}$$
$$= \frac{75}{100} = 0.75$$

3. $P(\text{E}) = \frac{1}{5} \quad P(\text{F}) = \frac{1}{7}$

(i) Since events are independent,

$$\therefore P(\text{E} \cap \text{F}) = P(\text{E}) \times P(\text{F}) = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35}$$

(ii) $$P(\text{E} \cup \text{F}) = P(\text{E}) + P(\text{F}) - P(\text{E} \cap \text{F}) = \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \frac{11}{35}$$

4. (i) $$P(\text{student does not study Biology}) = \frac{21}{56} = \frac{3}{8}$$

(ii) Number of students who study at least 2 subjects = 26

$$P(\text{student studying 2 subjects at least does not study Biology}) = \frac{4}{26} = \frac{2}{13}$$

(iii) There are 56 students in the class.

$$P(\text{both students picked randomly study Physics}) = \frac{\binom{28}{2}}{\binom{56}{2}} = \frac{378}{1540} = \frac{27}{110}$$

(iv) 25 students study Chemistry. $\text{C} \cap \text{B} = 13$ students studying both.

P(one of the two students picked studying Chemistry studies Biology) $= \frac{13}{25}$

$\Big[$ P(Biology, not biology) **or** P(Not biology, biology) i.e. $\left(\frac{13}{25} \times \frac{12}{24} + \frac{12}{25} \times \frac{13}{24}\right) = \frac{13}{25}$ $\Big]$

5. (i) $P(\text{C}) = \frac{20}{36} = \frac{5}{9}$

$P(\text{D}) = \frac{9}{36} = \frac{1}{4}$

$P(\text{C} \cap \text{D}) = \frac{5}{36}$

Then

$$P(\text{C}) \,.\, P(\text{D}) = \frac{5}{9} \times \frac{1}{4} = \frac{5}{36} = P(\text{C} \cap \text{D})$$

Hence C and D are independent.

(ii) (a) $$P(\text{X}|\text{Y}) = \frac{P(\text{X} \cap \text{Y})}{P(\text{Y})}$$

$$P(\text{X} \cap \text{Y}) = P(\text{Y}) \,.\, P(\text{X}|\text{Y}) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

(b) $$P(\text{Y}|\text{X}) = \frac{P(\text{X} \cap \text{Y})}{P(\text{X})}$$

$$= \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$$

6. (i) $P(A \text{ qualifies for 5000 m race}) = \frac{3}{5}$

$P(A \text{ qualifies for 10 000 m race}) = \frac{1}{4}$

$\therefore P(A \text{ qualifies for both races}) = \frac{3}{5} \times \frac{1}{4}$

$= \frac{3}{20} = 0.15$

(ii) $P(\text{exactly one of the athletes qualifies for 5000 m})$

$= P(A \text{ qualifies and } B \text{ does not}) \text{ or } P(A \text{ does not qualify \& } B \text{ does})$

$= \left(\frac{3}{5} \times \frac{1}{3}\right) + \left(\frac{2}{3} \times \frac{2}{5}\right)$

$= \frac{3}{15} + \frac{4}{15}$

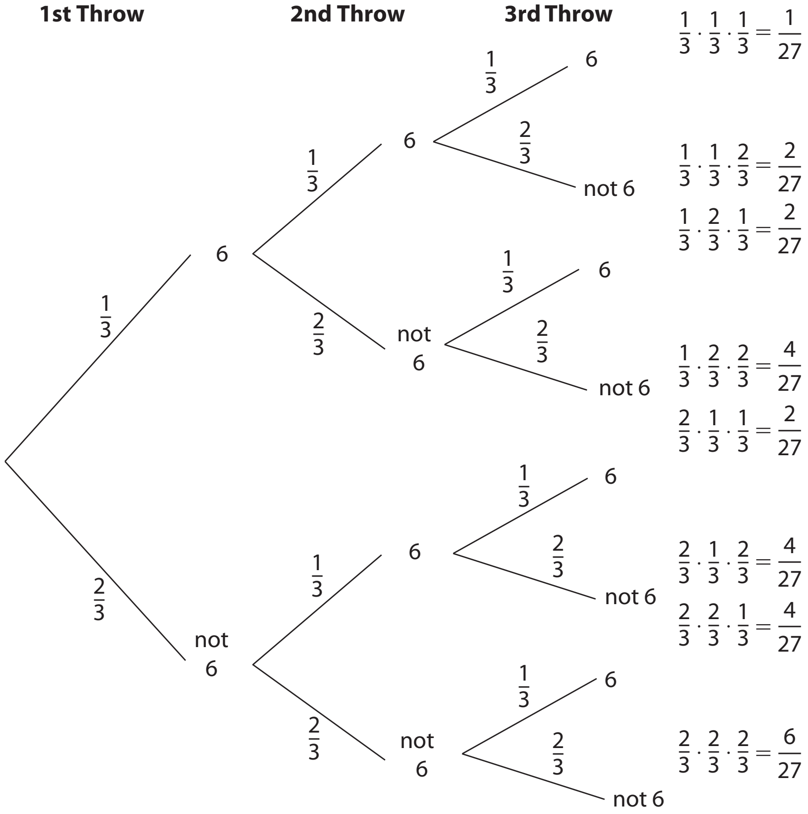
$= \frac{7}{15}$

(iii) $P(\text{athlete } A \text{ qualifies for 10 000 m}) = \frac{1}{4}$

$P(\text{athlete } B \text{ qualifies for 10 000 m}) = \frac{2}{5}$

$P(\text{both athletes qualify for 10 000 m race}) = \frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}$

7. (i)



At least 1 six in three throws means 1 six, or 2 sixes, or 3 sixes.

$\therefore P(\text{at least one six in 3 throws}) = \frac{1}{27} + \frac{2}{27} + \frac{2}{27} + \frac{4}{27} + \frac{2}{27} + \frac{4}{27} + \frac{4}{27} = \frac{19}{27}$

(ii) Given:

$$P(\text{A}) = \frac{2}{3} \qquad P(\text{A} \cup \text{B}) = \frac{3}{4} \qquad P(\text{A} \cap \text{B}) = \frac{5}{12}$$

To find $P(\text{B})$:

$$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$$

$$\frac{3}{4} = \frac{2}{3} + P(\text{B}) - \frac{5}{12}$$

$$\frac{3}{4} - \frac{2}{3} + \frac{5}{12} = P(\text{B})$$

$$\frac{6}{12} = P(\text{B})$$

$$\therefore P(B) = \frac{1}{2}$$

8. Expected value of payout

Payout (x)	**Probability (P)**	$x \times P$
€50	$\frac{1}{4}$	$12\frac{1}{2}$
€10	$\frac{1}{4}$	$2\frac{1}{2}$
€5	$\frac{1}{3}$	$1\frac{2}{3}$
€20	$\frac{1}{6}$	$3\frac{1}{3}$

$$\sum x\,.\,P(x) = 12.5 + 2.5 + 1.6666 + 3.3333$$
$$= €20$$

∴ Expected value of the payout is €20.
But it costs €25 to spin the spinner, so you expect to lose €5.
This game is not fair since expected payout does not equal zero.

9. (i) $n = 6 \qquad P(\text{six}) = \frac{1}{6} \qquad P(\text{not } 6) = \frac{5}{6}$

P(two sixes in first 6 rolls)

$$\therefore = \binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4$$

$$= 15 \times \frac{1}{36} \times \frac{625}{1296}$$

$$= \frac{3125}{15\,552} = 0.2$$

(ii) P(second 6 on sixth roll) **and** P(a six in the first 5 rolls)

$$= \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4$$

$$= 5 \times \frac{1}{6} \times \frac{625}{1296}$$

$$= \frac{3125}{7776}$$

$$= 0.40187$$

$$P(\text{a six on 6th roll}) = \frac{1}{6}$$

$\therefore P(\text{a second 6 on the 6th roll})$

$$= 0.40187 \times \frac{1}{6}$$

$$= 0.0669$$

$$= 0.067$$

10. (i) Given: $P(E) = \frac{2}{3} \quad P(E|F) = \frac{2}{3} \quad P(F) = \frac{1}{4}$

To find $P(E \cap F)$:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\therefore \frac{2}{3} = \frac{P(E \cap F)}{\frac{1}{4}}$$

$$\therefore P(E \cap F) = \frac{2}{3} \times \frac{1}{4}$$

$$= \frac{2}{12} = \frac{1}{6}$$

(ii) $P(F|E) = \dfrac{P(F \cap E)}{P(E)}$

$$P(F|E) = \frac{\frac{1}{6}}{\frac{2}{3}}$$

$$= \frac{1}{6} \cdot \frac{3}{2} = \frac{3}{12}$$

$$= \frac{1}{4}$$

Yes, E and F are independent events as $P(E \cap F) = P(E) \times P(F)$.

Revision Exercise 7 (Extended-Response Questions)

1. (i) Possible paths are:

ABEH and ACEH

(ii) Paths from A:

ABDGL ABDGM

ABDHM ABDHN

ABEHM ABEHN

ABEJN ABEJP

ACEJN ACEJP

ACFJP ACFJN

ACEHN ACFKQ

ACFKP ACEHM

$P(\text{marble passes through H or J})$

$$= \frac{12}{16} = \frac{3}{4}$$

(iii) $P(\text{marble lands at N})$

$$= \frac{6}{16} = \frac{3}{8}$$

(iv) $P(\text{two marbles from A land at P}) = \frac{1}{16}$

Both go separately but there is only 1 way.

2. (i) $P(\text{success}) = 0.7, \quad P(\text{failure}) = 0.3$

$P(\text{1st goal on 3rd attempt}) = P(\text{not goal}) \,.\, P(\text{not goal}) \,.\, P(\text{goal})$

$= 0.3 \times 0.3 \times 0.7$

$= 0.063$

(ii) $P(\text{score exactly 3 goals in 5 attempts})$

$$= \binom{5}{3}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$$

$$= 1\cancel{0} \cdot \frac{343}{1000} \cdot \frac{9}{10\cancel{0}} = \frac{3087}{10\,000}$$

$= 0.3087$

$= 0.309$

(iii) $P(\text{two goals in six attempts})$

$$= \binom{6}{2}\left(\frac{7}{10}\right)^2\left(\frac{3}{10}\right)^4$$

$$= 15 \times \frac{49}{100} \times \frac{81}{10\,000}$$

$= 0.059$

$P(\text{a goal on seventh attempt})$

$$= \frac{7}{10}$$

$\therefore P(\text{third goal on seventh attempt})$

$$= 0.059 \times \frac{7}{10}$$

$= 0.0416$

$= 0.042$

3. (a) Given $P(\text{A}) = \frac{13}{25}, \quad P(\text{B}) = \frac{9}{25}, \quad P(\text{A}|\text{B}) = \frac{5}{9}$

(i) To find $P(\text{A and B})$, i.e. $P(\text{A} \cap \text{B})$;

$$P(\text{A}|\text{B}) = \frac{P(\text{A} \cap \text{B})}{P(\text{B})}$$

$$\frac{5}{9} = \frac{P(\text{A} \cap \text{B})}{\frac{9}{25}}$$

$$\therefore \quad P(\text{A} \cap \text{B}) = \frac{\cancel{5}^1}{9} \times \frac{9}{\cancel{25}_5} = \frac{9}{45}$$

$$= \frac{1}{5}$$

(ii)
$$P(\text{B}|\text{A}) = \frac{P(\text{B} \cap \text{A})}{P(\text{A})}$$

$$= \frac{\frac{1}{5}}{\frac{13}{25}}$$

$$\therefore \quad \frac{1}{\cancel{5}_1} \times \frac{\cancel{25}^5}{13} = \frac{5}{13}$$

(iii) $P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

$$= \frac{13}{25} + \frac{9}{25} - \frac{1}{5}$$

$$= \frac{17}{25}$$

(b) $P(6) = p$

$P(1) = P(2) = P(3) = P(4) = P(5) = \dfrac{1-p}{5}$

With a fair dice, all throws 1–6 have a probability of $\frac{1}{6}$.

Number of possible outcomes with 2 dice = 36.
Scores totalling 7 are (3, 4), (4, 3), (5, 2), (2, 5), (6, 1), (1, 6);
all independent of p.

$$\therefore P(\text{rolling a total of 7}) = \frac{6}{36} = \frac{1}{6}$$

4. (i) The chart is filled in below.

	Girl	Boy	Total
Basketball	6	8	14
Does not play basketball	9	5	14
Total	15	13	28

(ii) $P(G) = \dfrac{15}{28}$

(iii) $P(B) = \dfrac{14}{28} = \dfrac{1}{2}$

(iv) $P(B') = 1 - P(B) = \dfrac{1}{2}$

(v) $P(G|B) = \dfrac{6}{14} = \dfrac{3}{7}$

(vi) $P(B|G) = \dfrac{6}{15} = \dfrac{2}{5}$

(vii) $P(B|G') = \dfrac{8}{13}$

(viii) $P(B \cap G) = \dfrac{6}{28} = \dfrac{3}{14}$

5. (i) Bag has 4 red, 6 green counters.
4 counters drawn at random.
P(all counters drawn are green)

$$= \frac{\binom{6}{4}}{\binom{10}{4}} = \frac{15}{210} = \frac{1}{14}$$

(ii) P(at least one counter of each colour is drawn)
$\therefore P(1R, 3G)$ or $P(2R, 2G)$ or $P(3R, 1G)$

$$\therefore \frac{\binom{4}{1}\binom{6}{3}}{\binom{10}{4}} + \frac{\binom{4}{2}\binom{6}{2}}{\binom{10}{4}} + \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}}$$

$$\therefore \frac{4 \times 20}{210} + \frac{6 \times 15}{210} + \frac{(4)(6)}{210}$$

$$= \frac{194}{210} = \frac{97}{105}$$

$$\therefore P(\text{one at least of each colour is drawn}) = \frac{97}{105}$$

(iii) P(at least 2 green counters drawn)

$$\therefore P(2R, 2G) + P(1R, 3G) + P(\text{all 4 green})$$

$$= \frac{\binom{4}{2}\binom{6}{2}}{210} + \frac{\binom{4}{1}\binom{6}{3}}{210} + \frac{\binom{6}{1}}{210}$$

$$= \frac{90}{210} + \frac{80}{210} + \frac{15}{210}$$

$$= \frac{185}{210}$$

$$= \frac{37}{42}$$

(iv) P(at least 2G drawn given that at least one of each colour is drawn)

Choices are:

1R, 3G or 2R, 3G

$$P = \frac{\binom{4}{1}\binom{6}{3}}{210} + \frac{\binom{4}{2}\binom{6}{2}}{210}$$

$$= \frac{4.20}{210} + \frac{6.15}{210}$$

$$= \frac{80 + 90}{210} = \frac{170}{210}$$

$$= \frac{17}{21}$$

The two events are not independent since the answers in (iii) and (iv) are different.

6. (i) $P(G|P) = \frac{48}{64} = \frac{3}{4}$

(ii) $P(G \cap P) = \frac{48}{150} = \frac{8}{25}$

(iii) $P(G \cup P) = \frac{48 + 16 + 24}{150} = \frac{44}{75}$

(iv) No, as $P(G) . P(P) = \frac{72}{150} \times \frac{64}{150} = \frac{128}{625}$ which is not the same as $P(G \cap P) = \frac{8}{25}$.

(v) No, as it is possible for a person to be getting good reception (G) while using Provider A(P).

7. (a) (i) Since $\sum$ probabilities $= 1$

$\therefore 0.1 + a + b + 0.2 + 0.1 = 1$

$\therefore a + b = 0.6$

(ii) $\sum x . P(x) = 2.9$

$\therefore 0.1 + 2a + 3b + 0.8 + 0.5 = 2.9$

$\therefore 2a + 3b = 2.9 - 1.4$

$\therefore 2a + 3b = 1.5$

Solve:

$$\begin{aligned} a + b &= 0.6 \\ 2a + 3b &= 1.5 \end{aligned} \qquad \begin{aligned} \cancel{2a} + 3b &= 1.5 \\ \cancel{2a} + 2b &= 1.2 \quad \text{(subtract)} \\ b &= 0.3 \end{aligned}$$

$a + b = 0.6$

$a + 0.3 = 0.6$

$\therefore \quad a = 0.3$

$\therefore \quad a = 0.3, \quad b = 0.3$

(b) 16 girls 8 boys

12 study french

let girl studying french $= x$

let boy studying french $= y$

$\therefore x + y = 12$ (i)

$P(\text{girl study F}) = \dfrac{x}{16}$ $P(\text{boy study F}) = \dfrac{y}{8}$

$\therefore \dfrac{x}{16} = \dfrac{3}{2}\left(\dfrac{y}{8}\right)$ (ii)

$x + y = 12,$ so $\therefore x = 12 - y$

$\therefore \dfrac{12 - y}{16} = \dfrac{3y}{16}$ so $\therefore 12 - y = 3y$

$\therefore 4y = 12$ $\therefore y = 3$ (boy)

Hence, $x = 12 - 3 = 9$ (girl)

$\therefore$ 3 boys and 9 girls study french.

8. (i) The spinner since scores are added.

(ii) Ann: Dice

Outcome (x)	1	2	3	4	5	6
Probability (P)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$x \times P(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1

$\therefore \sum x \,.\, P(x) = 3.5$

Jane: Spinners

Outcome (x)	1	2	3
Probability (P)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$x \times P(x)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$
$2[x \,.\, P(x)]$	$\frac{2}{3}$	$\frac{4}{3}$	2

$\therefore \sum x \,.\, P(x) = 4$

Spinners have a better chance of reaching 20 points first as expected outcome is 4, whereas for the dice it is 3.5.

9. (i) $P(\text{H}) = \dfrac{1}{2}$ $P(\text{T}) = \dfrac{1}{2}$

$$P(\text{3H, 2 tails}) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \times \frac{1}{4} \times \frac{1}{8}$$

$$= 10 \times \frac{1}{32} = \frac{10}{32} = \frac{5}{16}$$

i.e. 16 outcomes with 5 showing 3H's, 2 tails.

Note: You can fully write out the outcomes also.

(ii) If the 5 coins are tossed 8 times:

$\therefore$ Probability (3H, 2T) $= \dfrac{5}{16}$

$\therefore P(\text{not getting 3H, 2T}) = \dfrac{11}{16}$

$\therefore P(\text{getting 3H, 2T exactly 4 times})$

$= \binom{8}{4}\left(\frac{5}{16}\right)^4\left(\frac{11}{16}\right)^4$

$= 70 \cdot \frac{625 \times 14\,641}{4\,294\,967\,296}$

$= 0.1491$

$= 0.149$

10. Given 10 $\boxed{\text{R}}$, 15 $\boxed{\text{G}}$, 8 $\boxed{\text{R}}$, 12 $\boxed{\text{G}}$

E = event $\square$ is drawn.

F = event that green shape is drawn.

$\therefore P(\text{E}) = \frac{25}{45}$

$\therefore P(\text{F}) = \frac{27}{45}$

(i) $P(\text{E} \cap \text{F}) = P(\text{a square that is green})$

$= \frac{15}{45} = \frac{1}{3}$

(ii) $P(\text{E} \cap \text{F}) = P(\text{square drawn } \textit{or} \text{ a green shape drawn})$

$= \frac{10 + 15 + 12}{45} = \frac{37}{45}$

(iii) Yes, evens E and F are independent as $P(\text{E} \cap \text{F}) = P(\text{E}) \times P(\text{F})$.

$P(\text{E} \cap \text{F}) = \frac{1}{3}$ and $P(\text{E}) \times P(\text{F})$

$= \frac{25}{45} \times \frac{27}{45}$

$= \frac{675}{2025} = \frac{1}{3}$

(iv) No, E and F are not mutually exclusive events as

$P(\text{E} \cup \text{F}) \neq P(\text{E}) + P(\text{F}) \left(\text{i.e.} \frac{37}{45} \neq \frac{25}{45} + \frac{27}{45}\right)$

Chapter 8

Exercise 8.1

1. Not a function because the input 2 has two different outputs, ie. 5 and 10.

2. (i) Is a function

(ii) Not a function because the input -2 has two different outputs, ie. 1 and 5.

(iii) Is a function

3. (i) Yes

(ii) No, as inputs a, c each have two different outputs.

(iii) No, as input 9 has two different outputs, ie. 14 and 20.

(iv) Yes

4. Rule: $y = 2x - 4$

$x = -1 \Rightarrow y = 2(-1) - 4 = -6$

$x = 0 \quad \Rightarrow y = 2(0) - 4 = -4$

$x = 1 \quad \Rightarrow y = 2(1) - 4 = -2$

$x = 2 \quad \Rightarrow y = 2(2) - 4 = 0$

$x = 3 \quad \Rightarrow y = 2(3) - 4 = 2 \qquad \Rightarrow$ couples $= (-1, -6), (0, -4), (1, -2), (2, 0), (3, 2)$

$\Rightarrow$ Range $= \{-6, -4, -2, 0, 2\}$

5. $f(x) = 3x - 2$

(i) $f(2) = 3(2) - 2 = 4$

(ii) $f(-3) = 3(-3) - 2 = -11$

(iii) $f(k) = 3k - 2$

(iv) $f(2k - 1) = 3(2k - 1) - 2 = 6k - 3 - 2 = 6k - 5$

6. $g(x) = (x - 2)^2$

(i) $g(4) = (4 - 2)^2 = 2^2 = 4$

(ii) $g(-4) = (-4 - 2)^2 = (-6)^2 = 36$

(iii) $g(8) = (8 - 2)^2 = (6)^2 = 36$

(iv) $g(a) = (a - 2)^2 = a^2 - 4a + 4$

7. $f(x) = 3x - 4$

$f(k) = 3k - 4$

$f(2k) = 3(2k) - 4 = 6k - 4$

Hence, $f(k) + f(2k) = 0$

$\Rightarrow 3k - 4 + 6k - 4 = 0$

$\Rightarrow 9k - 8 = 0 \quad \Rightarrow 9k = 8 \quad \Rightarrow k = \frac{8}{9}$

8. $f(x) = 4x$ and $g(x) = x + 1$

$f(3) = 4(3) = 12 \quad g(3) = 3 + 1 = 4$

Hence, $g(3) + k[f(3)] = 8$

$\Rightarrow 4 + k(12) = 8$

$\Rightarrow 12k = 4 \quad \Rightarrow \quad k = \frac{4}{12} = \frac{1}{3}$

9. $f(x) = 2x^2 - 1$ and $g(x) = x + 2$

(i) Solve $f(x) = 3 \Rightarrow 2x^2 - 1 = 3$

$\Rightarrow 2x^2 - 4 = 0$

$\Rightarrow \quad x^2 - 2 = 0 \quad \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

(ii) Solve $g(x) = f(3)$

$\Rightarrow x + 2 = 2(3)^2 - 1 = 17 \quad \Rightarrow x = 17 - 2 = 15$

(iii) $f(x) = g(x)$

$\Rightarrow 2x^2 - 1 = x + 2$

$\Rightarrow 2x^2 - x - 3 = 0$

$\Rightarrow (2x - 3)(x + 1) = 0$

$\Rightarrow 2x - 3 = 0 \quad$ OR $\quad x + 1 = 0$

$\Rightarrow \quad 2x = 3 \quad$ OR $\quad x = -1$

$\Rightarrow x = 1\frac{1}{2}, -1$

10. $f(x) = 1 + \frac{2}{x}$

(i) $f(-4) = 1 + \frac{2}{-4} = 1 - \frac{1}{2} = \frac{1}{2}$

$f\left(\frac{1}{5}\right) = 1 + \frac{2}{\frac{1}{5}} = 1 + 10 = 11$

(ii) $f(x) = 2$

$\Rightarrow 1 + \frac{2}{x} = 2$

$\Rightarrow \quad \frac{2}{x} = 1 \Rightarrow x = 2$

(iii) $kf(2) = f\left(\frac{1}{2}\right)$

$\Rightarrow k\left[1 + \frac{2}{2}\right] = 1 + \frac{2}{\frac{1}{2}}$

$\Rightarrow \quad 2k = 1 + 4 = 5 \quad \Rightarrow k = \frac{5}{2}$

11. $g(x) = 1 - 4x$

(i) $g(k + 1) = 1 - 4(k + 1) = 1 - 4k - 4 = -4k - 3$

(ii) Solve $g(k + 1) = g(-3)$

$\Rightarrow \quad -4k - 3 = 1 - 4(-3) = 13$

$\Rightarrow -4k = 13 + 3 = 16 \Rightarrow k = \frac{16}{-4} = -4$

12. $g(x) = 3x - 2$

(i) $g(-x) = 6$

$\Rightarrow 3(-x) - 2 = 6 \quad \Rightarrow \quad -3x = 8 \quad \Rightarrow \quad x = \frac{8}{-3} = -\frac{8}{3}$

(ii) $g(2x) = 4$

$\Rightarrow 3(2x) - 2 = 4 \quad \Rightarrow \quad 6x - 2 = 4 \;\Rightarrow\; 6x = 6 \;\Rightarrow\; x = 1$

(iii) $\frac{1}{g(x)} = 6 \;\Rightarrow \frac{1}{3x - 2} = \frac{6}{1}$

$\Rightarrow 18x - 12 = 1 \;\Rightarrow\; 18x = 13 \Rightarrow x = \frac{13}{18}$

13. (i) $f(x) = x^2 - 2x \;\Rightarrow x^2 - 2x = 3$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = -1, x = 3$

(ii) $g(x) = x^2 - x - 6 \;\Rightarrow g(x) = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow x = -2, 3$

(iii) $h(x) = x + \dfrac{1}{x} = 2$

$$\Rightarrow x.x + \frac{1}{x}\cdot x = 2.x$$

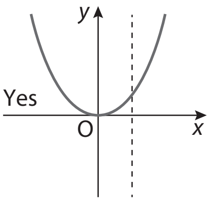
$$\Rightarrow x^2 + 1 = 2x$$

$$\Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)(x-1) = 0 \qquad \Rightarrow x = 1$$

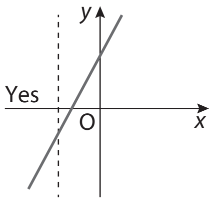
14. Use the vertical line test to determine if each of the following is the graph of a function where $x \in R$.

(i)



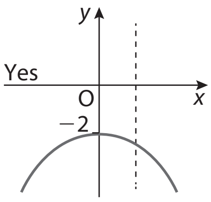
Yes

(ii)



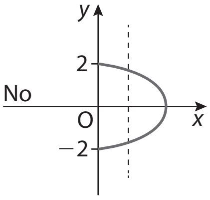
Yes

(iii)



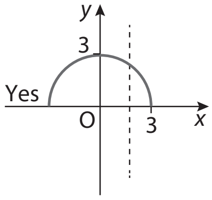
Yes

(iv)



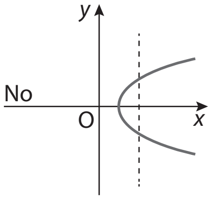
No

(v)



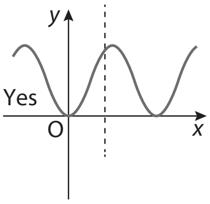
Yes

(vi)



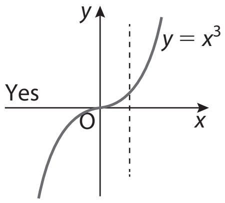
No

(vii)



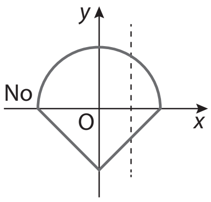
Yes

(viii)



Yes

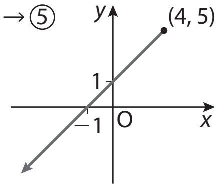
(ix)



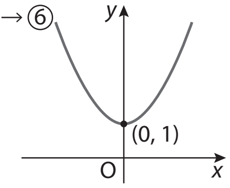
No

15. The graphs and the ranges of six relations are given below. Connect each graph to its correct range.

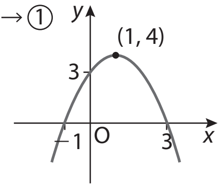
Ⓐ→⑤



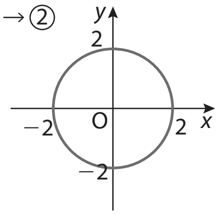
Ⓑ→⑥



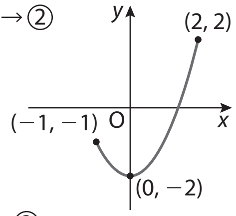
Ⓒ→①



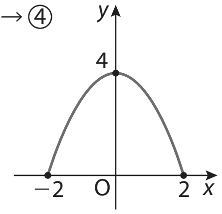
Ⓓ→②



Ⓔ→②



Ⓕ→④



Ⓔ→②

Ⓒ→① Range = (−∞, 4]   Ⓓ→② Range = [−2, 2]   ③ Range = (−∞, 2]

Ⓕ→④ Range = [0, 4]   Ⓐ→⑤ Range = (−∞, 5]   Ⓑ→⑥ Range = [1, ∞)

16. (i) Domain = R; Range = [−2, ∞]

(ii) Domain = [−∞, 2]; Range = R

(iii) Domain = (−2, 3); Range = (0, 9)

(iv) Domain = (−3, 1); Range = (−6, 2)

(v) Domain = (−4, 0); Range = (0, 4)

(vi) Domain = R; Range = [−∞, 4]

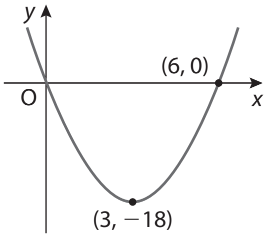
17. (i), (iii), (iv) and (vi) are functions

18. $f(x) = kx(x - 6) = k(x^2 - 6x)$

Point $(3, -18) \Rightarrow f(3) = k[(3)^2 - 6(3)] = -18$

$\Rightarrow k[9 - 18] = -18$

$\Rightarrow -9k = -18 \quad \Rightarrow k = \dfrac{-18}{-9} = 2$



19. $f(x) = x^2 + px + q$

$\Rightarrow f(3) = (3)^2 + p(3) + q = 4$

$\Rightarrow 9 + 3p + q = 4 \qquad \Rightarrow 3p + q = -5$

and $f(-1) = (-1)^2 + p(-1) + q = 4$

$\Rightarrow 1 - p + q = 4 \qquad \Rightarrow -p + q = 3$

Hence, $3p + q = -5$

and $\underline{p - q = -3}$

Add $\Rightarrow 4p = -8 \qquad \Rightarrow p = -2$

$\Rightarrow 3(-2) + q = -5$

$\Rightarrow -6 + q = -5 \qquad \Rightarrow q = 1$

Solve $x^2 - 2x + 1 = 0$

$\Rightarrow (x - 1)(x - 1) = 0 \qquad \Rightarrow x = 1$

20. $f(x) = x^2 + bx + c$

(i) $(-3, 0) \Rightarrow f(-3) = (-3)^2 + b(-3) + c = 0$

$\Rightarrow 9 - 3b + c = 0 \quad \Rightarrow -3b + c = -9$

(ii) $(0, -3) \Rightarrow f(0) = (0) + b(0) + c = -3$

$\Rightarrow c = -3$

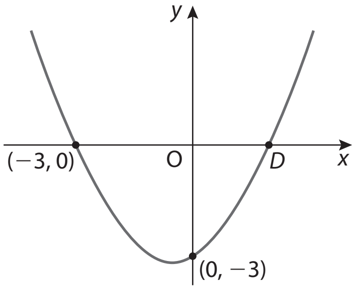
$\Rightarrow -3b - 3 = -9$

$\Rightarrow -3b = -6 \quad \Rightarrow b = \dfrac{-6}{-3} = 2$

(iii) $f(x) = x^2 + 2x - 3 = 0$

$\Rightarrow (x + 3)(x - 1) = 0$

$\Rightarrow x = -3, \quad x = 1 \quad \Rightarrow \quad D = (1, 0)$



Exercise 8.2

1. $f(x) = x^2 + 1$ and $g(x) = 2x - 1$

(i) $f(3) = (3)^2 + 1 = 10$

(ii) $gf(3) = g(10) = 2(10) - 1 = 19$

(iii) $g(3) = 2(3) - 1 = 5$

(iv) $fg(3) = f(5) = (5)^2 + 1 = 26$

(v) $f^2(3) = f(f(3)) = f(10) = (10)^2 + 1 = 101$

(vi) $g^2(3) = g(g(3)) = g(5) = 2(5) - 1 = 9$

(vii) $gf(-4) = g[(-4)^2 + 1] = g(17) = 2(17) - 1 = 33$

(viii) $fg\left(\frac{1}{2}\right) = f\left[2\left(\frac{1}{2}\right) - 1\right] = f(0) = (0)^2 + 1 = 1$

2. $f(x) = 2x + 1$ and $g(x) = 4x - 3$

(i) $f(3) = 2(3) + 1 = 7$

(ii) $gf(3) = g[7] = 4(7) - 3 = 25$

(iii) $fg(-2) = f[4(-2) - 3] = f(-11) = 2(-11) + 1 = -21$

(iv) $gf(x) = g(2x + 1) = 4(2x + 1) - 3 = 8x + 4 - 3 = 8x + 1$

Solve $fg(x) = 19$.

$\Rightarrow f(4x - 3) = 19$

$\Rightarrow 2(4x - 3) + 1 = 19$

$\Rightarrow 8x - 6 + 1 = 19$

$\Rightarrow 8x - 5 = 19$

$\Rightarrow 8x = 24 \quad \Rightarrow x = \frac{24}{8} = 3$

3. $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

(i) $fg(-2) = f[(-2)^2 + 2] = f(6) = 2(6) - 1 = 11$

(ii) $gf\left(\frac{1}{2}\right) = g\left[2\left(\frac{1}{2}\right) - 1\right] = g(0) = (0)^2 + 2 = 2$

(iii) $fg(x) = f(x^2 + 2) = 2(x^2 + 2) - 1 = 2x^2 + 4 - 1 = 2x^2 + 3$

(iv) $gf(x) = g(2x - 1) = (2x - 1)^2 + 2 = 4x^2 - 4x + 1 + 2 = 4x^2 - 4x + 3$

Solve $gf(x) = fg(x)$.

$\Rightarrow 4x^2 - 4x + 3 = 2x^2 + 3$

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x - 2) = 0 \quad \Rightarrow x = 0, 2$

4. $f(x) = 2^{x-1}$ and $g(x) = 3 + 4x$

(i) $fg(x) = f(3 + 4x) = 2^{3+4x-1} = 2^{4x+2}$

(ii) $gf(x) = g(2^{x-1}) = 3 + 4(2^{x-1}) = 3 + 2^2 \cdot 2^{x-1}$

$= 3 + 2^{2+x-1}$

$= 3 + 2^{x+1}$

5. $f(x) = 3x^2$ and $g(x) = 2x + 1$

$fg(x) = f(2x + 1) = 3(2x + 1)^2 = 3(4x^2 + 4x + 1) = 12x^2 + 12x + 3$

$\Rightarrow \quad fg(a) = 12a^2 + 12a + 3 = g(1) = 2(1) + 1 = 3$

$\Rightarrow 12a^2 + 12a + 3 = 3$

$\Rightarrow \quad 12a^2 + 12a = 0$

$\Rightarrow \quad a^2 + a = 0$

$\Rightarrow \quad a(a + 1) = 0 \quad \Rightarrow a = 0, -1$

6. $f(x) = 2x + 3 \qquad g(x) = 2x - 3$

(i) $fg(x) = f(2x - 3) = 2(2x - 3) + 3$

$= 4x - 6 + 3 = 4x - 3$

$gf(x) = g(2x + 3) = 2(2x + 3) - 3$

$= 4x + 6 - 3 = 4x + 3$

(ii) $fg(x) \times gf(x) = (4x - 3)(4x + 3)$

$= 16x^2 + 12x - 12x - 9$

$= 16x^2 - 9$

$\Rightarrow$ least value $= 16(0)^2 - 9 = 0 - 9 = -9$

7. $f(x) = 2x + 1$ and $g(x) = 3x + c$

(i) $gf(x) = g(2x + 1) = 3(2x + 1) + c = 6x + 3 + c$

$fg(x) = f(3x + c) = 2(3x + c) + 1 = 6x + 2c + 1$

$gf(x) = fg(x)$

$\Rightarrow \quad 6x + 3 + c = 6x + 2c + 1$

$\Rightarrow \quad -c = -2 \quad \Rightarrow c = 2$

(ii) $f^2(m) = ff(m) = f(2m + 1) = 2(2m + 1) + 1 = 4m + 2 + 1 = 4m + 3$

Hence, $f^2(m) = m$

$\Rightarrow 4m + 3 = m$

$\Rightarrow 3m = -3 \quad \Rightarrow \quad m = -1$

8. $f(x) = s + tx \quad g(x) = x^2 - 4$ and $h(x) = 3x + 1$

$$\begin{aligned}\Rightarrow hgf(x) &= hg(s + tx)\\ &= h[(s + tx)^2 - 4]\\ &= h[s^2 + 2stx + t^2x^2 - 4]\\ &= 3(s^2 + 2stx + t^2x^2 - 4) + 1\\ &= 3s^2 + 6stx + 3t^2x^2 - 12 + 1\\ &= 3t^2x^2 + 6stx + 3s^2 - 11\end{aligned}$$

Solve $hgf(x) = 4(3x^2 + 3x - 2)$.

$\Rightarrow 3t^2x^2 + 6stx + 3s^2 - 11 = 12x^2 + 12x - 8$

$\Rightarrow 3t^2 = 12$ and $6st = 12$

$\Rightarrow t^2 = 4 \qquad \Rightarrow st = 2$

$\Rightarrow t = 2$ as $t \in N \qquad \Rightarrow s(2) = 2$

$\Rightarrow 2s = 2 \quad \Rightarrow s = 1$

9. $f(x) = \cos x$ and $g(x) = x + \dfrac{\pi}{6}$

$$\Rightarrow fg\left(\frac{\pi}{6}\right) = f\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = f\left(\frac{2\pi}{6}\right) = f\left(\frac{\pi}{3}\right)$$

$$\Rightarrow f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

10. $f(x) = x^2 - x + 10 \quad g(x) = 5 - x$ and $h(x) = \log_2 x$

(i) $hf(x) = h(x^2 - x + 10) = \log_2 (x^2 - x + 10)$

$hg(x) = h(5 - x) = \log_2 (5 - x)$

(ii) Solve $hf(x) - hg(x) = 3$.

$\Rightarrow \log_2 (x^2 - x + 10) - \log_2 (5 - x) = 3 = \log_2 8$

$$\Rightarrow \log_2 \frac{x^2 - x + 10}{5 - x} = \log_2 8$$

$$\Rightarrow \frac{x^2 - x + 10}{5 - x} = \frac{8}{1} \Rightarrow x^2 - x + 10 = 40 - 8x$$

$\Rightarrow x^2 + 7x - 30 = 0$

$\Rightarrow (x + 10)(x - 3) = 0$

$\Rightarrow x = -10, 3 \quad \Rightarrow x = 3$ as $x > 0$

11. $f(x) = 2x + 3$

(i) $f^2(x) = ff(x) = f(2x + 3) = 2(2x + 3) + 3$

$= 4x + 6 + 3 = 4x + 9$

$= 2^2x + 3(2^2 - 1)$

(ii) $f^3(x) = fff(x) = f(4x + 9) = 2(4x + 9) + 3$

$= 8x + 18 + 3 = 8x + 21$

$= 2^3x + 3(2^3 - 1)$

(iii) $f^4(x) = ffff(x) = f(8x + 21) = 2(8x + 21) + 3$

$= 16x + 42 + 3 = 16x + 45$

$= 2^4x + 3(2^4 - 1)$

$f^n(x) = 2^nx + 3(2^n - 1)$

12. $f(x) = x^2 + 1$ and $g(x) = 1 - 2x$

$gf(x) = g(x^2 + 1) = 1 - 2(x^2 + 1) = 1 - 2x^2 - 2 = -2x^2 - 1$

$fg(x) = f(1 - 2x) = (1 - 2x)^2 + 1 = 1 - 4x + 4x^2 + 1 = 4x^2 - 4x + 2$

$\Rightarrow gf(x) \neq fg(x)$; composition of functions is not commutative.

13. $f(x) = \frac{1}{2}\left(\frac{1}{x} + 1\right)$ and $g(x) = \frac{1}{2x - 1}$

$$fg(x) = f\left(\frac{1}{2x-1}\right) = \frac{1}{2}\left(\frac{1}{\frac{1}{2x-1}} + 1\right) = \frac{1}{2}(2x - 1 + 1) = \frac{1}{2}(2x) = x$$

$$gf(x) = g\left[\frac{1}{2}\left(\frac{1}{x} + 1\right)\right] = \frac{1}{2 \cdot \frac{1}{2}\left(\frac{1}{x} + 1\right) - 1}$$

$$= \frac{1}{\frac{1}{x} + 1 - 1} = \frac{1}{\frac{1}{x}} = x$$

$\Rightarrow$ Yes, $fg(x) = gf(x)$ as both are equal to x.

14. $p(x) = (3x - 4)^3$ and $f(x) = 3x, g(x) = x - 4, h(x) = x^3$

$hgf(x) = hg(3x) = h(3x - 4) = (3x - 4)^3 = p(x)$

15. (i) $h(x) = (3x - 1)^2 = fg(x)$

$\Rightarrow f(x) = x^2$ and $g(x) = 3x - 1$

(ii) $h(x) = \frac{1}{5x + 3} = gf(x)$

$\Rightarrow f(x) = 5x + 3$ and $g(x) = \frac{1}{x}$

(iii) $h(x) = \sin^2(3x) = fgk(x)$

$\Rightarrow f(x) = x^2,\quad g(x) = \sin x$ and $k(x) = 3x$

(iv) $b(x) = \cos(\sqrt{2x}) = hgf(x)$

$\Rightarrow f(x) = 2x,\quad g(x) = \sqrt{x}$ and $h(x) = \cos x$

16. $f(x) = 2^{x-1}$ and $g(x) = 3 + 4x$

$\Rightarrow fg(x) = f(3 + 4x) = 2^{3+4x-1} = 2^{4x+2}$

Hence, solve $fg(x) = 64$

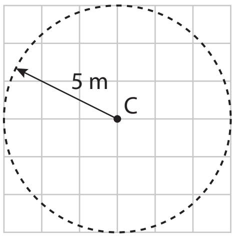
$\Rightarrow 2^{4x+2} = 64 = 2^6$

$\Rightarrow 4x + 2 = 6 \quad \Rightarrow 4x = 4 \quad \Rightarrow x = 1$

17. $f(r) = \frac{5}{4}t$

(i) $f(A) = \pi r^2$

(ii) $f(A) = \pi\left(\frac{5}{4}t\right)^2$



18. (i) $f(x) =$ €$0.04x$ $\Rightarrow$ this function represents 4% of sales

(ii) $g(x) =$ €$(x - 4000)$ $\Rightarrow$ value of sales in excess of €4000

$fg(x) = f(x - 4000) = 0.04(x - 4000) =$ average weekly commission

$fg(8000) = 0.04(8000 - 4000) = 0.04(4000) =$ €160

Exercise 8.3

1. (a) (i) *f* is a function; only one arrow from each element in A.
(ii) *f* is not injective; Two elements in A map to the same element in B.
(iii) *f* is not surjective; an element in B is not in the range of *f*.

(b) (i) *g* is a function; only one arrow from each element in A.
(ii) *g* is injective; each element in A maps to only one element in B.
(iii) *g* is surjective; each element in B is the image of some element in A.

There is an exact one-to-one correspondence between the elements in A and B; hence bijective.

2. (i) Yes, h is a function; only one arrow from each element in A.
(ii) No, h is not injective; Two elements in A map to the same element in B.
(iii) No, h is not surjective; an element in B is not in the range of h.
(iv) Not both injective and surjective.

3. (a) Surjective (b) Injective (c) Bijective

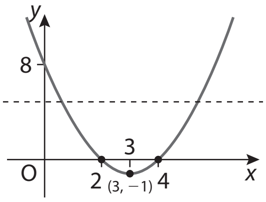
4. (i) Yes
(ii) No
(iii) Yes
Not a one-to-one correspondence

5. (i) (*a*), (*b*), (*d*), (*e*) and (*f*) are functions
(ii) Only (*b*) and (*e*) are injective functions

6. Injective because any horizontal line will intersect the curve at most once.
Surjective because any horizontal line will intersect the curve at least once.

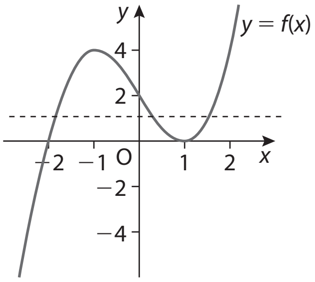
7. (i) Yes
(ii) Yes

8. (i) $y \geqslant -1$
(ii) R
(iii) Range not equal to codomain
(iv) Codomain : $y \geqslant -1$
(v) Horizontal line intersects the curve more than once.
(vi) $x \geqslant 3$ or $x \leqslant 3$



9. (i) No; a horizontal line will intersect the curve more than once.
(ii) Yes; as range and codomain are equal
(iii) $x \geqslant 2$ or $x \leqslant 2$

10. (i) Not injective
(ii) Is surjective
No; not both injective and surjective



11. A vertical line will intersect the curve more than once.
$y \geqslant 0$ or $y \leqslant 0$

12. (i) N
(ii) N
(iii) all even numbers
(iv) Codomain and range not equal
(v) Yes
(vi) Codomain should be the set of even positive numbers.

13. (i) No; a horizontal line will intersect the graph more than once.
(ii) Yes; a horizontal line will intersect the graph at least once.
(iii) $\pi \leqslant x \leqslant 3\pi$

14. (i) Yes, as a horizontal line will intersect the graph at least once.
(ii) No, as a horizontal line will intersect the graph more than once.
(iii) $x \geqslant 0,\ y \geqslant 0$

15. (i) $y > 0$

(ii) Yes, as a horizontal line will intersect the graph at most once.

(iii) Yes, as a horizontal line will intersect the graph at least once.

(iv) Because it is both injective and surjective.

Exercise 8.4

1. $y = x - 4 \quad \Rightarrow \quad x = y + 4$

$$\therefore f^{-1}(x) = x + 4$$

2. $y = 2x - 3 \quad \Rightarrow 2x = y + 3$

$$\Rightarrow \quad x = \frac{y+3}{2}$$

$$\therefore f^{-1}(x) = \frac{x+3}{2}$$

3. $y = 5x + 3 \quad \Rightarrow 5x = y - 3$

$$\Rightarrow \quad x = \frac{y-3}{5}$$

$$\therefore f^{-1}(x) = \frac{x-3}{5}$$

4. $y = 3x \quad \Rightarrow x = \frac{y}{3}$

$$\therefore f^{-1}(x) = \frac{x}{3}$$

5. $y = \frac{2x}{5} \quad \Rightarrow 2x = 5y$

$$\Rightarrow \quad x = \frac{5y}{2}$$

$$\therefore f^{-1}(x) = \frac{5x}{2}$$

6. $y = \frac{4x-3}{2} \quad \Rightarrow 4x - 3 = 2y$

$$\Rightarrow \quad 4x = 2y + 3$$

$$\Rightarrow \quad x = \frac{2y+3}{4}$$

$$\therefore f^{-1}(x) = \frac{2x+3}{4}$$

7. $y = \frac{x-6}{x} \quad \Rightarrow \quad xy = x - 6$

$$\Rightarrow \quad xy - x = -6$$

$$\Rightarrow x(y-1) = -6$$

$$\Rightarrow \quad x = \frac{-6}{y-1}$$

$$\therefore f^{-1}(x) = \frac{-6}{x-1}$$

8. $y = \frac{3x}{x-1} \quad \Rightarrow \quad xy - y = 3x$

$$\Rightarrow xy - 3x = y$$

$$\Rightarrow x(y-3) = y$$

$$\Rightarrow \quad x = \frac{y}{y-3}$$

$$\therefore f^{-1}(x) = \frac{x}{x-3}$$

9. $$y = \frac{10 - 2x}{3} \Rightarrow 10 - 2x = 3y$$
$$\Rightarrow \quad -2x = 3y - 10$$
$$\Rightarrow \quad 2x = 10 - 3y$$
$$\Rightarrow \quad x = \frac{10 - 3y}{2}$$
$$\therefore f^{-1}(x) = \frac{10 - 3x}{2}$$

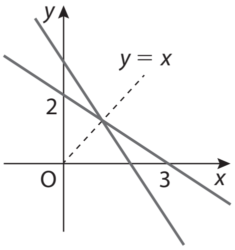
10. $$y = 4x + 5 \quad \Rightarrow \quad 4x = y - 5$$
$$\Rightarrow \quad x = \frac{y - 5}{4}$$
$$\therefore f^{-1}(x) = \frac{x - 5}{4}$$
$$ff^{-1}(x) = f\left(\frac{x - 5}{4}\right) = 4\left(\frac{x - 5}{4}\right) + 5 = x - 5 + 5 = x$$
$$f^{-1}f(x) = f^{-1}(4x + 5) = \frac{4x + 5 - 5}{4} = \frac{4x}{4} = x$$

Hence, $f^{-1}f(x) = ff^{-1}(x)$

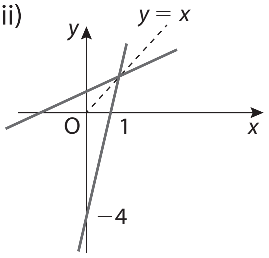
11. $$y = \frac{x}{3} - 2 \quad \Rightarrow \frac{x}{3} = y + 2$$
$$\Rightarrow \quad x = 3(y + 2)$$
$$\therefore f^{-1}(x) = 3(x + 2)$$

Hence, $ff^{-1}(x) = f[3(x + 2)] = \dfrac{3(x + 2)}{3} - 2 = x + 2 - 2 = x$

12. (i)



(ii)



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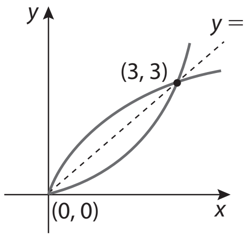

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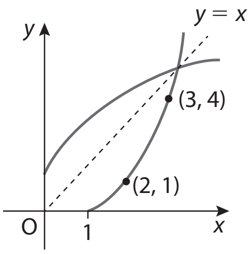

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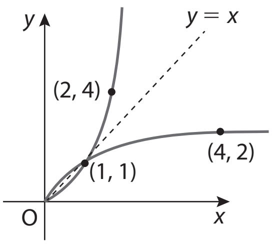

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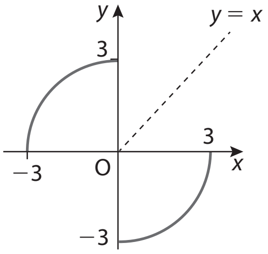

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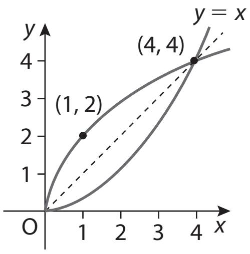
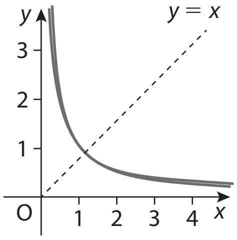
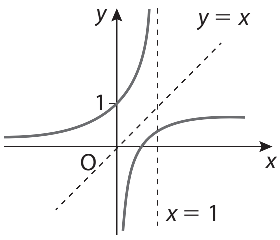


(viii)



(ix)



13. Line ℓ: $\underset{(x_1, y_1)}{(-4, 0)}$ and $\underset{(x_2, y_2)}{(2, 2)}$

(i) Slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2 - 0}{2 + 4} = \dfrac{2}{6} = \dfrac{1}{3}$

Equation of line ℓ: $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y - 0 = \frac{1}{3}(x + 4)$$

$$\Rightarrow \quad 3y = x + 4$$

$$\Rightarrow \quad x - 3y + 4 = 0$$

Line m: $\underset{(x_1, y_1)}{(0, -4)}$ and $\underset{(x_2, y_2)}{(2, 2)}$

Slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2 + 4}{2 - 0} = \dfrac{6}{2} = 3$

Equation of line m: $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y + 4 = 3(x - 0)$$

$$\Rightarrow \quad y + 4 = 3x$$

$$\Rightarrow 3x - y - 4 = 0$$

(ii) Line ℓ: $3y = x + 4$ Line m: $3x - y - 4 = 0$

$$\Rightarrow \quad y = \frac{x + 4}{3} \qquad \Rightarrow f^{-1}(x) = y = 3x - 4$$

$$\Rightarrow f(x) = \frac{x + 4}{3}$$

Hence, $ff^{-1}(x) = f(3x - 4) = \dfrac{3x - 4 + 4}{3} = \dfrac{3x}{3} = x$

$\Rightarrow$ Equation of ℓ is the inverse of the equation of m.

14. $g(x) = \dfrac{1}{x - 2}$ and $f(x) = \dfrac{1 + kx}{x}$

$$\Rightarrow gf(x) = x \quad \Rightarrow gf(x) = g\left(\frac{1 + kx}{x}\right) = \frac{1}{\frac{1 + kx}{x} - 2}$$

$$= \frac{1}{\frac{1 + kx - 2x}{x}} = x$$

$$\Rightarrow \frac{x}{1 + kx - 2x} = x$$

$$\Rightarrow \frac{1}{1 + kx - 2x} = 1$$

$$\Rightarrow 1 + kx - 2x = 1$$

$$\Rightarrow \quad x(k - 2) = 0$$

$$\Rightarrow \quad k = 2$$

15. $f(x) = 2x - 3$ and $g(x) = x - 4$

(i) $gf(x) = g(2x - 3) = 2x - 3 - 4 = 2x - 7$

$y = 2x - 7 \quad \Rightarrow \quad 2x = y + 7$

$$\Rightarrow \quad x = \frac{y + 7}{2}$$

$$\Rightarrow [gf(x)]^{-1} = \frac{x + 7}{2}$$

(ii) $y = 2x - 3$ and $y = x - 4$

$\Rightarrow 2x = y + 3 \quad \Rightarrow \quad x = y + 4$

$\Rightarrow \quad x = \dfrac{y+3}{2} \quad \Rightarrow \quad g^{-1}(x) = x + 4$

$\Rightarrow f^{-1}(x) = \dfrac{x+3}{2}$

$\Rightarrow f^{-1}g^{-1}(x) = f^{-1}(x + 4)$

$= \dfrac{x+4+3}{2} = \dfrac{x+7}{2} = [gf(x)]^{-1} \quad \therefore \text{YES}$

16. $f(x) = \dfrac{x+3}{2} = y \quad \Rightarrow \quad f(0) = \dfrac{0+3}{2} = 1\tfrac{1}{2} \quad \Rightarrow \text{Points } \left(0, 1\tfrac{1}{2}\right), (5, 4)$

$\Rightarrow x + 3 = 2y \qquad f(5) = \dfrac{5+3}{2} = 4$

$\Rightarrow x = 2y - 3$

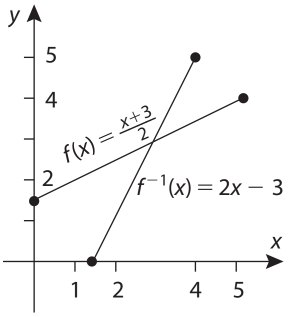
$\Rightarrow f^{-1}(x) = 2x - 3$

Points $\left(1\tfrac{1}{2}, 0\right), (4, 5)$

Domain of $f^{-1}(x) = \left\{1\tfrac{1}{2}, 4\right\}$

$= \text{Range of } f$

Range of $f^{-1}(x) = \{0, 5\}$



17. (i) $f(x) = y = x^2 + 4x - 6$

$\Rightarrow y = x^2 + 4x + 4 - 6 - 4$

$\Rightarrow y = (x + 2)^2 - 10$

$\Rightarrow (x + 2)^2 = y + 10$

$\Rightarrow \quad x + 2 = \sqrt{y + 10}$

$\Rightarrow \quad x = -2 + \sqrt{y + 10}$

$\therefore f^{-1}(x) = -2 + \sqrt{x + 10}, \quad x \geqslant -10$

(ii) $f(x) = y = x^2 - 2x - 5$

$\Rightarrow y = x^2 - 2x + 1 - 5 - 1$

$\Rightarrow y = (x - 1)^2 - 6$

$\Rightarrow (x - 1)^2 = y + 6$

$\Rightarrow \quad x - 1 = \sqrt{y + 6}$

$\Rightarrow \quad x = 1 + \sqrt{y + 6}$

$\therefore f^{-1}(x) = 1 + \sqrt{x + 6}, \quad x \geqslant -6$

(iii) $f(x) = y = x^2 - 8x - 3$

$\Rightarrow y = x^2 - 8x + 16 - 3 - 16$

$\Rightarrow y = (x - 4)^2 - 19$

$\Rightarrow (x - 4)^2 = y + 19$

$\Rightarrow \quad x - 4 = \sqrt{y + 19}$

$\Rightarrow \quad x = 4 + \sqrt{x + 19}$

$\therefore \quad f^{-1}(x) = 4 + \sqrt{x + 19}, \quad x \geqslant -19$

(iv) $f(x) = y = x^2 + 8x + 20$

$\Rightarrow y = x^2 + 8x + 16 + 4$

$\Rightarrow y = (x + 4)^2 + 4$

$\Rightarrow (x + 4)^2 = y - 4$

$\Rightarrow \quad x + 4 = \sqrt{y - 4}$

$\Rightarrow \quad x = -4 + \sqrt{y - 4}$

$\therefore f^{-1}(x) = -4 + \sqrt{x - 4}, \quad x \geqslant 4$

18. $f(x) = y = \dfrac{3 - x}{2} \quad \Rightarrow 2y = 3 - x$

$$\Rightarrow \quad x = 3 - 2y$$

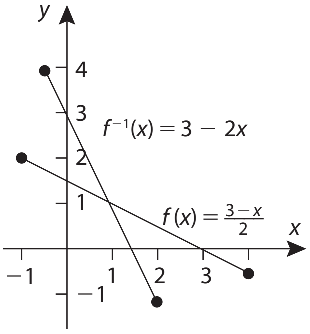
$$\therefore f^{-1}(x) = 3 - 2x$$

2 points on $f(x) = (-1, 2)$ and $\left(4, -\dfrac{1}{2}\right)$

2 points on $f^{-1}(x) = (2, -1)$ and $\left(-\dfrac{1}{2}, 4\right)$

Domain of $f^{-1}(x) = \left\{-\dfrac{1}{2}, 2\right\}$

Range of $f^{-1}(x) = \{-1, 4\}$



19. $A \leqslant 3 \quad$ [OR $(-\infty, 3)$]

20. $b = 0$

$g(x) = y = 1 - x^2$

$\Rightarrow x^2 = 1 - y$

$\Rightarrow \ x = \sqrt{1 - y}$

$\therefore g^{-1}(x) = \sqrt{1 - x}, x \leqslant 1$

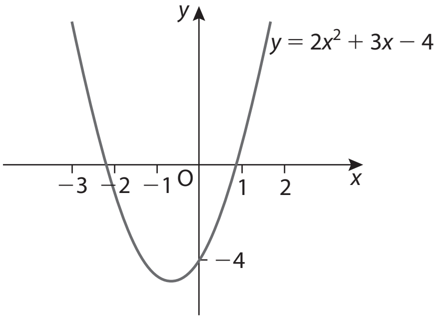
Domain of $g^{-1}(x) = \{1, -3\}$ or $\{-3, 1\}$

Range of $g^{-1}(x) = \{0, 2\}$

Exercise 8.5

1. Graph the function $f(x) = 2x^2 + 3x - 4$ in the domain $-3 \leqslant x \leqslant 2$

x	**$2x^2 + 3x - 4$**	**y**
-3	$18 - 9 - 4$	5
-2	$8 - 6 - 4$	-2
-1	$2 - 3 - 4$	-5
0	$0 + 0 - 4$	-4
1	$2 + 3 - 4$	1
2	$8 + 6 - 4$	10

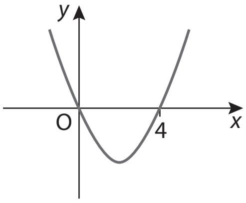


2. (i) $f(x) = x^2 - 4x$

$\Rightarrow y = x(x - 4) = 0$

$\Rightarrow x = 0, x = 4$

Points on x-axis $(0, 0), (4, 0)$



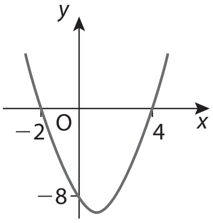
(ii) $f(x) = x^2 - 2x - 8$

$\Rightarrow y = (x + 2)(x - 4) = 0$

$\Rightarrow x = -2, x = 4$

Points on x-axis $(-2, 0), (4, 0)$

Points on y-axis $(0, -8)$



(iii) $f(x) = -x^2 + 2x + 3$

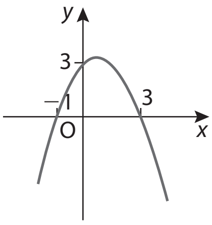
$\Rightarrow \ y = -(x^2 - 2x - 3) = 0$

$= -(x + 1)(x - 3) = 0$

$\Rightarrow x = -1, x = 3$

Points on x-axis $(-1, 0), (3, 0)$

Points on y-axis $(0, 3)$



3. (i) $x^2 - 4x + 2$

$= x^2 - 4x + 4 + 2 - 4$

$= (x - 2)^2 - 2$

(ii) $x^2 - 12x + 36 = (x - 6)^2$

(iii) $-x^2 + 8x - 12$

$= -(x^2 - 8x + 12)$

$= -(x^2 - 8x + 16 + 12 - 16)$

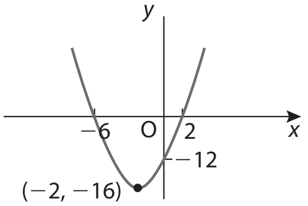
$= -[(x - 4)^2 - 4] = -(x - 4)^2 + 4$

4. $y = x^2 + 4x - 12$

$= x^2 + 4x + 4 - 12 - 4$

$= (x + 2)^2 - 16$

$(x + 2)^2 - 16$; Intersects x-axis at $(-6, 0)$ and $(2, 0)$; Turning point $= (-2, -16)$



5. $y = x^2 + 4x - 5$

On x-axis $\Rightarrow y = 0$

$\Rightarrow x^2 + 4x - 5 = 0$

$\Rightarrow (x + 5)(x - 1) = 0$

$\Rightarrow x = -5, x = 1$

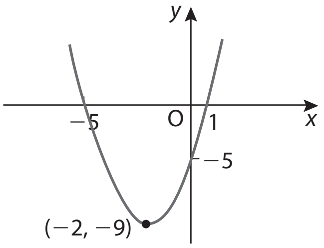
On x-axis $\Rightarrow$ $(-5, 0)$ and $(1, 0)$

On y-axis $\Rightarrow$ $(0, -5)$

$y = x^2 + 4x - 5 = x^2 + 4x + 4 - 5 - 4$

$= (x + 2)^2 - 9$

$\Rightarrow$ Turning point $= (-2, -9)$



6. $x^2 + 3x - 10 = x^2 + 3x + \frac{9}{4} - 10 - \frac{9}{4}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{49}{4}$

$\Rightarrow$ Turning point $= \left(-\frac{3}{2}, -\frac{49}{4}\right)$

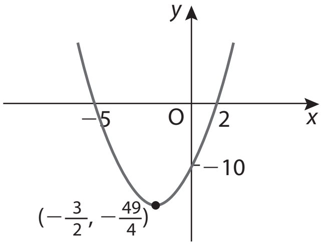
On x-axis, $y = 0 \Rightarrow x^2 + 3x - 10 = 0$

$\Rightarrow (x + 5)(x - 2) = 0$

$\Rightarrow x = -5, x = 2$

Points on x-axis $(-5, 0)$, $(2, 0)$

Point on y-axis $(0, -10)$



7. $y = ax^2 + c$

Point $(-1, 4) \Rightarrow 4 = a(-1)^2 + c \Rightarrow a + c = 4$

Point $(0, 8) \Rightarrow 8 = a(0) + c \Rightarrow c = 8$

$\Rightarrow a + 8 = 4 \Rightarrow a = -4$

8. (i) $y = x^2$

(ii) $y = a(x - 1)(x - 3) = a(x^2 - 4x + 3)$

Point $(0, 3)$ $\Rightarrow 3 = a(0 - 0 + 3)$

$\Rightarrow 3 = 3a \Rightarrow a = 1$

$\Rightarrow$ Equation: $y = x^2 - 4x + 3$

(iii) $y = a(x + 3)(x - 1) = a(x^2 + 2x - 3)$

Point $(-1, 5)$ $\Rightarrow 5 = a[(-1)^2 + 2(-1) - 3]$

$\Rightarrow 5 = a(1 - 2 - 3)$

$\Rightarrow 5 = -4a \Rightarrow a = -\frac{5}{4}$

$\Rightarrow$ Equation: $y = -\frac{5}{4}(x + 3)(x - 1)$

9. $y = (x - 4)^2 - 3$ $\Rightarrow$ Turning point $= (4, -3)$ minimum

$\Rightarrow$ Graph Ⓑ

10. $y = 3 - (x - 4)^2$ $\Rightarrow$ Turning point $= (4, 3)$ maximum

$\Rightarrow$ Graph Ⓓ

11. Turning point $= (-1, 3) \Rightarrow y = k(x + 1)^2 + 3$

Point $(0, 4)$ $\Rightarrow 4 = k(0 + 1)^2 + 3$

$\Rightarrow 1 = k\,.\,(1) \Rightarrow k = 1$

$\Rightarrow$ Equation: $y = (x + 1)^2 + 3$ or $y = x^2 + 2x + 4$

12. (a) (i) $y = (x + 1)(x + 2)(x - 3) = 0$

$\Rightarrow x = -1, -2, 3$

Intersects x-axis at $(-1, 0)$, $(-2, 0)$ and $(3, 0)$

(ii) y-axis $\Rightarrow x = 0$ $\Rightarrow y = (0 + 1)(0 + 2)(0 - 3) = -6$ $\Rightarrow$ point $(0, -6)$

(b) (i) $y = x(x - 6)(x + 3) = 0$

$\Rightarrow x = 0, 6, -3$

Intersects x-axis at $(0, 0)$, $(6, 0)$, $(-3, 0)$

(ii) y-axis $\Rightarrow x = 0$ $\Rightarrow y = 0(0 - 6)(0 + 3) = 0$ $\Rightarrow$ point $(0, 0)$

(c) (i) $y = (x - 1)(x + 2)^2 = 0$

$\Rightarrow x = 1, x = -2$

Intersects x-axis at $(1, 0)$ and $(-2, 0)$

(ii) On y-axis $\Rightarrow x = 0$ $\Rightarrow y = (0 - 1)(0 + 2)^2 = -4$ $\Rightarrow$ point $(0, -4)$

(d) (i) $y = x(x^2 - 9) = x(x + 3)(x - 3) = 0$

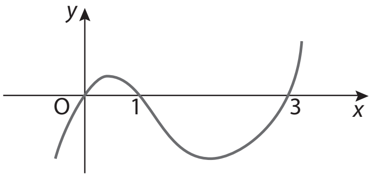
$\Rightarrow x = 0, -3, 3$

Intersects x-axis at $(0, 0)$, $(-3, 0)$, $(3, 0)$

(ii) On y-axis $\Rightarrow x = 0$ $\Rightarrow y = 0[(0)^2 - 9] = 0$ $\Rightarrow$ point $(0, 0)$

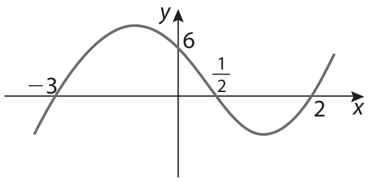
13. (i) $y = x(x - 1)(x - 3)$

Intersects x-axis at $(0, 0)$, $(1, 0)$ and $(3, 0)$; y-axis at $(0, 0)$



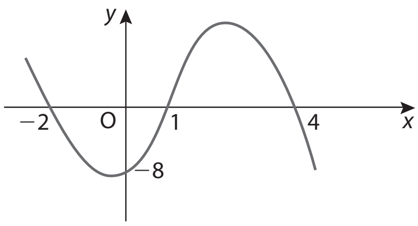
(ii) $y = (x - 2)(x + 3)(2x - 1)$

Intersects x-axis at $(-3, 0)$, $\left(\frac{1}{2}, 0\right)$ and $(2, 0)$; y-axis at $(0, 6)$



(iii) $y = -(x - 1)(x + 2)(x - 4)$

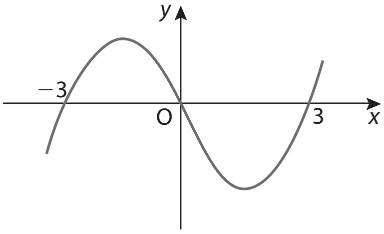
Intersects x-axis at $(-2, 0)$, $(1, 0)$ and $(4, 0)$; y-axis at $(0, -8)$



(iv) $y = x^3 - 9x = x(x^2 - 9)$

$= x(x + 3)(x - 3)$

Intersects x-axis at $(-3, 0)$, $(0, 0)$ and $(3, 0)$; y-axis at $(0, 0)$



14. (i) Graph Ⓑ (ii) Graph Ⓒ (iii) Graph Ⓑ (iv) Graph Ⓑ

15. $y = x^3 - x^2$ and graph C

$y = 1 - x^2$ and graph A

$y = x - x^2$ and graph B

$y = \frac{-3}{4}x + 3$ and graph F

$y = x^2 + 3x$ and graph E

$y = 9x - x^3$ and graph D

16. (i) $f(3) = -27$

(ii) Maximum Turning point $= (-1, 5)$

(iii) Roots $x = -2.8$, $x = 1.8$, $x = 3.9$

(iv) $f(x)$ is decreasing $\Rightarrow -1 < x < 3$

(v) Line $y = 10$ intersects the graph at one point only.

(vi) Line $y = -10$ intersects the graph at three points.

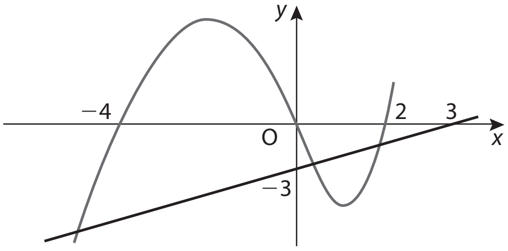
(vii) $f(x) = k$ has three real roots

$\Rightarrow -27 < k < 5$

17. $y = x(x - 2)(x + 4)$

Points on x-axis $= (-4, 0), (0, 0), (2, 0)$

$y = x - 3 \quad \Rightarrow 2$ points $(0, -3), (3, 0)$



3 intersection points

18. $y = k(x + 2)(x - 1)(x - 5)$

Point $(0, -4) \quad \Rightarrow \quad -4 = k(0 + 2)(0 - 1)(0 - 5)$

$\Rightarrow \quad 10k = -4 \quad \Rightarrow k = \frac{-4}{10} = \frac{-2}{5}$

$\Rightarrow y = \frac{-2}{5}(x + 2)(x - 1)(x - 5)$

19. (i) Length $= \ell \quad \Rightarrow$ perimeter $= x + \ell + x = 60$ m

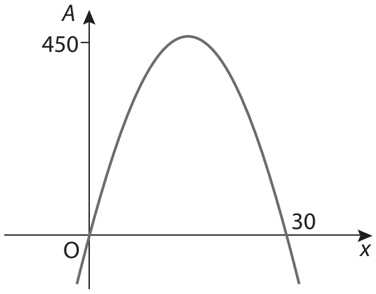
$\Rightarrow \ell = (60 - 2x)$ m

Area $A = x(60 - 2x)$ m^2

(ii) Points on x-axis $= (0, 0), (30, 0)$

(iii) Maximum occurs at $x = 15$

$\Rightarrow A = 15[60 - 2(15)] = 450$ m^2



20. (i) As −1 is a root and 2 is a double root, the equation of the curve can be written:

$y = k(x + 1)(x - 2)^2$.

This curve also contains the point (0, −2). Thus

$-2 = k(1)^2(-2)^2$

$-2 = 4k$

$k = -\frac{1}{2}$

Thus the curve is

$y = -\frac{1}{2}(x + 1)(x - 2)^2$

(ii) Because every output does not have a unique input, or because some horizontal lines cut the graph more than once.

(iii) Yes, because each element of the codomain is the image of some element in the domain.

(iv) The function is not bijective because it is not injective.

(v) $y = -\frac{1}{2}(x + 1)(x^2 - 4x + 4)$

$y = -\frac{1}{2}(x^3 - 3x^2 + 4)$

Then

$\frac{dy}{dx} = -\frac{1}{2}(3x^2 - 6x)$

Put $\frac{dy}{dx} = 0$:

$-\frac{1}{2}3x(x - 2) = 0$

$x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

When $x = 0$: $y = -2$. Thus (0, −2) is a stationary point.

When $x = 2$: $y = 0$. Thus (2, 0) is also a stationary point.

21. (a) (i) $y = 3x^3 - x^2$

$\frac{dy}{dx} = 9x^2 - 2x$

Put $\frac{dy}{dx} = 0$: $x(9x - 2) = 0$

$x = 0$ or $9x - 2 = 0$

$x = 0$ or $x = \frac{2}{9} = 0.22$

When $x = 0$: $y = 0$. Thus (0, 0) is a stationary point.

When $x = 0.22$: $y = -0.02$. Thus (0.22, −0.02) is a stationary point.

(ii) x-intercepts: Let $y = 0$.

$3x^3 - x^2 = 0$

$x^2(3x - 1) = 0$

$x^2 = 0$ or $3x - 1 = 0$

$x = 0$ or $x = 0.33$

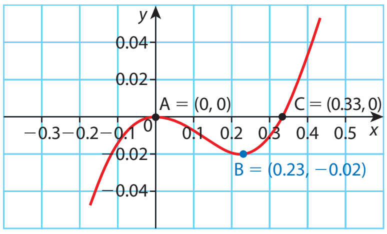
The curve crosses the x-axis at the points (0, 0) and (0.33, 0).

y-intercept: Let $x = 0$.

$x = 0$: $y = 0$

The curve crosses the y-axis at (0, 0).

The graph is shown below.



(b) (i) $y = x^3 + 6x^2 + 9x + 4$

$$\frac{dy}{dx} = 3x^2 + 12x + 9$$

Put $\frac{dy}{dx} = 0$: $3x^2 + 12x + 9 = 0$

$$x^2 + 4x + 3 = 0$$

$$(x + 3)(x + 1) = 0$$

$$x + 3 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = -3 \quad \text{or} \quad x = -1$$

When $x = -3$: $y = 4$. Thus $(-3, 4)$ is a stationary point.

When $x = -1$: $y = 0$. Thus $(-1, 0)$ is a stationary point.

(ii) x-intercepts: Let $y = 0$.

$x^3 + 6x^2 + 9x + 4 = 0$

Let $f(x) = x^3 + 6x^2 + 9x + 4$

$f(-1) = 0$

Thus $(x + 1)$ is a factor of $f(x)$.

$$\begin{array}{r l}
 & x^2 + 5x + 4 \\
x + 1 & \overline{)\,x^3 + 6x^2 + 9x + 4} \\
 & \underline{x^3 + x^2} \\
 & \phantom{x^3 +{}} 5x^2 + 9x \\
 & \phantom{x^3 +{}} \underline{5x^2 + 5x} \\
 & \phantom{x^3 + 5x^2 +{}} 4x + 4 \\
 & \phantom{x^3 + 5x^2 +{}} \underline{4x + 4} \\
 & \phantom{x^3 + 5x^2 + 4x + {}} 0
\end{array}$$

Thus $f(x) = (x + 1)(x^2 + 5x + 4)$

$f(x) = (x + 1)(x + 1)(x + 4)$

$f(x) = (x + 1)^2(x + 4)$

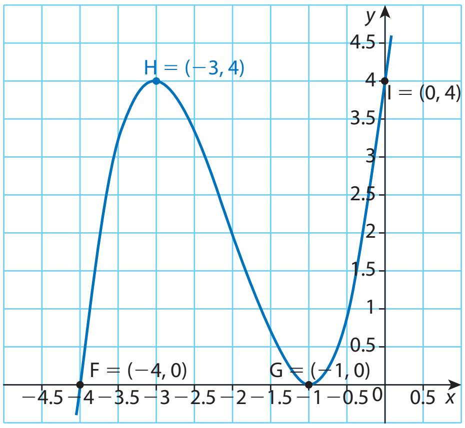
Thus the curve intersects the x-axis at -4 and at -1 (where there is a double root).

y-intercept: Let $x = 0$.

$x = 0$: $y = 4$

The curve crosses the y-axis at $(0, 4)$.

The graph is shown below.



22. $y = x^3 - 6x^2$

$$\frac{dy}{dx} = 3x^2 - 12x$$

Put $\frac{dy}{dx} = 0$: $3x^2 - 12x = 0$

$$x(x - 4) = 0$$
$$x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

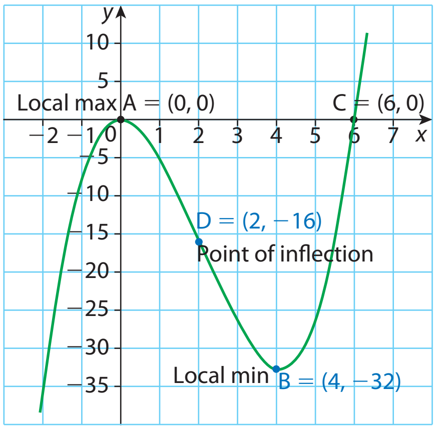
When $x = 0$: $y = 0$. Thus $(0, 0)$ is a stationary point.
When $x = 4$: $y = -32$. Thus $(4, -32)$ is a stationary point.
Also

$$\frac{d^2y}{dx^2} = 6x - 12$$

When $x = 0$: $\frac{d^2y}{dx^2} = -12 < 0$. Thus $(0, 0)$ is a local maximum point.

When $x = 4$: $\frac{d^2y}{dx^2} = 12 > 0$. Thus $(4, -32)$ is a local minimum point.

Also, to find the point of inflection, put $\frac{d^2y}{dx^2} = 0$:
$6x - 12 = 0$
$x = 2$
When $x = 2$: $y = -16$. Thus $(2, -16)$ is the point of inflection.
x-intercept: Let $y = 0$.
$x^3 - 6x^2 = 0$
$x^2(x - 6) = 0$
$x = 0$ or $x - 6 = 0$
$x = 0$ or $x = 6$
The curve crosses the x-axis at the points $(0, 0)$ and $(6, 0)$.
The graph is shown below.



23. (i) The three types of stationary point are: local maximum point, local minimum point and saddle point.

(ii) $y = 4x^3 - 3x^4$

$$\frac{dy}{dx} = 12x^2 - 12x^3$$

(a) The curve is increasing for

$$\frac{dy}{dx} > 0$$

$$12x^2 - 12x^3 > 0$$

$$12x^2(1 - x) > 0$$

$$1 - x > 0, \quad \text{and } x \neq 0$$

$$1 > x \quad (\text{or } x > 1)$$

(b) The curve is decreasing for

$$\frac{dy}{dx} < 0$$

$$12x^2(1 - x) < 0$$

$$1 - x < 0, \quad \text{and } x \neq 0$$

$$1 < x \quad (\text{or } x > 1)$$

(iii) Put $\frac{dy}{dx} = 0$: $12x^2(1 - x) = 0$

$x = 0$ or $1 - x = 0$

$x = 0$ or $x = 1$

When $x = 0$: $y = 0$ Thus $(0, 0)$ is a stationary point.

When $x = 1$: $y = 1$ Thus $(1, 1)$ is a stationary point.

x-intercept: Let $y = 0$.

$4x^3 - 3x^4 = 0$

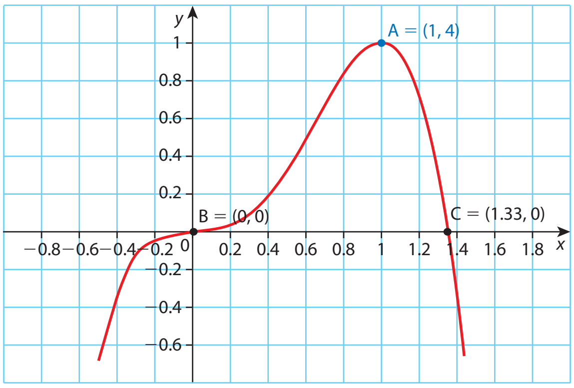
$x^3(4 - 3x) = 0$

$x = 0$ or $4 - 3x = 0$

$x = 0$ or $x = \frac{4}{3} = 1.33$

Thus the curve crosses the x-axis at $(0, 0)$ and $(1.33, 0)$.

The graph is shown below.



24. (i) $f(x) = x^2 - 6x + 18$

$= x^2 - 6x + 9 + 9$

$= (x - 3)^2 + 9$

(ii) $P = (0, 18)$, minimum point $Q = (3, 9)$

(iii) $y = x^2 - 6x + 18 \quad \cap \quad y = 41$

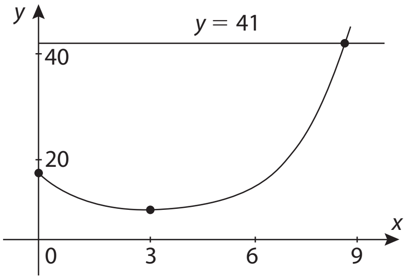
$\Rightarrow x^2 - 6x + 18 = 41$

$\Rightarrow x^2 - 6x - 23 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 4(1)(-23)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{128}}{2} = \frac{6 \pm 8\sqrt{2}}{2} = 3 \pm 4\sqrt{2}$$

$$\Rightarrow x = 3 + 4\sqrt{2} \quad (\text{as } x \geqslant 0)$$



Exercise 8.6

1. (i) Graph Ⓐ and $f(x) = a^x, a > 1$

Graph Ⓑ and $f(x) = a^x, 0 < a < 1$

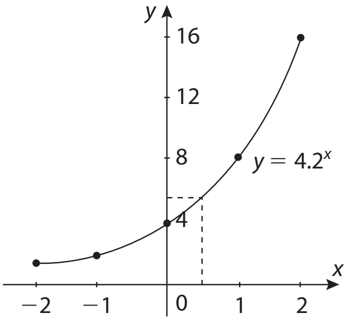
(ii) At P $\Rightarrow$ $x = 0 \Rightarrow f(x) = a^0 = 1 \Rightarrow$ P $= (0, 1)$

(iii) Ⓐ: $y = 0$; Ⓑ: $y = 0$

2.

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
4.2^x	1	2	4	8	16

$f(0.5) = 5.65 = 5.7$



3. A is $f(x) = 3.3^x$

B is $f(x) = 3^x$

C is $f(x) = 2^x$

4. (i) Point (0, 5) $\Rightarrow$ $f(0) = k.2^0 = 5$ OR $f(0) = k.3^0 = 5$

$\Rightarrow$ $k.1 = 5$ $\Rightarrow$ $k.1 = 5$

$\Rightarrow$ $k = 5$ $\Rightarrow$ $k = 5$

(ii) $f(x) = 5.2^x$ OR $f(x) = 5.3^x$

Point (2, 20) $\Rightarrow f(2) = 5.2^2 = 20$ (valid) $f(2) = 5.3^2 = 45$ (not valid)

Hence, function is $y = 5.2^x$

5. $y = a(2^x)$

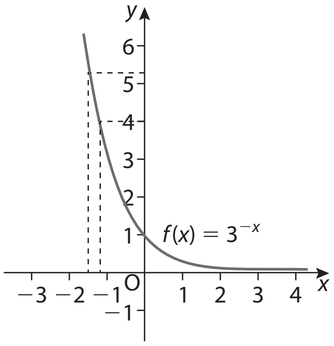
Point (1, 3) $\Rightarrow 3 = a(2^1) \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

6.

x	-2	-1	0	1	2	3
$f(x) = 3^{-x}$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

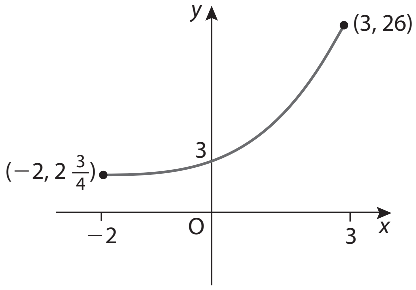
(i) $f(-1.5) = 5.2$

(ii) $f(x) = 4 \Rightarrow x = -1.25$



7.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
3.2^x	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24
$3.2^x + 2$	$2\frac{3}{4}$	$3\frac{1}{2}$	5	8	14	26



Range $= \left[2\frac{3}{4}, 26\right]$

8. $y = ae^x + b$

Point (0, 0) $\Rightarrow 0 = ae^0 + b = a\,.\,1 + b$

$\Rightarrow a + b = 0$

Point (1, 14) $\Rightarrow 14 = ae^1 + b = ae + b$

$\Rightarrow \;\; ae + b = 14$

$\underline{-a - b = \;\;0}$

$\Rightarrow \;\; ae - a = 14$

$\Rightarrow a(\text{e} - 1) = 14 \qquad \Rightarrow a = \dfrac{14}{\text{e} - 1}$

$a + b = 0 \;\Rightarrow b = -a \;\Rightarrow b = \dfrac{-14}{\text{e} - 1}$

9. (i) $y = 3 \times 4^x$

Point $(a, 6)$ $\Rightarrow 6 = 3 \times 4^a \Rightarrow 4^a = \dfrac{6}{3} = 2$

$\Rightarrow \qquad 4^a = (2^2)^a = 2^{2a} = 2^1$

$\Rightarrow \quad 2a = 1 \Rightarrow a = \dfrac{1}{2}$

(ii) Point $\left(-\frac{1}{2}, b\right)$ lies on $y = 3 \times 4^x$

$\Rightarrow b = 3 \times 4^{-\frac{1}{2}} = 3 \times \dfrac{1}{2} = \dfrac{3}{2}$

10. $y = 3^x$ and $y = x + 3$

$x = 0 \;\Rightarrow\; y = 3^0 = 1$ Point (0, 1)

$x = 1 \;\Rightarrow\; y = 3^1 = 3$ Point (1, 3)

$x = 2 \;\Rightarrow\; y = 3^2 = 9$ Point (2, 9)

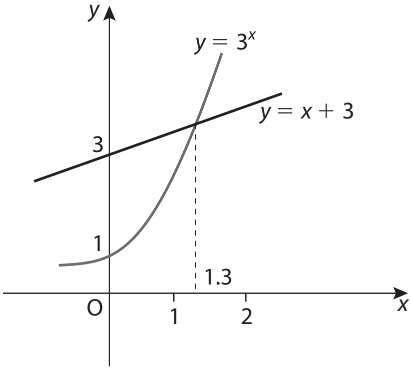
line: $y = x + 3$

$x = 0 \;\Rightarrow\; y = 0 + 3 = 3$ Point (0, 3)

$x = 1 \;\Rightarrow\; y = 1 + 3 = 4$ Point (1, 4)

$x = 2 \;\Rightarrow\; y = 2 + 3 = 5$ Point (2, 5)

$\Rightarrow$ Solution for $3^x = x + 3 \;\Rightarrow\; x = 1.3$



11. $f(x) = \log_2 x$

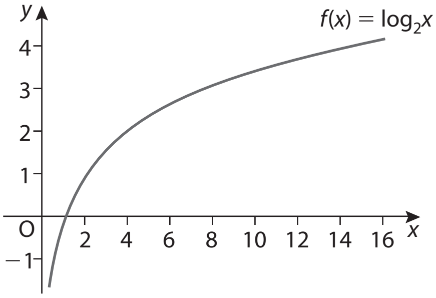
$x = 1 \;\Rightarrow\; f(1) = \log_2 1 = 0$, Point (1, 0)

$x = 2 \;\Rightarrow\; f(2) = \log_2 2 = 1$, Point (2, 1)

$x = 4 \;\Rightarrow\; f(4) = \log_2 4 = 2$, Point (4, 2)

$x = 8 \;\Rightarrow\; f(8) = \log_2 8 = 3$, Point (8, 3)

$x = 16 \;\Rightarrow\; f(16) = \log_2 16 = 4$, Point (16, 4)



12. $y = a\log_2 x + b$

Point $(8, 10) \quad \Rightarrow \quad 10 = a\log_2 8 + b = a(3) + b = 3a + b$

Point $(32, 14) \quad \Rightarrow \quad 14 = a\log_2 32 + b = a(5) + b = 5a + b$

$$\begin{array}{lrl} \text{hence} & 5a + b = 14 & \\ \text{and} & \underline{-3a - b = -10} & \\ & 2a \qquad = 4 & \Rightarrow a = 2 \end{array}$$

$3a + b = 10 \Rightarrow 3(2) + b = 10$

$\Rightarrow 6 + b = 10 \quad \Rightarrow \quad b = 4$

13. $y = \log_a x$

Point $(3, 1) \quad \Rightarrow \quad 1 = \log_a 3$

$\Rightarrow \quad a^1 = 3 \quad \Rightarrow \quad a = 3$

14. (c): $f(x) = \log_3(x - 3)$

Test $(4, 0) \Rightarrow \; f(4) = \log_3(4 - 3) = \log 1 = 0$ True

Test $(6, 1) \Rightarrow \; f(6) = \log_3(6 - 3) = \log_3 3 = 1$ True

15. Ⓑ: $y = \log_5(x - 2)$

Test $(3, 0) \quad \Rightarrow \quad \log_5(3 - 2) = \log_5 1 = 0$ True

Test $(7, 1) \quad \Rightarrow \quad \log_5(7 - 2) = \log_5 5 = 1$ True

16. $y = \log_3(x - 4)$

Point $(q, 2) \quad \Rightarrow \quad \log_3(q - 4) = 2 \quad \Rightarrow \quad q - 4 = 3^2$

$\Rightarrow \quad q - 4 = 9 \quad \Rightarrow \quad q = 13$

17. $T = T_0 e^{\frac{t}{20}}$

(i) $t = 10 \quad \Rightarrow \quad 165 = T_0 e^{\frac{10}{20}}$

$\Rightarrow \quad T_0 e^{\frac{1}{2}} = 165$

$$\Rightarrow \quad T_0 = \frac{165}{e^{\frac{1}{2}}} = \frac{165}{1.6487} = 100.07 = 100$$

(ii) $T = 100^{\frac{t}{20}}$

$t = 24 \Rightarrow T = 100e^{\frac{24}{20}} = 100e^{1.2} = 100(3.32) = 332\ °\text{C}$

18. $A_t = A_0\, e^{-0.002t}$

(i) $600 = A_0\, e^{-0.002(1000)}$

$\Rightarrow 600 = A_0\, e^{-2}$

$$\Rightarrow A_0 = \frac{600}{e^{-2}} = 4433.43 = 4433$$

(ii) $2216.5 = 4433\ e^{-0.002t}$

$$\Rightarrow \quad \frac{2216.5}{4433} = e^{-0.002t}$$

$\Rightarrow \quad 0.5 = e^{-0.002t}$

$\Rightarrow \quad \ln(0.5) = \ln e^{-0.002t}$

$\Rightarrow -0.693147 = -0.002t(\ln e) = -0.002t$

$$\Rightarrow t = \frac{0.693147}{0.002} = 346.57 = 347 \text{ years}$$

19. $N = 200 - A\,\mathrm{e}^{-\frac{t}{20}}$

(i) $t = 10$ years $\Rightarrow 91 = 200 - A\,\mathrm{e}^{-\frac{10}{20}}$

$\Rightarrow A\,\mathrm{e}^{-0.5} = 109$

$\Rightarrow A = \dfrac{109}{\mathrm{e}^{-0.5}} = 179.7 = 180$

(ii) $t = 0 \Rightarrow N = 200 - 180\mathrm{e}^{-\frac{0}{20}} = 200 - 180\mathrm{e}^{0} = 200 - 180 = 20$

(iii) $N = 200 - 180\mathrm{e}^{-\frac{t}{20}}$

$\Rightarrow$ as time increases, x-axis ($y = 0$) is an asymptote for $\mathrm{e}^{-\frac{t}{20}}$

$\Rightarrow N = 200 - 180(0) = 200$

20. $m = m_0\mathrm{e}^{-kt}$

$m = \dfrac{9}{10}m_0$ when $t = 10$ $\Rightarrow 0.9m_0 = m_0\mathrm{e}^{-k(10)}$

$\Rightarrow 0.9 = \mathrm{e}^{-10k}$

$\Rightarrow \ln(0.9) = \ln \mathrm{e}^{-10k}$

$\Rightarrow -0.10536 = -10k(\ln \mathrm{e}) = -10k$

$\Rightarrow 10k = 0.10536$

$\Rightarrow k = \dfrac{0.10536}{10} = 0.010536 = 0.0105$

$m = \dfrac{1}{2}m_0$ $\Rightarrow 0.5m_0 = m_0\mathrm{e}^{-0.0105t}$

$\Rightarrow 0.5 = \mathrm{e}^{-0.0105t}$

$\Rightarrow \ln(0.5) = \ln \mathrm{e}^{-0.0105t}$

$\Rightarrow -0.693147 = -0.0105t(\ln \mathrm{e})$

$\Rightarrow 0.0105t = 0.693147$

$\Rightarrow t = \dfrac{0.693147}{0.0105} = 66$ years

Exercise 8.7

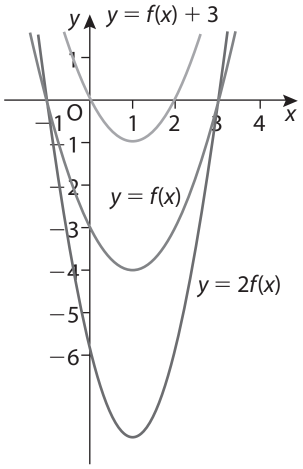
1. $y = f(x)$ [Shown]

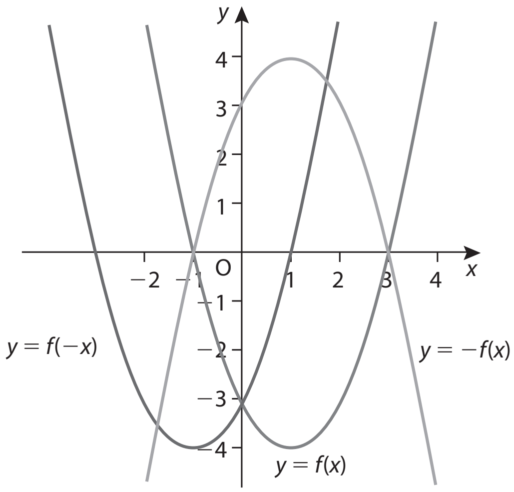
(i) $y = f(x) + 3$ [Shown]

(ii) $y = 2f(x)$ [Shown]

(iii) $y = -f(x)$ [Shown below]

(iv) $y = f(-x)$ [Shown below]

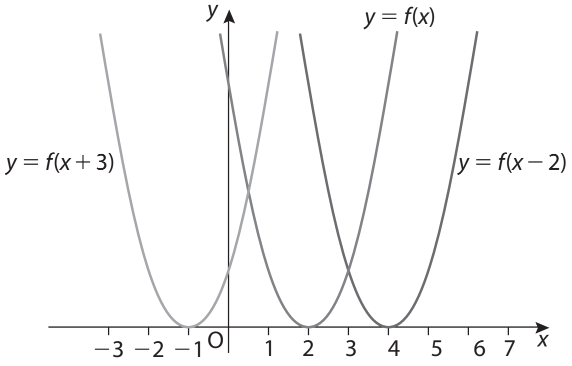




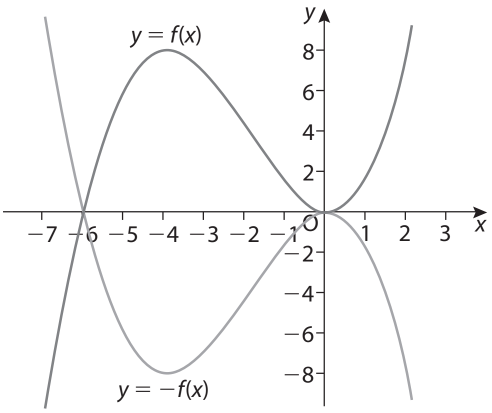
2. $g(x) = -f(x)$
$h(x) = f(x) + 3$

3. $y = f(x)$ [Shown]
(i) $y = f(x + 3)$ [Shown]
(ii) $y = f(x - 2)$ [Shown]

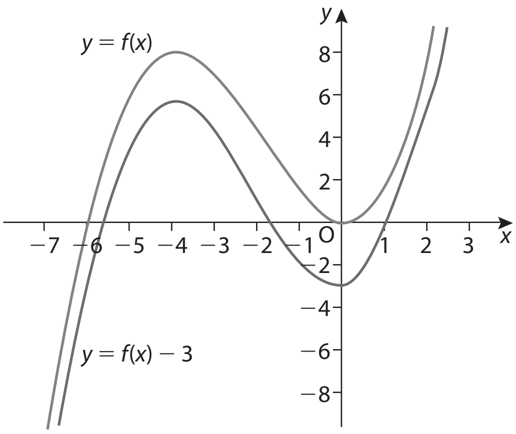
4. Graph Ⓓ



5. (i) $y = f(x)$ [Shown]
$y = -f(x)$ [Shown]



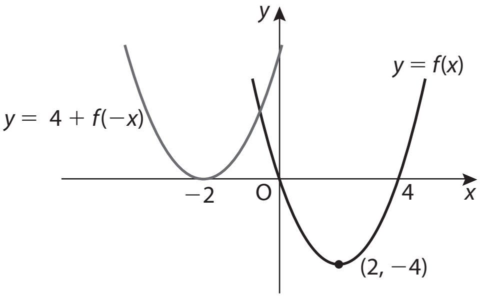
(ii) $y = f(x)$ [Shown]
$y = f(x) - 3$ [Shown]



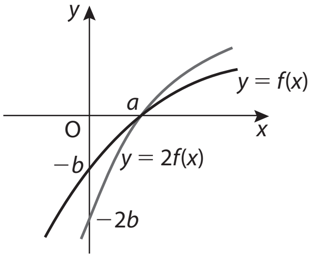
6. Graph Ⓒ

7. Graph Ⓐ

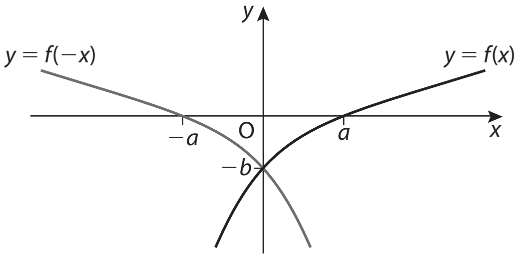
8. $y = f(x)$ [Shown]
$y = 4 + f(-x)$ [Shown]



9. (i) $y = f(x)$ [Shown]
$y = 2f(x)$ [Shown]



(ii) $y = f(x)$ [Shown]
$y = f(-x)$ [Shown]



10. (i) $f(x) = (x - 2)(x^2 + 1)$

(a) x-axis $\Rightarrow f(x) = 0 \Rightarrow (x - 2)(x^2 + 1) = 0$
$\Rightarrow x = 2$ only $\Rightarrow$ point $(2, 0)$

(b) y-axis $\Rightarrow x = 0 \Rightarrow f(0) = (0 - 2)[(0)^2 + 1] = (-2)(1) = -2$
$\Rightarrow$ point $(0, -2)$

(ii) Graph Ⓓ

Revision Exercise 8 (Core)

1. $f(x) = 2x - 3$ and $g(x) = x^2$
$\Rightarrow g(f(x)) = g(2x - 3) = (2x - 3)^2$
Solve $g(f(x)) = 9$.
$\Rightarrow (2x - 3)^2 = 9$
$\Rightarrow 4x^2 - 12x + 9 - 9 = 0$
$\Rightarrow 4x^2 - 12x = 0$
$\Rightarrow x^2 - 3x = 0$
$\Rightarrow x(x - 3) = 0 \Rightarrow x = 0$ OR $x = 3$

2. Ⓐ and $y = x^2 - 2$ because (i) Curve for $+x^2$ is "∪-shaped"
(ii) Point on y-axis is $(0, -2)$
Ⓑ and $y = 2 - x^2$ because (i) Curve for $-x^2$ is "∩-shaped"
(ii) Point on y-axis is $(0, 2)$
Ⓒ and $y = 2x$ because it's a linear graph

3. $f(x) = \dfrac{x}{x + 1}$

$f(1) = \dfrac{1}{1 + 1} = \dfrac{1}{2}$

$f(2) = \dfrac{2}{2 + 1} = \dfrac{2}{3}$

$f(3) = \dfrac{3}{3 + 1} = \dfrac{3}{4}$

$f(4) = \dfrac{4}{4 + 1} = \dfrac{4}{5}$

$f(5) = \dfrac{5}{5 + 1} = \dfrac{5}{6}$

$\Rightarrow$ Range $= \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}\right\}$

4. Curve meets x-axis at 0 and -7

$\Rightarrow \quad y = kx(x + 7)$

Point (4, 4) $\Rightarrow \; 4 = k4(4 + 7)$

$\Rightarrow \; 1 = 11k \;\Rightarrow\; k = \dfrac{1}{11}$

$\Rightarrow \; y = \dfrac{1}{11}x(x + 7) = \dfrac{x}{11}(x + 7)$

5. $g(x) = y = 5 + \dfrac{x}{2}$

$\Rightarrow \; 2y = 10 + x \quad \Rightarrow \quad x = 2y - 10$

Hence, $\; g^{-1}(x) = 2x - 10$

(i) $g^{-1}(-2) = 2(-2) - 10 = -4 - 10 = -14$

(ii) Solve $g(x) = g^{-1}(x)$.

$\Rightarrow \quad 5 + \dfrac{x}{2} = 2x - 10$

$\Rightarrow \quad 10 + x = 4x - 20$

$\Rightarrow \quad -3x = -30 \;\Rightarrow\; x = 10$

6. $y = a^x$

(i) C is on the y-axis $\Rightarrow \; x = 0 \Rightarrow \; y = a^0 = 1 \;\; \Rightarrow$ Point C = (0, 1)

(ii) B(2, 16) $\Rightarrow \; 16 = a^2 \;\; \Rightarrow \; a = \sqrt{16} = 4$

7. (i) $x = -2, 1, 3$

(ii) $x = 1.4 \quad$ or $\quad x = 2.8$

(iii) $x^3 - 2x^2 - 5x = 0 \;\Rightarrow x^2 - 2x - 5x + 6 = 0 + 6 = 6$

$\Rightarrow y = 6 \;\Rightarrow\; x = -1.5$ or $x = 0$

No: a horizontal line will cut the graph at more than one point;

Surjective: any horizontal line will cut the graph at least once.

8. $y = 2m^x$

(i) Point (3, 54) $\Rightarrow \; 54 = 2.3^3$

$\Rightarrow m^3 = 27 \;\Rightarrow\; m = (27)^{\frac{1}{3}} = 3$

(ii) $y = 2.3^x$

P is on the y-axis $\Rightarrow \; x = 0 \;\Rightarrow\; y = 2.3^0 = 2.1 = 2 \;\Rightarrow\;$ P(0, 2)

9. (i) $y = f(x) = x^3 + ax^2 + b$

$\dfrac{dy}{dx} = 3x^2 + 2ax$

Put $\dfrac{dy}{dx} = 0; \quad 3x^2 + 2ax = 0$

$x(3x + 2a) = 0$

$x = 0 \quad$ or $\quad 3x + 2a = 0$

$x = 0 \quad$ or $\quad x = -\dfrac{2a}{3}$

Thus the curve has a stationary point at $x = 0$.

(ii) The other stationary point is when $x = -\dfrac{2a}{3}$.

$-\dfrac{2a}{3} = -2$

$a = 3$

When $\; x = -2: \qquad y = (-2)^3 + 3(-2)^2 + b$

$y = b + 4$

The y co-ordinate of the turning point is 6. Thus

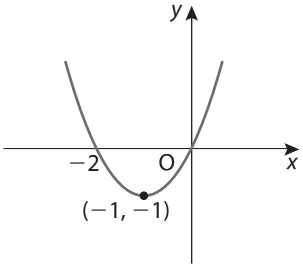
$b + 4 = 6$

$b = 2.$

10. $x^2 + 2x = x^2 + 2x + 1 - 1$

$= (x + 1)^2 - 1$

$\Rightarrow$ Turning point $= (-1, -1)$



11. (i) (a) Domain = R; Range is $y \geq -2$

(b) Domain is $x \leq 2$; Range is R

(c) Domain is $-4 \leq x \leq 0$; Range is $0 \leq y \leq 4$

(ii) (a) is the only function; a vertical line will intersect graphs (b) and (c) more than once.

12. A and $y = \left(\frac{1}{2}\right)^x$; B and $y = 3^{-x}$ OR $\left(\frac{1}{3}\right)^x$;

C and $y = 5^x$; D and $y = 2^x$

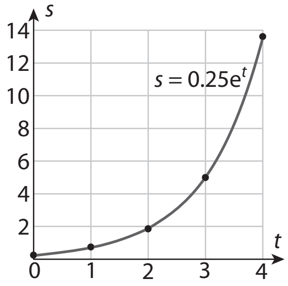
13. $s = 0.25e^t$

$v = \frac{ds}{dt} = 0.25e^t$

The table is completed below.

t	0	1	2	3	4
v	0.3	0.7	1.8	5.0	13.6

The graph is sketched below.



14. $f(x) = 10x$ and $g(x) = x + 3$

(i) $fg(x) = f(x + 3) = 10(x + 3) = 10x + 30$

$y = 10x + 30 \Rightarrow 10x = y - 30$

$\Rightarrow x = \frac{y - 30}{10}$

Hence, $(fg)^{-1}(x) = \frac{x - 30}{10}$

(ii) $fg(a) = b \Rightarrow fg(a) = 10a + 30 = b$

$(fg)^{-1}(b) = (fg)^{-1}(10a + 30) = \frac{10a + 30 - 30}{10} = \frac{10a}{10} = a$

15. $f(x) = x^2 + 3$ and $g(x) = x + 4$

(a) $fg(x) = f(x + 4) = (x + 4)^2 + 3 = x^2 + 8x + 16 + 3 = x^2 + 8x + 19$

$gf(x) = g(x^2 + 3) = x^2 + 3 + 4 = x^2 + 7$

(b) $fg(x) + gf(x) = 0 \Rightarrow x^2 + 8x + 19 + x^2 + 7 = 0$

$\Rightarrow 2x^2 + 8x + 26 = 0$

$\Rightarrow x^2 + 4x + 13 = 0$

No real roots if $b^2 - 4ac < 0$

$\Rightarrow (4)^2 - 4(1)(13) = 16 - 52 = -36 < 0$ … True

16. (i) Range of f is $y \geq 0$

(ii) $y = \dfrac{1}{2x-3} \quad \Rightarrow 2xy - 3y = 1$

$$\Rightarrow 2xy = 3y + 1$$

$$\Rightarrow \quad x = \frac{3y+1}{2y}$$

Hence, $g^{-1}(x) = \dfrac{3x+1}{2x}$

(iii) Domain of g: $x \in R, x \neq \dfrac{3}{2} \Rightarrow$ range of g^{-1}: $x \in R, x \neq \dfrac{3}{2}$

(iv) $fg(x) = f\left[\dfrac{1}{(2x-3)}\right] = \dfrac{1}{(2x-3)^2} = \dfrac{1}{4x^2 - 12x + 9}$

$fg(x) = 9 \Rightarrow \dfrac{1}{4x^2 - 12x + 9} = 9$

$\Rightarrow 36x^2 - 108x + 81 = 1$

$\Rightarrow 36x^2 - 108x + 80 = 0$

$\Rightarrow 9x^2 - 27x + 20 = 0$

$\Rightarrow (3x-5)(3x-4) = 0$

$\Rightarrow 3x = 5$ OR $3x = 4$

$\Rightarrow x = \dfrac{5}{3}$ OR $x = \dfrac{4}{3}$

Revision Exercise 8 (Advanced)

1. $f(x) = x - 1, g(x) = 2x^2 - x - 1$ and $h(x) = \log_3 x$

(i) $hf(x) = h(x-1) = \log_3(x-1)$

$hg(x) = h(2x^2 - x - 1) = \log_3(2x^2 - x - 1)$

(ii) Solve $hg(x) - hf(x) = 2$.

$\Rightarrow \log_3(2x^2 - x - 1) - \log_3(x-1) = 2$

$\Rightarrow \log_3 \dfrac{2x^2 - x - 1}{x-1} = 2$

$\Rightarrow \dfrac{2x^2 - x - 1}{x-1} = 3^2 = 9$

$\Rightarrow 2x^2 - x - 1 = 9x - 9$

$\Rightarrow 2x^2 - 10x + 8 = 0$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0 \quad \Rightarrow \quad x = 1$ OR $x = 4$

2. (a) and graph Ⓒ $\Rightarrow$ x-axis $(-4, 0)$, $(8, 0)$

(b) and graph Ⓑ $\Rightarrow$ y-axis $(0, 2)$

(c) and graph Ⓓ $\Rightarrow$ minimum point $(1, -10)$

(d) and graph Ⓐ $\Rightarrow$ x-axis $(-3, 0)$ and $(3, 0)$

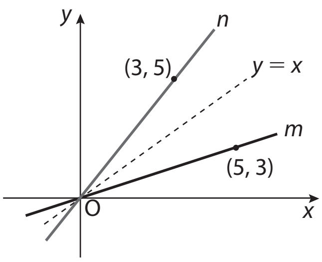
3. (i) Graph of line m: points $(0, 0)$, $(5, 3)$

Graph of line n; the inverse of m: points $(0, 0)$, $(3, 5)$

(ii) $f(x) = +\sqrt{16 - x^2}$

Domain of f: $-4 \leq x \leq 4$

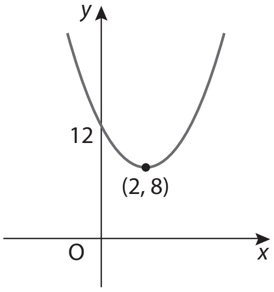
Range of f: $0 \leq y \leq 4$



4. $x^2 - 4x + 12 = x^2 - 4x + 4 + 8$

$= (x - 2)^2 + 8$

Turning point $= (2, 8)$



5. x cm cut from each corner

$\Rightarrow$ Length $= (24 - 2x)$ cm, Width $= (18 - 2x)$ cm, height $= x$ cm

Volume $V = (24 - 2x)(18 - 2x)(x)$

$18 - 2x > 0$

$\Rightarrow -2x > -18 \Rightarrow 2x < 18 \Rightarrow x < 9$

$\Rightarrow$ domain: $0 < x < 9$

6. (i) $y = a \,.\, b^x$

Point $(0, 2)$ $\Rightarrow 2 = a \,.\, b^0 = a \,.\, 1 \Rightarrow a \Rightarrow a = 2$

$y = 2 \,.\, b^x$

Point $(3, 54) \Rightarrow 54 = 2 \,.\, b^3$

$\Rightarrow b^3 = 27 \Rightarrow b = (27)^{\frac{1}{3}} = 3$

(ii) (a) $x = 1$

(b) Domain: $x \in R\backslash\{1\}$

(c) Range $= R \backslash \{0\}$

(d) Yes; a horizontal line will intersect the graph at most once.

7. (i) Area of canvas $= 2x^2 = \ell x = 9$

$\Rightarrow \ell x = 9 - 2x^2$

$\Rightarrow \ell = \dfrac{9 - 2x^2}{x} = \dfrac{9}{x} - \dfrac{2x^2}{x} = \left(\dfrac{9}{x} - 2x\right)$ cm

Volume $V = \left(\dfrac{9}{x} - 2x\right)(x)(x) = 9x - 2x^3$

(ii) On x-axis, $y = 0 \Rightarrow 9x - 2x^3 = 0$

$\Rightarrow x(9 - 2x^2) = 0$

$\Rightarrow x = 0, x^2 = 4.5$

$\Rightarrow x = \sqrt{4.5} = 2.1$

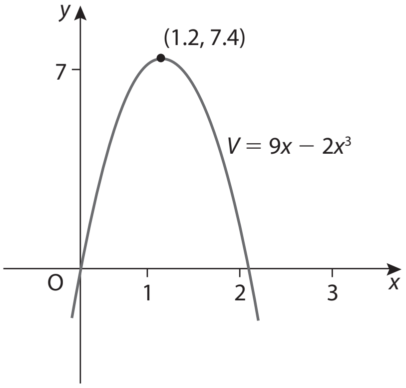
$x = 1 \Rightarrow y = 9(1) - 2(1)^3 = 7$

$x = 2 \Rightarrow y = 9(2) - 2(2)^3 = 2$

Points are $(0, 0), (1, 7), (2, 2), (2.1, 0)$

(iii) (a) $x = 1.2$

(b) $y = 7.4$ m^3 = largest volume.



8. $f(x) = 3x - 1$ and $g(x) = x^2 + 1$

(a) Range of g; $y \geq 1$

(b) $gf(x) = g(3x - 1) = (3x - 1)^2 + 1 = 9x^2 - 6x + 2$

$fg(x) = f(x^2 + 1) = 3(x^2 + 1) - 1 = 3x^2 + 2$

Solve $gf(x) = fg(x)$.

$\Rightarrow 9x^2 - 6x + 2 = 3x^2 + 2$

$\Rightarrow 6x^2 - 6x = 0$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \quad \text{OR} \quad x = 1$

(c) $|f(x)| = 8$

$|3x - 1| = 8$

$\Rightarrow +(3x - 1) = 8 \quad \text{OR} \quad -(3x - 1) = 8$

$\Rightarrow \quad 3x - 1 = 8 \quad \Rightarrow \quad -3x + 1 = 8$

$\Rightarrow \quad 3x = 9 \quad \Rightarrow \quad -3x = 7$

$\Rightarrow \quad x = 3 \quad \Rightarrow \quad x = -\frac{7}{3}$

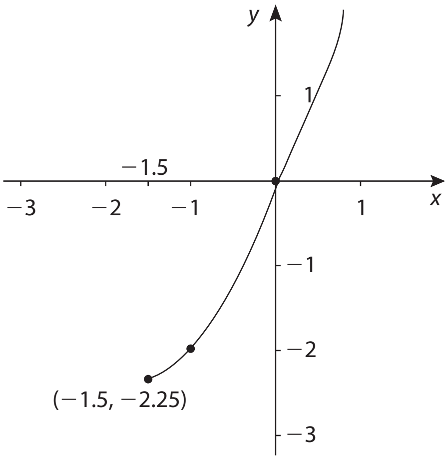
(d) $h(x) = x^2 + 3x$

One-to-one $\Rightarrow$ minimum point occurs

at $x = -1.5 \Rightarrow h(-1.5) = (-1.5)^2 + 3(-1.5)$

$= 2.25 - 4.5 = -2.25$

Hence, $q = -1.5 = -\frac{3}{2}$



9. (i) $y = k(x - 1)^2\,(x + t) = k(x - 1)^2\,(x - 5) \quad \Rightarrow t = -5$

Point (0, 10) $\quad \Rightarrow 10 = k(0 - 1)^2\,(0 - 5)$

$\Rightarrow 10 = -5k \quad \Rightarrow k = -2$

(ii) From part (i), the equation of the curve is

$y = -2(x - 1)^2(x - 5)$

(a) Under reflection in the x-axis, the image curve is found by replacing y by $-y$. The image curve is then

$-y = -2(x - 1)^2(x - 5)$

$y = 2(x - 1)^2(x - 5)$

(b) Under reflection in the y-axis, the image curve is found by replacing x by $-x$. The image curve is then

$y = -2(-x - 1)^2(-x - 5)$

$y = -2[-1(x + 1)]^2[-1(x + 5)]$

$y = 2(x + 1)^2(x + 5)$

10. (i) Suitable domain; $-5 \leq x \leq 5$

Corresponding range; [0, 5]

(ii) (a) Curve $y = 3^x$ cuts the y-axis at $x = 0 \Rightarrow y = 3^0 = 1$

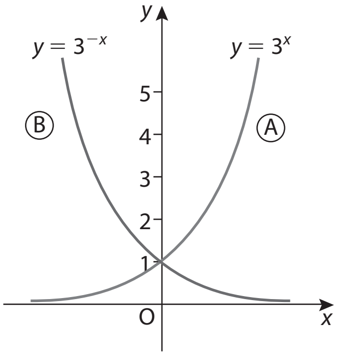
$\Rightarrow$ point $= (0, 1)$

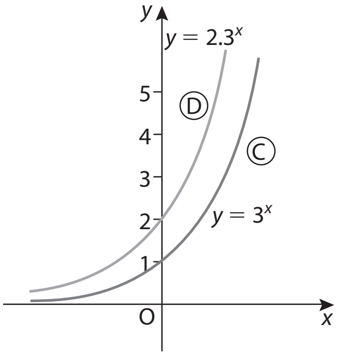
(b) Graph Ⓐ $\Rightarrow y = 3^x$

Graph Ⓑ $\Rightarrow y = 3^{-x}$

Graph Ⓒ $\Rightarrow y = 3^x$

Graph Ⓓ $\Rightarrow y = 2.3^x$





11. $P = Ae^{\frac{t}{20}} \Rightarrow$ when $t = 0 \Rightarrow P = A\,.\,e^0 = A$

From table $\Rightarrow A = 5$ when $t = 0$

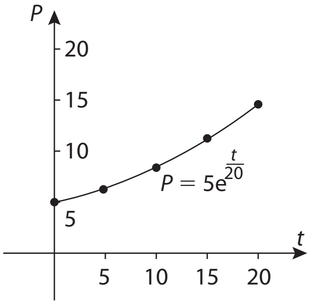
$P = 5e^{\frac{t}{20}} \Rightarrow t = 5 \Rightarrow P = 5\,.\,e^{\frac{5}{20}} = 5e^{0.25} = 6.4$

$t = 10 \Rightarrow P = 5\,.\,e^{\frac{10}{20}} = 8.2$

$t = 15 \Rightarrow P = 5\,.\,e^{\frac{15}{20}} = 10.6$

$t = 20 \Rightarrow P = 5\,.\,e^{\frac{20}{20}} = 13.6$

t	0	5	10	15	20
P	5	6.4	8.2	10.6	13.6



$10 = 5e^{\frac{t}{20}}$

$\Rightarrow 2 = e^{\frac{t}{20}}$

$\Rightarrow \ln(2) = \ln e^{\frac{t}{20}}$

$\Rightarrow 0.693 = \frac{t}{20}(\ln e) = \frac{t}{20}$

$\Rightarrow t = 20(0.693) = 13.86 = 13.9$ days

12. (i) $x^2 - 7x + 12 > 0 \quad \Rightarrow \quad (x - 4)(x - 3) > 0 \Rightarrow x \geq 4$ or $x \leq 3$

(ii) (a) A vertical line will intersect the graph once only.

(b) NO

(c) YES

(d) Injective if $\frac{\pi}{2} \leq x \leq 3\frac{\pi}{2}$

13. Graph Ⓒ Check (1, 3) in $\log_3 y = x \Rightarrow \log_3 3 = 1 = x \Rightarrow$ True

14. (i) $n > 0$ as there cannot be a negative number of predators, and there must be at least one predator.

(ii) $P = n^3 - 12n^2 - 60n + 850$

$$\frac{dP}{dn} = 3n^2 - 24n - 60$$

Put $\frac{dP}{dn} = 0$: $\quad 3n^2 - 24n - 60 = 0$

$$n^2 - 8n - 20 = 0$$

$$(n - 10)(n + 2) = 0$$

$$n - 10 = 0 \quad \text{or} \quad n + 2 = 0$$

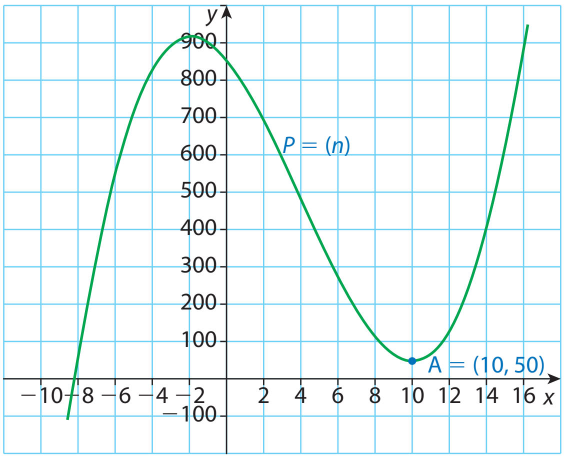
$$n = 10 \quad \text{or} \quad n = -2$$

$$\frac{d^2P}{dn^2} = 6n - 24.$$

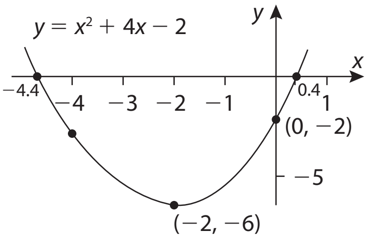
When $n = 10$, $\frac{d^2P}{dn^2} = 36 > 0$. Hence $n = 10$ gives a minimum for the population.

When $n = 10$, $P = 50$, which is the minimum population size.

(iii) The graph is shown below.



Revision Exercise 8 (Extended-Response Questions)



1. (a) $x^2 + 4x - 2 = x^2 + 4x + 4 - 2 - 4$

$= (x + 2)^2 - 6 = (x + a)^2 + b$

Hence, $a = 2, b = -6$

$\Rightarrow$ Turning point $= (-2, -6)$

(b) On y-axis, $x = 0 \Rightarrow y = (0)^2 - 4(0) - 2 = -2$

Point $(0, -2)$

On x-axis, $y = 0 \Rightarrow x^2 + 4x - 2 = 0$

$$\Rightarrow x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

$-2 - \sqrt{6} = -4.4$ and $-2 + \sqrt{6} = 0.4$

$\Rightarrow$ Points on x-axis: $(-4.4, 0)$ and $(0.4, 0)$

(c) Discriminant $= \sqrt{24}$; Since discriminant > 0, the curve will intersect the x-axis at two distinct points.

(d) $x^2 + 4x + k = 0$ has no real roots $\Rightarrow b^2 - 4ac < 0$

$\Rightarrow (4)^2 - 4(1)(k) < 0$

$\Rightarrow 16 - 4k < 0$

$\Rightarrow -4k < -16$

$\Rightarrow 4k > 16 \Rightarrow k > 4$

2. (a) Curve meets x-axis at $A(-10, 0), B(10, 0)$

$\Rightarrow y = a(x + 10)(x - 10) = a(x^2 - 100)$

$|OZ| = 9 \Rightarrow$ Point $Z = (0, 9)$

$\Rightarrow 9 = a((0)^2 - 100)$

$\Rightarrow 9 = -100a \Rightarrow a = \frac{-9}{100} = -0.09$

$\Rightarrow y = -0.09x^2 + 9 \Rightarrow b = 9$

(b) Point $C = (-7, 0) \Rightarrow y = -0.09(-7)^2 + 9 = 4.59$

$\Rightarrow |DE| = 4.59 - 1.8 = 2.79$

(c) $|OH| = 6.3 \Rightarrow y = 6.3$

$\Rightarrow -0.09x^2 + 9 = 6.3$

$\Rightarrow -0.09x^2 = 6.3 - 9 = -2.7$

$\Rightarrow x^2 = \frac{2.7}{0.09} = 30$

$\Rightarrow x = \sqrt{30}$

$\Rightarrow |FG| = 2\sqrt{30} = 2(5.477) = 10.954 = 10.95$ m

3. (a) $y = a \log_2 (x - b)$

Point $(5, 2) \Rightarrow 2 = a \log_2 (5 - b)$

$\Rightarrow \log_2 (5 - b) = \frac{2}{a}$

$\Rightarrow 5 - b = 2^{\frac{2}{a}} = (2^2)^{\frac{1}{a}} = 4^{\frac{1}{a}} \Rightarrow b = 5 - 4^{\frac{1}{a}}$

Point $(7, 4) \Rightarrow 4 = a \log_2 (7 - b)$

$\Rightarrow \log_2 (7 - b) = \frac{4}{a}$

$\Rightarrow 7 - b = 2^{\frac{4}{a}} = (2^2)^{\frac{2}{a}} = 4^{\frac{2}{a}} = \left(4^{\frac{1}{a}}\right)^2$

$\Rightarrow b = 7 - \left(4^{\frac{1}{a}}\right)^2$

$\Rightarrow 5 - 4^{\frac{1}{a}} = 7 - \left(4^{\frac{1}{a}}\right)^2$

$\Rightarrow \left(4^{\frac{1}{a}}\right)^2 - 4^{\frac{1}{a}} - 2 = 0$

Let $k = 4^{\frac{1}{a}} \Rightarrow k^2 - k - 2 = 0$

$\Rightarrow (k - 2)(k + 1) = 0$

$\Rightarrow k = 2 \quad \text{OR} \quad k = -1$ (Not Valid)

$\Rightarrow 4^{\frac{1}{a}} = 2$

$\Rightarrow (2^2)^{\frac{1}{a}} = 2^{\frac{2}{a}} = 2^1$

$\Rightarrow \dfrac{2}{a} = 1 \Rightarrow a = 2$

$\Rightarrow y = 2 \log_2 (x - b)$

Point (5, 2) $\Rightarrow 2 = 2 \log_2 (5 - b)$

$\Rightarrow \log_2 (5 - b) = \dfrac{2}{2} = 1$

$\Rightarrow 5 - b = 2^1 = 2 \Rightarrow b = 3$

(b) Statement (iii) is not true

4. (a) Graph Ⓓ

(b) (i) (a) $N_0 = 20\,000$

(b) Decrease by 20% $\Rightarrow N = 80\%$ of $20\,000 = 16\,000$

$t = 1 \Rightarrow 16\,000 = 20\,000\, e^{k(1)}$

$\Rightarrow \dfrac{16\,000}{20\,000} = 0.8 = e^k$

$\Rightarrow \ln 0.8 = \ln e^k = k \ln e = k\,.\,1 = k$

$\Rightarrow k = \ln (0.8) = -0.2231 = -0.223$

(b) (ii) $N = 20\,000\, e^{-0.223t}$

$N = 5000 \Rightarrow 5000 = 20\,000 e^{-0.223t}$

$\Rightarrow e^{-0.223t} = \dfrac{5000}{20\,000} = \dfrac{1}{4} = 0.25$

$\Rightarrow \ln e^{-0.223t} = \ln(0.25)$

$\Rightarrow -0.223t(\ln e) = -1.386$

$\Rightarrow 0.223t = 1.386$

$\Rightarrow \qquad t = \dfrac{1.386}{0.223} = 6.21 = 6.2$ years

5. $f(x) = x^3$ and $g(x) = \dfrac{1}{x - 3}$

(a) Range of $f = R$

(b) (i) $fg(x) = f\left(\dfrac{1}{x - 3}\right) = \left(\dfrac{1}{x - 3}\right)^3 = \dfrac{1}{(x - 3)^3}$

(ii) Solve $fg(x) = 64 \Rightarrow \dfrac{1}{(x - 3)^3} = 64$

$\Rightarrow (x - 3)^3 = \dfrac{1}{64}$

$\Rightarrow x - 3 = \sqrt[3]{\dfrac{1}{64}} = \dfrac{1}{4}$

$\Rightarrow x = 3 + \dfrac{1}{4} = \dfrac{13}{4}$

(c) (i) $g(x) = y = \dfrac{1}{x - 3}$

$\Rightarrow xy - 3y = 1$

$\Rightarrow xy \quad = 1 + 3y$

$\Rightarrow x \quad = \dfrac{1 + 3y}{y}$

Hence, $g^{-1}(x) = \dfrac{1 + 3x}{x}$

(ii) Range of g^{-1} = domain of $g = R, x \neq 3$

(iii) $gg^{-1}(x) = g\left(\dfrac{1+3x}{x}\right) = \dfrac{1}{\frac{1+3x}{x} - 3}$

$$= \dfrac{1}{\frac{1+3x-3x}{x}} = \dfrac{1}{\frac{1}{x}} = x$$

(iv) Graph of $g^{-1}(x)$ is not continuous at $x = 0$

6. $M = Ae^{-pt}$

(i) $A = €130\,000$

(ii) $t = 1 \Rightarrow €122\,000 = €130\,000e^{-p(1)}$

$$\Rightarrow e^{-p} = \frac{122\,000}{130\,000} = \frac{61}{65}$$

$$\Rightarrow \ln e^{-p} = \ln\frac{61}{65}$$

$$\Rightarrow -p(\ln e) = -0.0635$$

$$\Rightarrow p = 0.0635 = 0.064$$

(iii) 1 January 2006 to end of 2011 is 6 years

$\Rightarrow t = 6 \Rightarrow M = 130\,000e^{-0.064(6)}$

$= 130\,000e^{-0.384}$

$= €88\,547.085 = €88\,500$

7. (a) Graph Ⓑ: $y = \log_5(x-2)$

Check point $(3, 0) \Rightarrow \log_5(3-2) = \log_5 1 = 0$, True

Check point $(7, 1) \Rightarrow \log_5(7-2) = \log_5 5 = 1$, True

(b) (i) $y = 4^x \cap y = 3^{2-x}$

$\Rightarrow 4^x = 3^{2-x}$

$\Rightarrow \log_a 4^x = \log_a 3^{2-x}$

$\Rightarrow x\log_a 4 = (2-x)\log_a 3$

$\Rightarrow x\log_a 4 = 2\log_a 3 - x\log_a 3$

$\Rightarrow x\log_a 4 + x\log_a 3 = 2\log_a 3$

$\Rightarrow x(\log_a 4 + \log_a 3) = \log_a 3^2$

$\Rightarrow x(\log_a 4.3) = \log_a 9 \Rightarrow x = \dfrac{\log_a 9}{\log_a 12}$

(ii) $x = \dfrac{\log_a 9}{\log_a 12} = \log_{12} 9 = 0.884228$

$\Rightarrow y = 4^{0.884228} = 3.40 = 3.4$

8. (a) $y = x^2 - 4x + 5$

$= x^2 - 4x + 4 + 1$

$= (x-2)^2 + 1$

$\Rightarrow$ Turning point $= (2, 1)$

(b) $(x-2)^2 + 1 = y = f(x)$

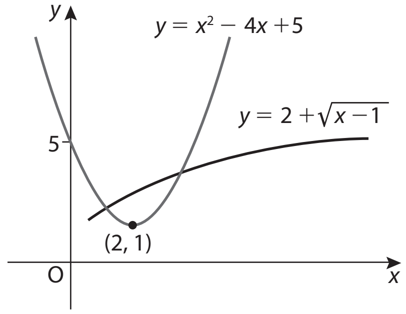
$\Rightarrow \quad (x-2)^2 = y - 1$

$\Rightarrow \quad x - 2 = \sqrt{y-1}$

$\Rightarrow \quad x = 2 + \sqrt{y-1}$

$\Rightarrow \quad f^{-1}(x) = 2 + \sqrt{x-1}$

(c)



9. $C = C_0e^{-kt}$

(i) $C_0 = 5$ kg/ha

$t = 1 \Rightarrow 2.8 = 5e^{-k(1)}$

$\Rightarrow e^{-k} = \frac{2.8}{5} = 0.56$

$\Rightarrow \text{In } e^{-k} = \text{In}(0.56)$

$\Rightarrow -k(\text{In } e) = -0.57981$

$\Rightarrow k = 0.5798$

(ii) $C = 5e^{-0.5798t}$

$\Rightarrow 0.2 = 5e^{-0.5798t}$

$\Rightarrow e^{-0.5798t} = \frac{0.2}{5} = 0.04$

$\Rightarrow \text{In } e^{-0.5798t} = \text{In}(0.04)$

$\Rightarrow -0.5798t\,(\text{In } e) = -3.218876$

$\Rightarrow \quad 0.5798t = 3.218876$

$\Rightarrow t = \frac{3.218876}{0.5798} = 5.551 = 5.6$ years

Chapter 9

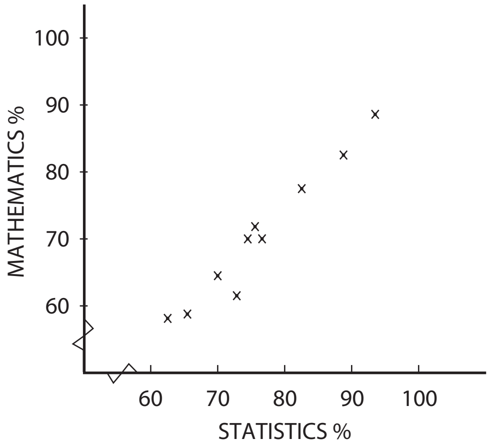
Exercise 9.1

1. (i) Since *y* increases as *x* increases graphs *C* and *E* show positive correlation.
(ii) Since *y* decreases as *x* increases graphs *A* and *F* show negative correlation.
(iii) In graphs *B* and *D*, the variables *x* and *y* show no linear pattern so we say there is no correlation.
(iv) Graph *A* shows a strong negative correlation, as the variables are in a straight line.
Graph *F* can be described as reasonally strong negative correlation.

2. (i) Graph *B* shows the strongest positive correlation with *y* increasing as *x* increases.
(ii) In graph *C* the variables *x* and *y* have a negative correlation with *y* decreasing as *x* increases.
(iii) The weakest correlation is shown in graph *D* as the points are more widely spread out.

3. (i) The correlation can be described as **strongly** positive.
(ii) The better grade a student gets in her mock exams, the better he/she tends to do in the final exam.

4. (i)



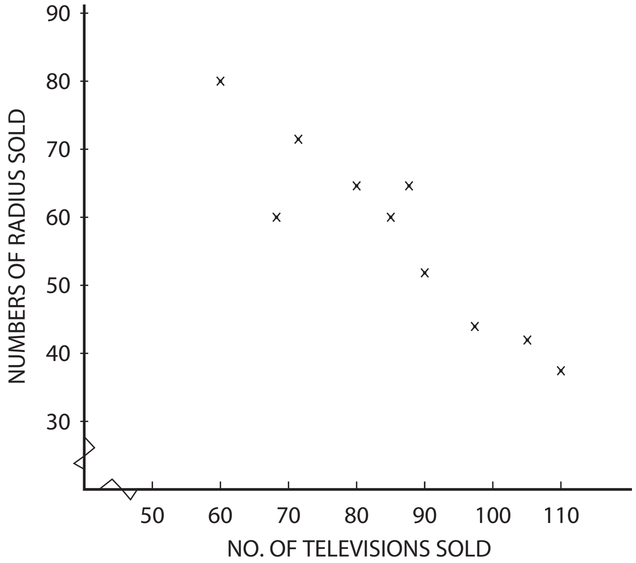
(ii) A strong positive correlation.
(iii) There is a tendency for those who do better at statistics to also do better at mathematics.

5. (i) Negative: The older the boat, it is likely its second-hand selling price decreases.
(ii) Positive: Generally, as children age they grow taller.
(iii) None.
(iv) Negative: The more time spent watching TV means there is less time for studying.
(v) Positive: There is a greater likelihood of accidents when there are higher numbers of vehicles travelling on a route.

6. (i) B: As boys get taller they generally require larger shoe sizes as their feet also increase in size.
(ii) C: There is no relationship between mens weight and time taken to complete a crossword puzzle.
(iii) A: As cars age, the selling price is reduced.
(iv) D: Students generally get similar grades in maths paper 1 and paper 2. There is a positive correlation.

7. (i) Reasonably strong negative correlation.
(ii) Yes, as the age of the bike increases, it causes the price to decrease.

8. (i)



(ii) A strong negative.

(iii) No, there is not a causal relationship. An increase in sales of one does not cause a decrease in sales of the other.

Exercise 9.2

1. $A = 0.6$
$B = -1$
$C = -0.4$
$D = 0.8$

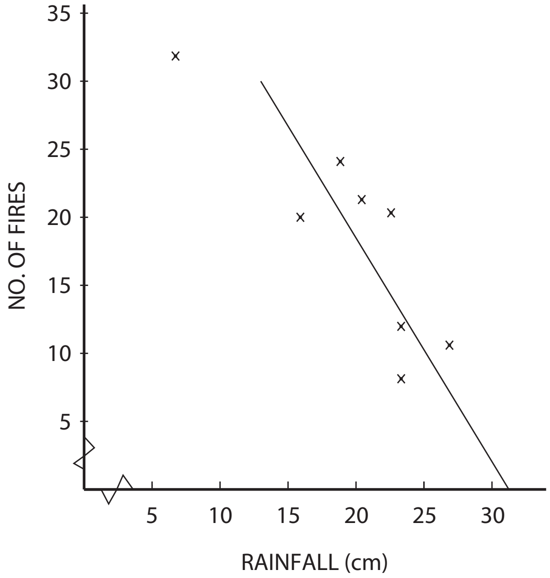
2. (i) 0.9 is strong positive correlation.
(ii) -0.8 is strong negative correlation.
(iii) 0 is no correlation.
(iv) -1 is perfect negative correlation.
(v) -0.1 is a very weak negative correlation.
(vi) 0.2 is a very weak positive correlation.

3. (i) Line of best fit.
(ii) Approximately an equal number of points lie on either side of the line.
(iii) Draw a line from the height (cm) axis at 150 cm to cut the line of best fit and read the answer on the weight (kg) axis.
solution: 55 kg.
(iv) Strong positive.

4. Solution: 0.86
Use your calculator methods (Appendix 1 p.178)

5. 0.86

6. (i)



(ii) Line of best fit

(iii) $r = -0.9$

(iv) From the graph, points (27.5, 8.0) (24, 12) are 2 points on the line of best fit.

The equation of the line of best fit is of the form

$y = mx + c$ or in this case

$y = a + bx$

slope of the line of best fit

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{12 - 8}{24 - 27.5}$$

$$= \frac{4}{-3.5} = -1.14$$

Equation of the line of best fit.

$$y - y_1 = m(x - x_1)$$

$$y - 12 = -1.14(x - 24)$$

$$y - 12 = -1.14(x) + 27.36$$

$$y = 39.36 - 1.1x$$

$$\therefore y = 39 - 1.1x$$

The equation of the line of best fit can be worked out using a calculator. Using this method, the solution was found to be

$$y = 41 - 1.1x$$

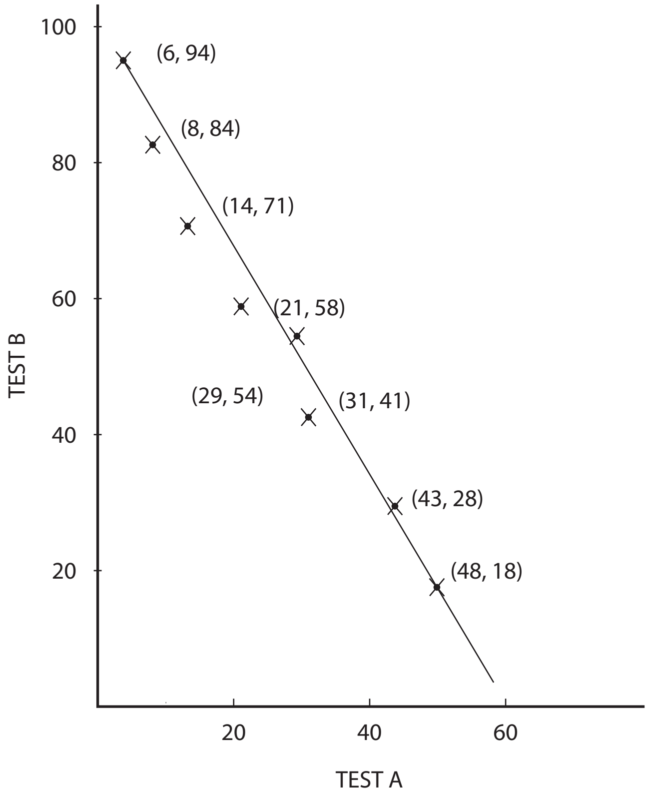
(v) Substitute $y = 25$ in the equation

$$25 = 41 - 1.1x$$

$\therefore 25 - 41 = -1.1x \quad \Rightarrow \quad x = 14.5$

$\therefore$ Approximately 15 fires

7. (i)



(ii) Strong negative correlation

(iii) Using two points on the line of best fit the slope is found using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point (48, 18) and (6, 94)

$$\therefore m = \frac{94 - 18}{6 - 48} = -1.8$$

Equation of line $\quad y - 18 = -1.8(x - 48)$

$$\therefore\ y = 104 - 1.8x$$

Using a calculator, the exact equation is

$$y = -1.7x + 98$$

(v) Using the graph, draw from score 18 on Test A to the line of best fit on the diagram and read off the solution on Test B axis.

$\therefore$ The student scored approx 68.

Alternatively: substitute $x = 18$

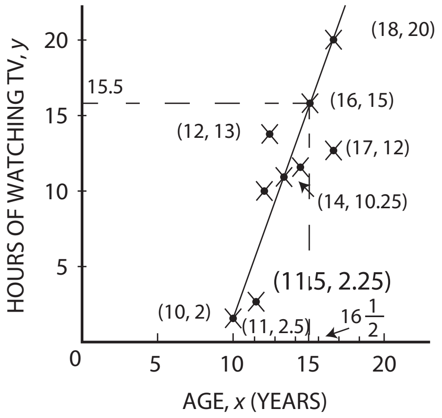
in the linear equation

$$y = -1.7(18) + 98$$

$$\therefore y = 67.4$$

8. Calculator value: $r = 0.85$

9. (i) & (ii)



(iii) Equation of the line of best fit

$y = 1.9x - 16$ (calculator)

(iv) Using the graph, draw a line from $16\frac{1}{2}$ on age axis and read where this cuts the line of best fit off the y axis.

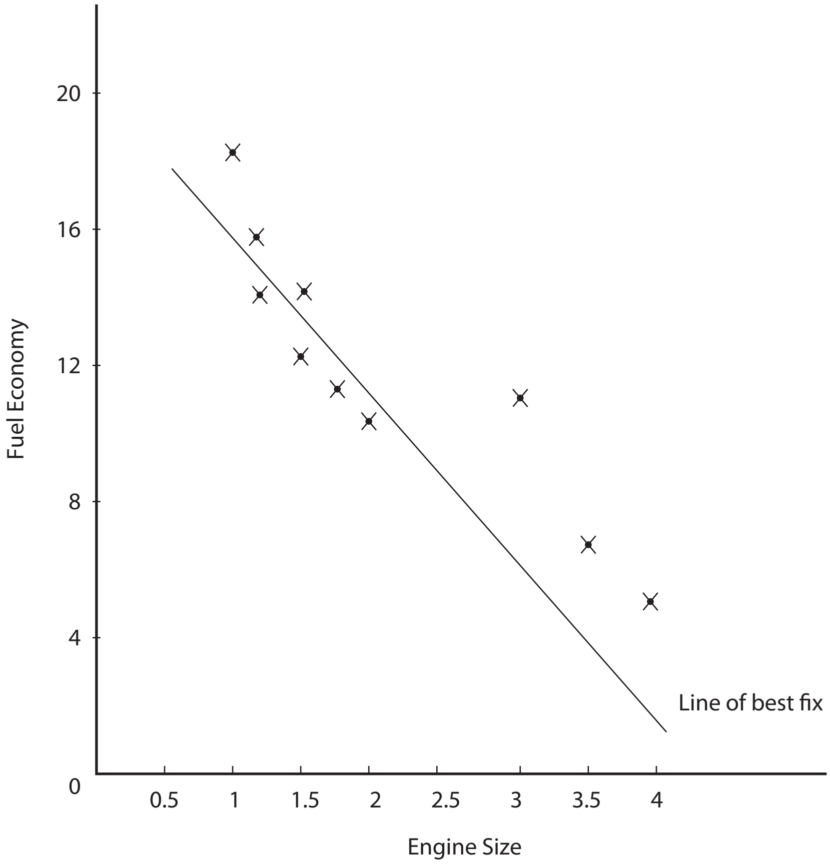
Solution 15.5 approximately.

Alternatively: substituting $x = 16\frac{1}{2}$ in the equation of the line of best fit $y = 1.9x - 16$ gives

$$y = 1.9(16.5) - 16$$
$$= 15.35 \text{ hours}$$

Solution = 15 hours (approx)

10.



(ii) Strong negative correlation

(iii) $r = -0.9250$ (Calculator)

(iv) Line of best fit

$y = -3x + 18$ (Calculator)

(Line of best fit is shown in diagram for part (i))

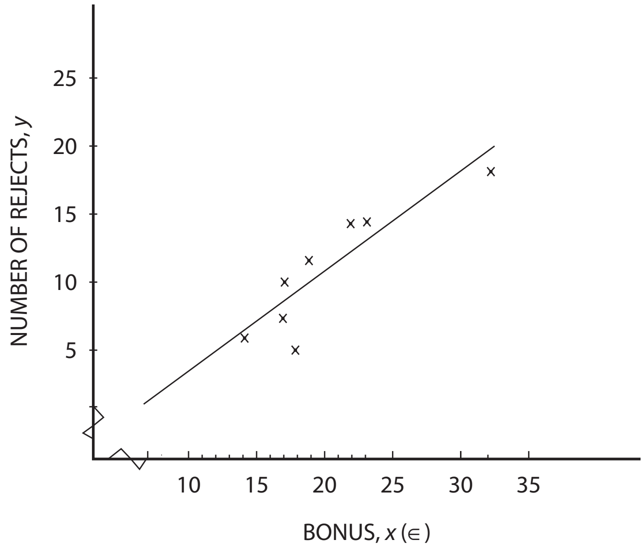
(v) Solution can be read from the graph showing the line of best fit or by substituting into the equation of the line of best fit.

Substitute engine size 5.7 litres

$$y = -3(5.7) + 18$$
$$= -17.1 + 18$$
$$= 0.9$$

This result shows the fuel economy of value less than 1, so this may not be reliable.

11. (i)



(ii) Fairly strong positive correlation

(iii) $r = 0.8591$ (calculator)

(iv) Using (12, 5) and (30, 16) from the line of best fit the slope $m = \frac{16-5}{30-12} = 0.61$

Eq. line is $y - 5 = 0.61(x - 12)$

$\therefore \quad y = 0.61 - 2.3$

Alternatively:

$y = 0.63x - 2.2$ (calculator)

(v) $y = 0.63x - 2.2$

$9 + 2.2 = 0.63x$

$11.2 = 0.63x$

$17.8 = x$

$\therefore$ max bonus should be set at €18 approx.

Exercise 9.3

1. (i) The percentage of all the values in the shaded area is 68%, as it is a characteristic of a normal distribution that 68% lie within one standard deviation of the mean.

(ii) Again, according to the Empirical Rule, 95% of values lie within two standard deviations of the mean.

(iii) Values between $-\sigma$ and $0 = \frac{1}{2}(68\%)$

Values between 0 and $2\sigma = \frac{1}{2}(95\%)$

$\therefore 34\% + 47\frac{1}{2}\% = 81\frac{1}{2}\%$

$\therefore 81\frac{1}{2}\%$ of values lie between $-\sigma$ and 2σ

(iv) $\mu = 60, \quad \sigma = 4$

$56 = 60 - 4 = \mu - \sigma$

$64 = 60 + 4 = \mu + \sigma$

There are 68% of all values in the range $[\mu - \sigma$ and $\mu + \sigma]$

$\therefore$ 68% of values lie between 56 and 64

2. $\mu = 72, \quad \sigma = 6$

$60 = 72 - 12 = \mu - 2\sigma$

$78 = 72 + 6 = \mu + \sigma$

(i) There are $\frac{1}{2}(68\%)$ of values in the range 72 to 78

$\therefore$ 34% of teenagers are that height.

(ii) The percentage of teenagers taller than 78 cm is

$50\% - 34\% = 16\%$

(iii) $\mu = 72, \quad \sigma = 6$

$60 = 72 - 12 = \mu - 2\sigma = \frac{1}{2}(95\%)$

$78 = 72 + 6 = \mu + \sigma = \frac{1}{2}(68\%)$

$\therefore 47\frac{1}{2}\% + 34\% = 81\frac{1}{2}\%$

$\therefore 81\frac{1}{2}\%$ of teenagers are between 60 cm and 78 cm in height

3. (i) $\mu = 55$ km/h, $\quad \sigma = 9$ km/h

given z-score $= -1$

$$\frac{x - 55}{9} = -1$$

$x - 55 = -9$

$x = 55 - 9$

$= 46$ km/h

(ii) Two standard deviations above the mean

$\Rightarrow$ z-score $= 2$

$\therefore \quad \frac{x - 55}{9} = 2$

$\therefore \quad x - 55 = 18$

$\therefore \quad x = 55 + 18$

$= 73$ km/h

(iii) Three standard deviations above the mean

$\Rightarrow \quad \frac{x - 55}{9} = 3$

$\therefore \quad x - 55 = 27$

$x = 55 + 27$

$= 82$ km/h

4. (i) $\mu = 60 \quad \sigma = 5$

Using z-score $= \frac{x - \mu}{\sigma}$

$\pm 1 = \frac{x - 60}{5}$

$\therefore \quad x - 60 = \pm 5(1)$

$\therefore \quad x - 60 = 5$ or $x - 60 = -5$

$\Rightarrow \quad x = 65 \qquad \Rightarrow \quad x = 60 - 5$

$= 55$

Hence, the range within which 68% of the distribution lies is

$55 < x < 65$

(ii) 95% will lie between ± 2σ of the mean

$\pm 2 = \frac{x-60}{5}$

$\therefore\ x - 60 = \pm 10$

$\therefore\ x - 60 = 10$ or $x - 60 = -10$

$\therefore\ x = 70$ or $\therefore\ x = 50$

$\therefore$ the range within 95% of the distribution lies is

$50 < x < 70$

5. (i) 68% of the sample will lie between $\pm 1\sigma$ of the mean

$\mu = 170,\quad \sigma = 8$

$z\text{-score} = \frac{x-\mu}{\sigma}$

$= \frac{x-170}{8} = \pm 1$

$\therefore\ x - 170 = \pm 8$

$\therefore\ x = 170 + 8$ or $x = 170 - 8$

$= 178$ or $x = 162$

$\therefore$ the limits within which 68% of the heights lie are [162, 178] cm.

(ii) 99.7% of the sample will lie between $\pm 3\sigma$ of the mean

Using z-score

$\frac{x-170}{8} = \pm 3$

$\therefore\ x - 170 = \pm 24$

$\therefore\ x - 170 = 24$ or $x - 170 = -24$

$\therefore\ x = 170 + 24$ $\quad x = 170 - 24$

$\therefore\ x = 194$ $\quad x = 146$

$\therefore$ 99.7% of the heights lie within the limits [146, 194] cm

6. (i) $35 - 23 = 12 \Rightarrow 2\sigma$ below the mean

$47 - 35 = 12 \Rightarrow 2\sigma$ above the mean

There are 95% of all values in the range $[\mu - 2\sigma$ and $\mu + 2\sigma]$

$\therefore$ 95% of all workers take 23 to 47 minutes to get to work.

(ii) Since approximately 47.5% of time values lie within μ plus two standard deviations of the mean

$\therefore\ 50\% - 47\frac{1}{2}\%$

$\Rightarrow$ approx 2.5% lie above 47 minutes

$\therefore$ Approx 2.5% of workers take more than 47 minutes to get to work.

(iii) 95% take 23 to 47 minutes to get to work

With 600 workers:

$\therefore\ 600 \times 95 = 570$ workers

7. (i) 68% of bulbs tested lie within $\pm 1\sigma$ of the mean

$\Rightarrow$ 68% of 12 000 = 8160 bulbs

$\therefore$ the lifetime of 8160 bulbs lie within one standard deviation of the mean.

(ii) $\mu = 620$ hrs $\quad \sigma = 12$ hours

$644 = \mu + 2\sigma$

$\therefore\ 47\frac{1}{2}\%$ of bulbs tested would lie in this range 620 to 644.

$\therefore\ 12\,000 \times 47\frac{1}{2}\% = 5700$ bulbs

(iii) $50\% - 47\frac{1}{2}\% = 2.5\%$ lie more than two standard deviations above the mean $2\frac{1}{2}\%$ of

$12\,000 = 300$ bulbs

8. $\mu = 134$ cm $\qquad \sigma = 3$ cm

Balls with rebound less than 128 cm rejected

The range 128 cm to 134 cm

is $134 - 2\sigma$ i.e. $\mu - 2\sigma$

$\therefore \frac{1}{2}(95\%)$ of balls lie in this range and are accepted

$\therefore 47\frac{1}{2}\%$ are accepted

$\therefore 50\% - 47\frac{1}{2}\% = 2\frac{1}{2}\%$ of balls are rejected

$2\frac{1}{2}\%$ of $1000 = 25$ balls

9. (i) The range 140 g to 180 g is

(a) $160 \pm 2\sigma$

$\therefore$ 95% of the portions have weights between 140 g and 180 g

(b) The range 130 g to 190 g is

$160 \pm 3\sigma$

$\therefore$ 99.7% of the weights lie in this range

(ii) The number of portions excepted to weigh between 140 g and 190 g is

$160 - 2\sigma \qquad$ to $\qquad 160 + 3\sigma$

$\therefore 47\frac{1}{2}\% \qquad + \qquad 49.75\%$

$= 97\frac{1}{4}\%$

Of a box with 100 portions approx 97 are expected to be of this weight

10. (i) $x = 84, \quad \mu = 80, \quad \sigma = 4$

$$z\text{-score} = \frac{x - \mu}{\sigma}$$

$$= \frac{84 - 80}{4} = 1$$

(ii) $x = 72, \quad \mu = 80, \quad \sigma = 4$

$$z\text{-score} = \frac{72 - 80}{4} = -2$$

(iii) $x = 86, \quad \mu = 80, \quad \sigma = 4$

$$z\text{-score} = \frac{86 - 80}{4} = 1.5$$

(iv) $x = 70, \quad \mu = 80, \quad \sigma = 4$

$$z\text{-score} = \frac{70 - 80}{4} = -2.5$$

11. (i) A *z*-score of 2 means a value which lies 2 standard deviations above the mean.

(ii) A *z*-score of -1.5 means a value which lies $1\frac{1}{2}$ standard deviations below the mean.

12. (i) Karl's mark is 1.8 standard deviations above the mean which was 70 marks.

Tanya's mark is 0.6 standard deviations below that same mean of 70 marks.

(ii) Karl's *z*-score $= 1.8$, his mark, *x*,

$\mu = 70$ marks $\quad \sigma = 15$ marks

$$\text{Using } z = \frac{x - \mu}{\sigma}$$

$$1.8 = \frac{x - 70}{15}$$

$\therefore x - 70 = 15(1.8)$

$\therefore \quad x = 70 + 27$

$= 97$ marks

Tanya's z-score is -0.6, her mark is x, $\mu = 70$ marks, $\sigma = 15$ marks

$$\therefore \; -0.6 = \frac{x - 70}{15}$$

$\therefore x - 70 = 15(-0.6)$

$\therefore x = 70 - 9$

$= 61$ marks

13. Weight:

$x = 48$ kg, $\quad \mu = 44$ kg, $\quad \sigma = 8$ kg

Use z-score formula $z = \dfrac{x - \mu}{\sigma}$

$$\therefore z = \frac{48 - 44}{8} = 0.5$$

Height: $\quad x = 160$ cm, $\quad \mu = 175$ cm, $\quad \sigma = 10$ cm

$$z = \frac{160 - 175}{10} = \frac{-15}{10} = -1.5$$

14. (i) Anna's score for **Maths**

Mark(x) $= 80$, $\quad \mu = 75$ mark, $\quad \sigma = 12$ mark

$$z\text{-score} = \frac{80 - 75}{12} = \frac{5}{12} = 0.417$$

$\therefore$ maths z-score $= 0.417$

Anna's score for **History**

mark $= 70$, $\quad \mu = 78$, $\quad \sigma = 10$

$$z\text{-score} = \frac{70 - 78}{10} = \frac{-8}{10} = -0.8$$

$\therefore$ Anna's history z-score $= -0.8$

(ii) Anna performed best in maths as she is found to have a higher z-score in the subject.

(iii) Ciara's history z-score $= 0.5$

$$\therefore \quad 0.5 = \frac{x - 78}{10}$$

$\therefore \; x - 78 = 10(0.5)$

$\therefore \quad x = 78 + 5$

$= 83$

$\therefore$ Ciara got 83 marks in history

15. (i) A z-score of 1.8 in a maths test means that Sarah-Jane's mark was 1.8 standard deviations above the mean.

(ii) $x = 80$, $\quad \sigma = 12$, $\quad$ find μ

Using z-score

$$1.8 = \frac{80 - \mu}{12}$$

$\therefore \; 80 - \mu = 12(1.8)$

$\therefore \; 80 - \mu = 21.6$

$\therefore \quad -\mu = -80 + 21.6$

$-\mu = -58.4$

$\therefore \mu = 58.4 =$ mean

(iii) Senan scores 50 in the same test

i.e. $\quad x = 50$, $\quad \mu = 58.4$, $\quad \sigma = 12$

$$z\text{-score} = \frac{50 - 58.4}{12} = -0.7$$

$\therefore \quad$ Senan's z-score $= -0.7$

16. Paper 1:

(i) Sarah's French mark = 59, $\mu = 45$, $\sigma = 8$

z-score $= \frac{59 - 45}{8} = \frac{14}{8} = 1.75$

(ii) To do equally well on Paper 2, Sarah would need a z-score of 1.75

Paper 2:

marks $= x$, $\mu = 56$, $\sigma = 12$

z-score = 1.75

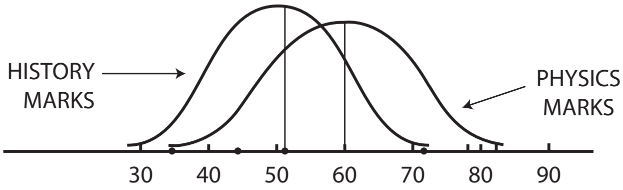
$\therefore \quad 1.75 = \frac{x - 56}{12}$

$\therefore x - 56 = 12(1.75)$

$\therefore x = 56 + 21$

$\therefore x = 77$ marks

17. (i)



HISTORY: 34 − 70

PHYSICS: 36 − 84

(ii) Kelly: History

$x = 64$, $\mu = 42$, $\sigma = 6$

z-score $= \frac{64 - 52}{6} = \frac{12}{6} = 2$

Kelly: Physics

$x = 72$, $\mu = 60$, $\sigma = 8$

z-score $= \frac{72 - 60}{8} = \frac{12}{8} = 1.5$

So yes, Kelly did better in history so her claim to be better at history is supported.

18. Beach 1: $\mu = 8$ mm, $\sigma = 1.4$ mm

z-score when $x = 10$ mm long

$z = \frac{10 - 8}{1.4} = \frac{2}{1.4} = 1.428$

$\therefore z$-score = 1.43

Beach 2: $\mu = 9$ mm, $\sigma = 0.8$ mm

z-score when $x = 10$ mm long

$z = \frac{10 - 9}{0.8} = \frac{1}{0.8} = 1.25$

$\therefore z$-score = 1.25

$\therefore$ it can be concluded that Alison's claim is correct.

Exercise 9.4

1. (i) $P(Z \leqslant 1.2) = 0.8849$

(ii) $P(Z \geqslant 1) = 1 - P(Z \leqslant 1)$

$= 1 - 0.8413$

$= 0.1587$

(iii) $P(Z \leqslant -1.92)$

$= 1 - P(Z \geqslant 1.92)$

(because the curve is symmetrical, we find the area to the left of 1.92)

$\therefore P(Z \leqslant -1.92) = 1 - P(Z \geqslant 1.92)$

$= 1 - 0.9726$

$= 0.0274$

(iv) $P(-1.8 \leqslant Z \leqslant 1.8)$

$$\begin{aligned} \text{Area to the left of } 1.8 &= 0.9641 \\ \text{Area to the right of } -1.8 &= 1 - P(Z \leqslant 1.8) \\ &= 1 - 0.9641 \\ &= 0.0359 \\ \therefore \text{Area shaded portion is } &\ 0.9641 - 0.0359 \\ &= 0.9282 \end{aligned}$$

2. $P(Z \leqslant 1.42) = 0.9222$

3. $P(Z \leqslant 0.89) = 0.8133$

4. $P(Z \leqslant 2.04) = 0.9793$

5. $P(Z \geqslant 2) = 0.9722$

6. $P(Z \geqslant 1.25) = 0.8944$

7. $P(Z \geqslant 0.75) = 0.7723$

8. $P(Z \leqslant -2.3)$

Use the fact the curve is symmetrical

$$\begin{aligned} \therefore P(Z \leqslant -2.3) &= 1 - P(Z \geqslant 2.3) \\ &= 1 - 0.9893 \\ &= 0.0107 \end{aligned}$$

9. $$\begin{aligned} P(Z \leqslant -1.3) &= 1 - P(Z \geqslant 1.3) \\ &= 1 - 0.9032 \\ &= 0.0968 \end{aligned}$$

10. $P(Z \leqslant -2.13) = \text{left tail}$

$$\begin{aligned} \therefore P(Z \leqslant -2.13) &= 1 - P(Z \geqslant 2.13) \\ &= 1 - 0.9834 \\ &= 0.0166 \end{aligned}$$

11. $P(Z \leqslant 0.56) = 0.7123$

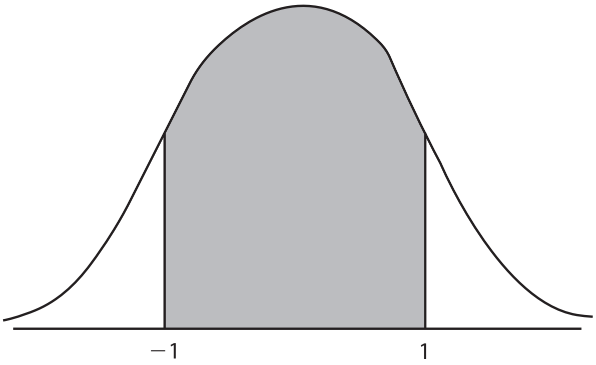
12. $P(-1 \leqslant Z \leqslant 1)$

(i) Area to left of $1 = 0.8413$

(ii) Area to right of $-1 = 1 - 0.8413 = 0.1587$

Then subtract (ii) from (i)

$$\begin{aligned} \text{Shaded area} &= 0.8413 - 0.1587 \\ &= 0.6833 \end{aligned}$$



13. $P(-1.5 \leqslant Z \leqslant 1.5)$

Area to left of $1.5 = 0.9332$

$$\begin{aligned} \text{Area to right of } -1.5 &= 1 - P(Z \leqslant 1.5) \\ &= 1 - 0.9332 \\ &= 0.0668 \end{aligned}$$

$$\begin{aligned} \therefore \text{shaded portion} &= 0.9332 - 0.0668 \\ &= 0.8664 \end{aligned}$$

14. $P(0.8 \leqslant Z \leqslant 2.2)$

$$\begin{aligned} \text{Area to left of } 2.2 &= 0.9861 \\ \text{Area to right of } 0.8 &= 0.7881 \\ \therefore \text{Area (ii)} - \text{Area (i)} &= 0.9861 - 0.7881 \\ &= 0.1980 \end{aligned}$$

15. $P(-1.8 \leqslant Z \leqslant 2.3)$

$$\begin{aligned} \text{Area to left of } 2.3 &= 0.9893 \\ \text{Area to right of } -1.8 &= 1 - P(Z \leqslant 1.8) \\ &= 1 - 0.9641 \\ &= 0.0359 \\ \therefore \text{Area (ii)} - \text{Area (i)} &= 0.9893 - 0.0359 \\ &= 0.9534 \end{aligned}$$

16. $P(-0.83 \leqslant Z \leqslant 1.4)$

$$\begin{aligned} \text{Area to left of } 1.4 &= 0.9192 \\ \text{Area to right of } -0.83 &= 1 - P(Z \leqslant 0.83) \\ &= 1 - 0.7967 \\ \therefore \quad &= 0.2033 \\ \therefore \quad 0.9192 - 0.2033 &= 0.7159 \end{aligned}$$

17. $P(Z \leqslant z_1) = 0.8686$

$\therefore z_1 = 1.12$

18. $P(Z \leqslant z_1) = 0.6331$

$\therefore z_1 = 0.34$

19. $P(-z_1 \leqslant Z \leqslant z_1) = 0.6368$

$z_1 = 0.91$

20. $P(-z_1 \leqslant Z \leqslant z_1) = 0.8438$

$\therefore z_1 = 1.42$

21. $\mu = 50 \qquad \sigma = 10$

(i) $P(X \leqslant 60)$

$$z\text{-score} = \frac{60 - 50}{10} = \frac{10}{10} = 1$$

$\therefore P(X \leqslant 1) = 0.8413$

(ii) $P(X \leqslant 55)$

$$z\text{-score} = \frac{55 - 50}{10} = \frac{5}{10} = 0.5$$

$\therefore P(X \leqslant 0.5) = 0.6915$

(iii) $P(X \geqslant 45)$

$$z\text{-score} = \frac{45 - 50}{10} = \frac{-5}{10} = -\frac{1}{2}$$

$\therefore P(X \geqslant -0.5) = 0.6915$

22. (i) $z\text{-score} = \dfrac{60 - 55}{25} = \dfrac{-6}{25}$

$\therefore P(X \geqslant -0.24)$

$= 0.5948$

(ii) $P(X \leqslant 312)$

$$z\text{-score} = \frac{312 - 300}{25} = \frac{12}{25}$$

$= 0.48$

$\therefore P(X \leqslant 0.48) = 0.6844$

23. (i) $\mu = 250, \qquad \sigma = 40$

$P(X \geqslant 300)$

$z\text{-score} = \frac{300 - 250}{40} = \frac{50}{40}$

$\therefore P(X \geqslant 1.25)$

$= 1 - 0.8944$

$= 0.1056$

(ii) $P(X \leqslant 175)$

$z\text{-score} = \frac{175 - 250}{40} = \frac{75}{40}$

$\therefore P(X \leqslant 1.875)$

$= 1 - 0.9699$

$= 0.0301$

24. (i) $\mu = 50 \qquad \sigma = 8$

$P(52 \leqslant X \leqslant 55)$

$z\text{-score} = \frac{52 - 50}{8} = \frac{2}{8} = 0.25$

$z\text{-score} = \frac{55 - 50}{8} = \frac{5}{8} = 0.625$

$\therefore P(0.25 \leqslant X \leqslant 0.625)$

$= 0.7357 - 0.5987$

$= 0.1370$

(ii) $P(48 \leqslant X \leqslant 54)$

$z\text{-score} = \frac{48 - 50}{8} = \frac{-2}{8} = -0.25$

$z\text{-score} = \frac{54 - 50}{8} = \frac{4}{8} = 0.5$

$\therefore P(-0.25 \leqslant X \leqslant 0.5)$

$P(X \leqslant 0.5) = 0.6915$

$P(-0.25 \leqslant X) = 1 - P(X \leqslant 0.25)$

$= 1 - 0.5987$

$= 0.4013$

$\therefore P(-0.25 \leqslant X \leqslant 0.5) = 0.6915 - 0.4013$

$= 0.2902$

25. $\mu = 100, \qquad \sigma = 80$

(i) $P(85 \leqslant X \leqslant 112)$

$z\text{-score} = \frac{85 - 100}{80} = \frac{-15}{80} = -0.1875$

$z\text{-score} = \frac{112 - 100}{80} = \frac{12}{80} = 0.15$

$\therefore P(-0.1875 \leqslant X \leqslant 0.15)$

$= P(X \leqslant 0.15) = 0.5596$

$P(-0.1875 \leqslant X) = 1 - P(X \leqslant 0.1875)$

$= 1 - 0.5753$

$= 0.4247$

$\therefore P(85 \leqslant X \leqslant 112) = 0.5596 - 0.4247$

$= 0.1349$

(ii) $P(105 \leqslant X \leqslant 115)$

$$z\text{-score} = \frac{105 - 100}{80} = \frac{5}{80} = 0.0625$$

$$z\text{-score} = \frac{115 - 100}{80} = \frac{15}{80} = 0.1875$$

$\therefore P = P(0.0625 \leqslant X \leqslant 0.1875)$

$\therefore P(105 \leqslant X \leqslant 115) = P(0.0625 \leqslant X \leqslant 0.1875)$

$= 0.5753 - (0.5239)$

$= 0.0514$

26. $\mu = 200 \qquad \sigma = 20$

(i) $P(190 \leqslant X \leqslant 210)$

$$z\text{-score} = \frac{190 - 200}{20} = \frac{-10}{20} = -0.5$$

$$z\text{-score} = \frac{210 - 200}{20} = \frac{10}{20} = 0.5$$

$\therefore P(-0.5 \leqslant X \leqslant 0.5)$

$= 0.6915 - (1 - 0.6915)$

$= 0.6915 - 0.3085$

$= 0.3830$

(ii) $P(185 \leqslant X \leqslant 205)$

$$z\text{-score} = \frac{185 - 200}{20} = \frac{15}{20} = -0.75$$

$$z\text{-score} = \frac{205 - 200}{20} = \frac{5}{20} = 0.25$$

$\therefore P = P(-0.75 \leqslant X \leqslant 0.25)$

$= 0.5987 - (1 - 0.7734)$

$= 0.5987 - 0.2266$

$= 0.3721$

27. (i) $x = 240, \qquad \mu = 210, \qquad \sigma = 20$

$$z\text{-score} = \frac{240 - 210}{20} = \frac{30}{20} = 1.5$$

$P(x > 240) = P(z > 1.5)$

$= 1 - 0.9332$

$= 0.0668$

(ii) $P(\text{bulb last} \leqslant 200 \text{ hrs})$

$$z\text{-score} = \frac{200 - 210}{20} = \frac{-10}{20} = -0.5$$

$\therefore P(z \leqslant -0.5)$

$= 1 - 0.6915$

$= 0.3085$

28. (i) $\mu = 101$ cm, $\qquad \sigma = 5$ cm, $\qquad x = 103$ cm

$P(\text{customer has chest measurement} < 103 \text{ cm})$

= writing expression in z-scores

$$z\text{-score} = \frac{103 - 101}{5} = \frac{2}{5} = 0.4$$

$\therefore P = P(z < 0.4)$

$= 0.6554$

$\therefore P(\text{chest} < 103 \text{ cm}) = 0.6554$

(ii) $P(\text{chest size} \geqslant 98 \text{ cm})$

$= z\text{-score of } \dfrac{98 - 101}{5} = -\dfrac{3}{5}$

$= -0.6$

$\therefore P(z \geqslant -0.6)$

$= 0.7257$

(iii) $P(\text{chest measurement between 95 cm and 100 cm})$

$= z\text{-score of } \dfrac{95 - 101}{5} \text{ and } \dfrac{100 - 101}{5}$

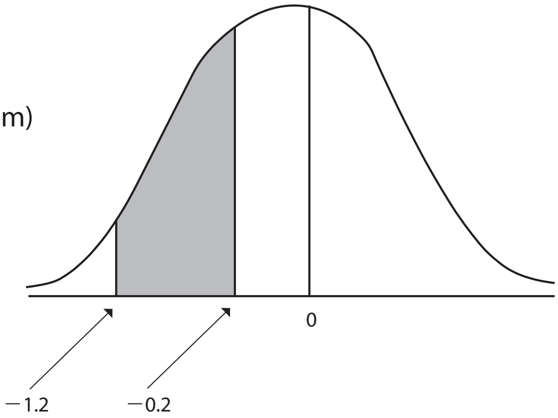
$\therefore z = \dfrac{-6}{5} \text{ and } z = \dfrac{-1}{5}$

$= -1.2 \text{ and } z = -0.2$

$\therefore P(-1.2 \leqslant z \leqslant -0.2)$

$\therefore 0.8849 - 0.5793$

$= 0.3056$



29. (i) $\mu = 12$ $\sigma = 2$

$P(\text{postman takes longer than 17 mins})$ is changed to z-scores

$z\text{-score} = \dfrac{17 - 12}{2} = \dfrac{5}{2} = 2\dfrac{1}{2}$

$\therefore P(z > 2.5) = 1 - P(z < 2.5)$

$= 1 - 0.9938$

$= 0.0062$

(ii) $P(\text{taking less than 10 mins})$

$z\text{-score} = \dfrac{10 - 12}{2} = \dfrac{-2}{2} = -1$

$\therefore P = P(z < -1) = 1 - 0.8413$

$= 0.1587$

(iii) $P(\text{taking between 9 and 13 mins})$

1st get $P(\text{taking 9 mins})$ and

then get $P(\text{taking 13 mins})$

$z\text{-score} = \dfrac{9 - 12}{2} = \dfrac{-3}{2} = -1.5$

$z\text{-score} = \dfrac{13 - 12}{2} = 0.5$

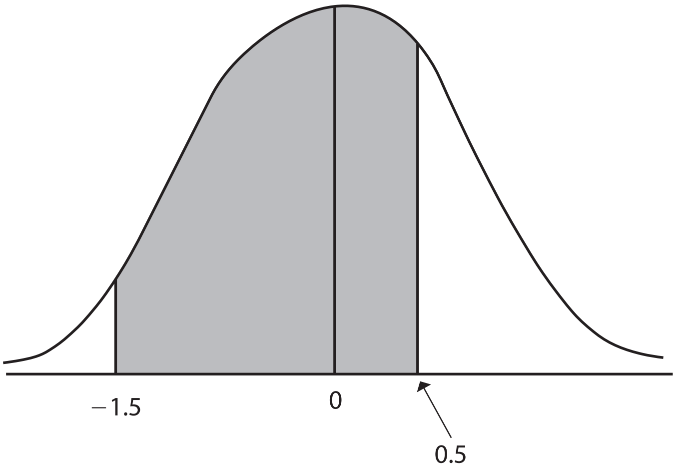
$\therefore P(\text{between 9 and 13 mins})$

$= P(-1.5 \leqslant z \leqslant 0.5)$

$= 0.9332 - (1 - 0.6915)$

$= 0.9332 - 0.3085$

$= 0.6247$



30. $\mu = 53$ $\sigma = 15$

To find $P(\text{bill between €47 and €74})$:

$z\text{-score} = \dfrac{47 - 53}{15} = \dfrac{-6}{15} = -0.4$

$z\text{-score} = \dfrac{74 - 53}{15} = \dfrac{21}{15} = 1.4$

$\therefore P(\text{bill between €47 and €74})$:

$= P(-0.4 \leqslant z \leqslant 1.4)$

$= 0.6554 - (1 - 0.9192)$

$= 0.6554 - 0.0808$

$= 0.5746$

31. (i) $\mu = 165 \qquad \sigma = 3.5$ cm

P(a student is less than 160 cm high)

$z\text{-score} = \dfrac{160 - 165}{3.5} = \dfrac{-5}{3.5} = -1.428$

$\therefore P = P(x < 160 \text{ cm})$

$= P(z < 1.428)$

$= 1 - 0.9236$

$= 0.0764$

$\therefore P$(a student is less than 160 cm high)

$= 0.0764$

(ii) P(student with height between 168 cm and 174 cm)

$z\text{-score:} \dfrac{168 - 165}{3.5} = \dfrac{3}{3.5} = 0.857$

$z\text{-score:} \dfrac{174 - 165}{3.5} = \dfrac{9}{3.5} = 2.571$

$\therefore P$(a student with height between 168 cm and 174 cm)

$= P(0.857 \leqslant z \leqslant 2.571)$

$= 1 - 0.8051 - 0.0051$

$= 0.1949 - 0.0051$

$= 0.1898 = 18.98\%$

= approx. 19% of student from this group would satisfy the condition of having a height between 168 cm and 174 cm.

32. Given:

$x = 500, \quad \mu = 151 \text{ mm}, \quad \sigma = 15 \text{ mm}$

(i) P(having leaves greater than 185 mm long)

$z\text{-score:} = \dfrac{185 - 151}{15} = \dfrac{34}{15} = 2.266$

$\therefore P(z > 2.266)$

$= 1 - 0.9881$

$= 0.0119$

Of the 500 laurel leaves, then

500×0.0119

$= 5.95$

$= 6$ leaves

measure greater than 185 mm long.

(ii) z-score for a leaf 120 mm long

$= \dfrac{120 - 151}{15} = -2.066$

z-score for a leaf 155 mm long

$= \dfrac{155 - 151}{15} = 0.26$

$\therefore P$(leaves between 120 and 155 mm)

is $P(-2.06 \leqslant z \leqslant 0.26)$

$= 0.9808 - 0.3936$

$= 0.5872.$

Of the 500 leaves, then 500×0.5872 leaves have lengths between 120 mm and 155 mm.

$= 500 \times 0.5872$

$= 293.6$ leaves

$= 294$ leaves

33. Given: $\mu = 300$ grams, $\sigma = 60$ grams

(i) P(weight less than 295 grams) shows

$$z\text{-score} = \frac{295 - 300}{6} = \frac{-5}{6} = -0.833$$

$\therefore P(z < -0.833)$

$= 1 - P(z > 0.833)$

$= 1 - 0.7967$

$= 0.2033$

Out of the 1000 packages then 1000×0.2033 weigh less than 295 grams

(ii) To find the number of packages between 306 and 310 grams, write the weights in z-scores.

$$z\text{-scores: } \frac{306 - 300}{6} = \frac{6}{6} = 0.1$$

$$z\text{-score: } \frac{310 - 300}{6} = \frac{10}{6} = 1.66$$

$\therefore P$(a packet of weight between 306 and 310 grams)

$= P(1 \leqslant z \leqslant 1.66)$

$= (1 - 0.8413) - (1 - 0.9527)$

$= 0.1587 - 0.0475$

$= 0.1112$

$\therefore 1000 \times 0.1112$

$= 111$ packets weight between 306 and 310 grams.

34. (i) $\mu = 60\%$ $\sigma = 10\%$

(a) P(mark less than 45%)

has z-score

$$= \frac{45 - 60}{10} = -\frac{15}{10} = -1.5$$

$P(z < -1.5)$

$= 1 - P(z > 1.5)$

$= 1 - 0.9332$

$= 0.0668$

(b) P(mark is between 50% and 75%)

has z-score

$$\frac{50 - 60}{10} = -\frac{10}{10} = -1$$

$$\frac{75 - 60}{10} = \frac{15}{10} = 1.5$$

$\therefore P(-1 \leqslant z \leqslant 1.5)$

$= 0.8413 - (1 - 0.9332)$

$= 0.8413 - 0.668$

$= 0.7745$

$\therefore P$(a randomly selected student scored between 50% and 75% in Geography)

$= 0.7745$ ($= 77.45\%$)

(ii) P(attaining more than 90%) will give a special award.

Let x be the number of students attaining more than 90% so

$\therefore z$-score

$$= \frac{x - 60}{10} = 0.9$$

From the tables, a z-score of 0.900 is given by 1.29,

i.e. 0.9015

$$\therefore \frac{x - 60}{10} = 1.29$$

$\therefore x - 60 = 10(1.29)$
$\therefore x - 60 = 12.90$
$\therefore x = 72.9\%$
$= 73\%$

$\therefore$ the percentage mark student need in order to get a special award is more than 73% in Geography.

Revision Exercise 9 (Core)

1. On left-hand side of 0,

between -2σ and 0 there is $\frac{1}{2}(95\%) = 47.5\%$

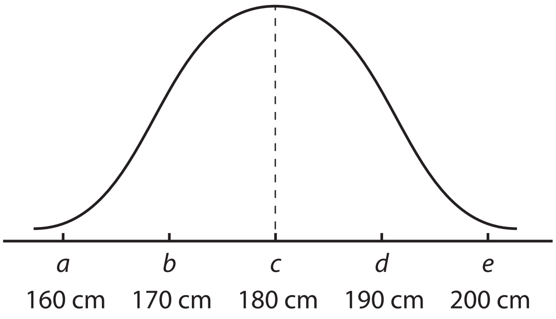
Between 0 and 1σ, there is $\frac{1}{2}(68\%) = 34\%$

$\therefore$ shaded region under curve $= 81.5\%$

2. (i) *B* – positive correlation
(ii) *A* – negative correlation
(iii) *C* – no correlation
(iv) *A* – negative correlation
(v) *B* – correlation coefficient of approx 0.7

3. $\mu = 180$ cm $\sigma = 10$ cm

(i)

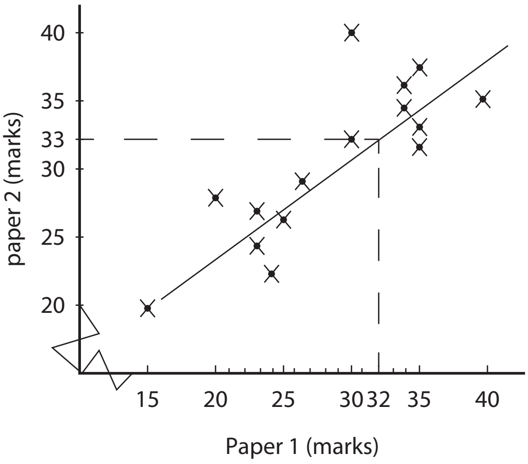


(ii) $z\text{-score} = \frac{190 - 180}{10} = \frac{10}{10} = 1$

$\therefore z = 1$

(iii) 34% of sample have height between 180 and 190.
$\therefore 50\% - 34\% = 16\%$
$\therefore$ 16% have height greater than 190 cm

4. (i)



(ii) Strong positive
(iii) Line on graph

(iv) Taking two points on the line of best fit

(25, 27.5) (40, 37.5)

$$\text{slope } m = \frac{37.5 - 27.5}{40 - 25} = \frac{10}{15} = 0.666$$

$$\Rightarrow m = 0.7$$

Eq. of line

$y - 27.5 = 0.7(x - 25)$

$y - 27.5 = 0.7x - 17.5$

$\therefore\ y = 0.7x + 10$

Using calculator line of best fit is

$y = 0.713x + 9.74$

(v) Drawing in the line from $x = 32$

on the graph gives $y =$ approx 33.

or

Substituting $x = 32$ into the equation of the line of best fit

$$\begin{aligned} x &= 32 \\ \therefore y &= 0.713(32) + 9.74 \\ &= 22.816 + 9.74 \\ &= 32.556 \end{aligned}$$

$\therefore$ score is 33 marks

5. $\mu = 175$ cm

$x = 160 + 15$ $\quad$ $x = 190 - 15$

$= 175$ $\quad$ $= 175$

$\therefore 160 = 175 - 1\sigma$

$\therefore 190 = 175 + 1\sigma$

Given 95% of students have heights between 160 and 190

i.e. $\mu \pm 2\sigma$

$\therefore 2\sigma = 15$

$\therefore \sigma = 7.5$

6.
$$\begin{aligned} P(Z > 0.93) &= 1 - P(Z \leq 0.93) \\ &= 1 - 0.8238 \\ &= 0.1762 \end{aligned}$$

7. (i) Correlation is a measure of the strength of the linear relationship between two sets of variables.

(ii) (a) $r = 0.916$ (calculator)

(b) It is very likely that a student who has done well in test 1 will also done well in test 2.

8. (i) Since 95% of a sample lies between $\pm 2\sigma$ of the mean, then diagram (i) has a 95% probability that a bamboo cane will have length falling in the shaded area.

(ii) Here in diagram (ii), $\frac{1}{2}$(95%) is shaded so the probability of a bamboo cane having a length falling in the shaded area = 47.5%

9. (i) Simon's French test:

$x = 76$ marks, $\mu = 68$ marks, $\sigma = 10$ marks

$$z\text{-score} = \frac{78 - 68}{10} = \frac{8}{10} = 0.8$$

(ii) Simon's German test:

$x = 78$ marks, $\mu = 70$ marks, $\sigma = 12$ marks

$$z\text{-score} = \frac{78 - 70}{12} = \frac{8}{12} = 0.66$$

(iii) Simon did better in his French test

10. There may be a strong positive correlation between house prices and car sales but that does not imply that one increase **causes** the other.

Revision Exercise 9 (Advanced)

1. $P(-1 \leq Z \leq 1.24)$

$$P(Z \leq 1.24) = 1 - 0.8925$$
$$= 0.1075$$
$$P(-1 \leq Z) = 0.8413$$
$$\therefore P(-1 \leq Z \leq 1.24) = 0.8413 - 0.1075$$
$$= 0.7338$$

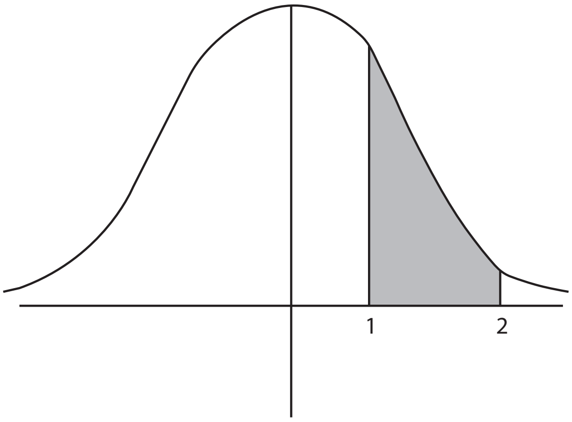
2. (i) $P(1 < Z < 2)$

$P(Z < 2)$ i.e. left side of 2 is 0.9772

$P(Z > 1) = 0.8413$

$\therefore P(1 < Z < 2) = 0.9772 - 0.8413$

$= 0.1359$



3. (i) $x = 60, \quad \mu = 48, \quad \sigma = 8$

$$z\text{-score} = \frac{60 - 48}{8} = \frac{12}{8} = 1\frac{1}{2}$$

$\therefore P(Z > 1.5) = 1 - 0.9332$

$= 0.0668$

(ii) $$z\text{-score} = \frac{35 - 48}{8} = \frac{-13}{8} = -1.625$$

$\therefore P(Z < -1.625)$

$= 1 - 0.9484$

$= 0.0516$

$= 0.052$

4. $P(-k \leq Z \leq k) = 0.8438$

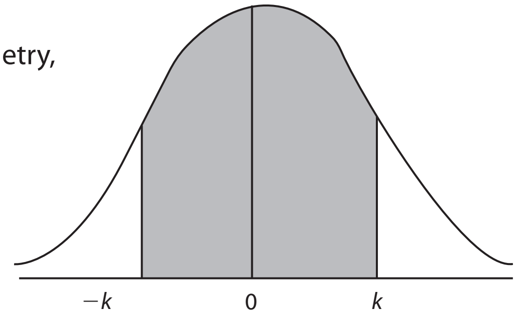
Since this is a normal distribution, and because of symmetry,

$$P(0 < Z \leq k) = \frac{1}{2}(0.8438)$$
$$= 0.4219$$

$\therefore P(-k \leq Z \leq k) = 0.5 + 0.4219$

$= 0.9419$ (formulae & tables p 36 & 37)

$\therefore Z = 1.42$



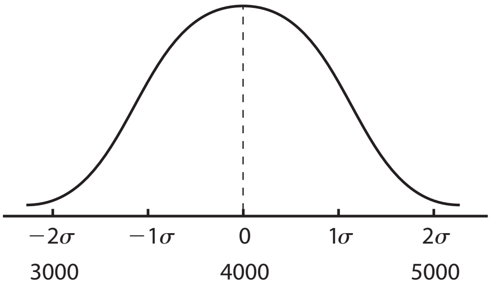
5. (i) $x = 3000$ hours, $\mu = 4000$ hrs, $\sigma = 500$ hrs

$$z\text{-score} = \frac{3000 - 4000}{500} = \frac{-1000}{500} = -2$$

$\therefore \frac{1}{2}(95\%) = 47.5\%$ of bulbs last between 3000 and 4000 hours

$\therefore 50\% - 47.5\%$ last less than 3000 hours

$\therefore 2.5\%$ last less than 3000 hrs



(ii) The probability that a tube will last between 3000 and 5000 hours i.e. $\mu \pm 2\sigma = 0.95$

(iii) $2\frac{1}{2}$% of the tubes will be expected to be working after 5000 hours. In a batch of 10 000 tubes $= 250$

6. (i) $r = 0.959$ (calculator)

(ii) This value show a very strong positive correlation between the number of employees and the units produced.

7. Tree 1:

$x = 7$ cm, $\mu = 5$ cm, $\sigma = 1$ cm

$$z\text{-score} = \frac{7-5}{1} = \frac{2}{1} = 2$$

Tree 2:

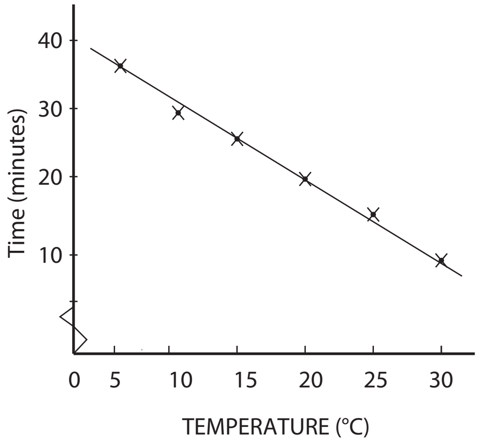
$x = 7$ cm, $\mu = 8$ cm, $\sigma = 1.5$ cm

$$z\text{-score} = \frac{7-8}{1.5} = \frac{-1}{1.5} = -0.666$$

$$= -0.67$$

Mr. Cross is correct since $z = -0.67$ has a greater chance of happening on the normal curve than $z = 2$.

8. (i)



(ii) Strong negative correlation

(iii) Two points on line of best fit are (10, 29) and (30, 8)

$$\text{slope} = \frac{8-29}{30-10} = \frac{-21}{20}$$

$\therefore m = -1.05$

Equation of line is $y = mx + c$

$\therefore\ 29 = -1.05(10) + c$

$29 = -10.5 + c$

$\therefore c = 39.5$

$\therefore y = -1.05x + 39.5$

Using calculator

$y = -1.12x + 41.6$ is the equation of the line of best fit.

(iv) When temp is $= 0$°C

$0 = -1.12x + 41.6$

$$\therefore \quad x = \frac{41.6}{1.12}$$

$= 37.5$

$= 38$ minutes

(v) $r = -1$ (calculator)

9. (i) $\mu = 135$ cm, $x = 120$ cm, $\sigma = 10$ cm

$z\text{-score} = \frac{120 - 135}{10} = \frac{-15}{10} = -1\frac{1}{2}$

$\therefore$ David's height is $-$ 1.5σ below the mean

(ii) $\mu = 180$ cm, $\sigma = 18$ cm, z-score $= -1.5$

$z\text{-score} = \frac{x - \mu}{\sigma}$

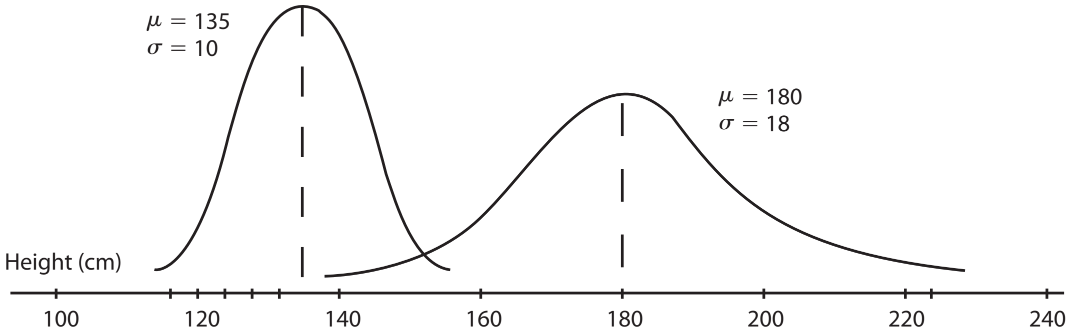
$\therefore -1.5 = \frac{x - 180}{18}$

$\therefore x - 180 = -1.5(18)$

$\therefore x = 180 - 27$

$= 153$ cm tall

(iii)



Revision Exercise 9 (Extended Response)

1. (i) (a) $\mu = 20$ mm, $\sigma = 3$ mm

17 mm $= \mu - 1\sigma$

23 mm $= \mu + 1\sigma$

$\therefore$ 17 $-$ 23 mm $= 20 \pm 1\sigma$

68% of a normal distribution lies within this area (Empirical rule)

(b) 14 mm $= \mu - 2\sigma$

23 mm $= \mu + \sigma$

$\therefore$ 14 mm $= 2\sigma$ below mean $= 47.5\%$

$\therefore$ 23 mm $= 1\sigma$ above mean $= 34\%$

$\therefore$ the percentage of nails measured 14 mm $-$ 23 mm is 47.5% + 34% = 81.5%

(ii) 17 $= 1\sigma$ below mean $= 34\%$

26 $= 2\sigma$ above mean $= 47.5\%$

$\therefore$ 81.5% are of 17 $-$ 26 mm nails.

When 10 000 are measured

$\therefore \frac{81.5}{100} \times 10\,000 = 8150$ nails

(iii) 23 mm $= 1\sigma$ above μ

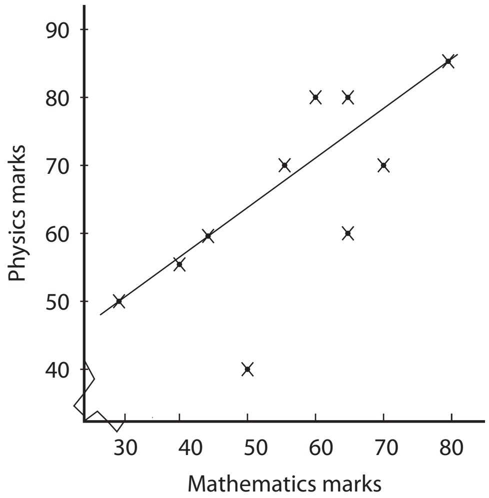
$= 34\%$

50% of all nails are $>$ 20 (mean length)

$\therefore$ 50% $-$ 34% of nails are more than 23 mm long

$= 16\%$

2. (i)



(ii) Equation of line of best fit

$y = 0.7x + 25$ (calculator)

Using two points:

(30, 50) (80, 85)

$$\text{slope} = \frac{85 - 50}{80 - 30} = \frac{35}{50} = 0.7$$

$$y - 50 = 0.7(x - 30)$$
$$y - 50 = 0.7x - 21$$
$$y = 0.7x - 21 + 50$$
$$y = 0.7x + 29$$

(iii) $r = 0.737$ (calculator)

(iv) There is fairly strong positive correlation between the mathematic and physics results of the students.

3. (i) $\mu = 176$ and $\sigma = 7$. Let X be the height of a randomly selected man.

$x = 190$: $z = \frac{190 - 176}{7} = 2$

$$P(X > 190) = P(Z > 2)$$
$$= 1 - P(Z \leqslant 2)$$
$$= 1 - 0.9772$$
$$= 0.0228$$

(ii) $P(\text{both}) = P(\text{1st and 2nd})$

$$= (0.0228)(0.0228)$$
$$= 0.00052$$

(iii) Single Male: success: $p = 0.0228$, $q = 0.9772$

Let X be the number of successes in n trials. Then

$$P(X = r) = \binom{n}{r}(0.0228)^r(0.9772)^{n-r}$$

(iv) Let X be the number of males taller than 190 cm from a group of 10 males.

$$P(X \geqslant 2) = 1 - P(X < 2)$$
$$= 1 - P(X = 0) - P(X = 1)$$
$$= 1 - \binom{10}{0}(0.9772)^{10} - \binom{10}{1}(0.0228)(0.9772)^9$$
$$= 1 - 0.7940 - 0.1853$$
$$= 0.0207$$

4. $\mu = 60$ yrs, $\sigma = 8$ yrs

(i) (a) Abdul z-score:

$$z = \frac{x - \mu}{\sigma} = \frac{70 - 60}{8} = 1.25$$

(b) Marie z-score:

$$\frac{52 - 60}{8} = \frac{-8}{8} = -1$$

(c) George z-score:

$$\frac{60 - 60}{8} = 0$$

(d) Elsie z-score:

$$z = \frac{92 - 60}{8} = 4$$

(ii) $76 = 60 + 16 = \mu + 2\sigma$

$= 47.5\%$

Hence, the percentage of people more than 76 years is

$50\% - 47.5\% = 2.5\%$

(iii) Ezra

$$2.5 = \frac{x - 60}{8}$$

$\therefore x - 60 = 8(2.5)$

$\therefore x - 60 = 20$

$\therefore \quad x = 60 + 20$

$= 80$ years

(iv) $x = 40$ yrs

$$z = \frac{40 - 60}{8} = \frac{-20}{8} = -2.5$$

Since the z-score $= -2.5$ it is very unlikely as the probability will be less than 1%

5. (i) $r = -0.85$ approx

(ii) Outlier: age $= 37$, bpm $= 139$

(iii) Read from x (age value) $= 44$ to cut the line of best fit and read y (bpm value)

Solution (44, 180 bpm)

(iv) Possible points: (20, 200) (80, 150)

$$\text{slope} = \frac{150 - 200}{80 - 20} = \frac{-50}{60} = -0.833$$

$\therefore m = -0.8$

(v) Equation of the line of best fit

$y - y_1 = m(x - x_1)$

$y - 200 = -0.833(x - 20)$

$y - 200 = -0.833x + 20(0.833)$

$y = -0.8x + 16$

$= 200 + 16 - 0.8(\text{age})$

Replacing y with MHR

MHR $= 216 - 0.8$(age)

(vi)

Age	Old rule	New rule
20	200	200
50	170	176
70	150	160

For a younger person (20 years) the MHRs are roughly the same. For an older person (50 years or 70 years) the new rule gives a higher MHR reading.

(vii) At 65 years, the old rule gives MHR $= 155$ and the new rule gives MHR $= 164$. To get more benefit from exercise, he should increase his activity to 75% of 164 instead of 75% of 155.

Chapter 9

Chapter 10

Exercise 10.1

1. (i) The sample proportion, $\hat{p} = \frac{150}{500} = 0.3$

(ii) Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{500}}$

$= 0.04$

(iii) Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$\therefore 0.3 - 0.04 < p < 0.3 + 0.04$

$\therefore 0.26 < p < 0.34$

2. (i) The sample proportion, $\hat{p} = \frac{136}{\sqrt{400}}$

$= 0.34 = 34\%$

$\therefore$ 34% of computer shops are selling below the list price

(ii) Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{400}} = \frac{1}{20} = 0.05$

Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.34 - 0.05 < p < 0.34 + 0.05$

$= 0.29 < p < 0.39$

This means that the interval obtained works for 95% of the time and would give this result.

3. The sample proportion, $\hat{p} = \frac{36\,000}{10\,000} = 0.36$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{10\,000}} = \frac{1}{100} = 0.01$

95%, confidence interval $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.36 - 0.01 < p < 0.36 + 0.01$

$= 0.35 < p < 0.37$

4. The sample proportion, $\hat{p} = \frac{45}{150} = 0.3$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{150}} = 0.082$

Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.3 - 0.082 < p < 0.3 + 0.082$

$= 0.218 < p < 0.382$

5. Sample proportion, $\hat{p} = \frac{57}{80} = 0.713$

$\therefore$ Sample proportion *not* in favour $= 1 - 0.713 = 0.287$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{80}} = 0.111$

Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.287 - 0.111 < p < 0.287 + 0.111$

$= 0.176 < p < 0.398$

or $17.6\% < p < 39.8\%$

6. (i) Margin of error $= \frac{1}{\sqrt{n}}$

$$\frac{1}{\sqrt{n}} = 0.05 \quad \text{since } 5\% = 0.05$$
$$\left(\frac{1}{\sqrt{n}}\right)^2 = (0.05)^2$$
$$\therefore \frac{1}{n} = (0.05)^2$$
$$\therefore n = \frac{1}{(0.05)^2}$$
$$= 400 = \text{sample size}$$

(ii) Margin of error $= \frac{1}{\sqrt{n}}$ $3\% = 0.03$

$$\frac{1}{\sqrt{n}} = 0.03$$
$$\left(\frac{1}{\sqrt{n}}\right)^2 = (0.03)^2$$
$$\therefore \frac{1}{n} = (0.03)^2$$
$$\therefore n = \frac{1}{(0.03)^2}$$
$$= 1111 = \text{sample size}$$

(iii) Margin of error $= \frac{1}{\sqrt{n}}$, $1.5 = 0.015$

$$\frac{1}{\sqrt{n}} = 0.015$$
$$\left(\frac{1}{\sqrt{n}}\right)^2 = (0.015)^2$$
$$\therefore \frac{1}{n} = (0.015)^2$$
$$\therefore n = \frac{1}{(0.015)^2}$$
$$\therefore n = 4444 = \text{sample size}$$

7. Sample proportion, $\hat{p} = \frac{84}{200} = 0.42$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{200}} = 0.07$

Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$$= 0.42 - 0.07 < p < 0.42 + 0.07$$
$$= 0.35 < p < 0.49$$

8. Sample proportion, $\hat{p} = \frac{45}{300} = 0.15$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{300}} = 0.057$

(i) Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$$\therefore 0.15 - 0.057 < p < 0.15 + 0.057$$
$$= 0.093 < p < 0.207$$
$$= 0.09 < p < 0.21$$

(ii) If 100 samples were taken we would expect 95 of them to have defective items ranging between 9% and 21% (or between 27 items and 63 items)

(iii) If 200 such tests were performed we would expect 2 × 95 of them to have defective items $\therefore$ 190 defective items.

9. $n = 300$, the sample proportion $\hat{p} = \frac{45}{300} = 0.15$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.15 \pm 1.96\sqrt{\frac{0.15(1-0.15)}{300}}$

$= 0.15 \pm 1.96(0.0206)$
$= 0.15 \pm 0.0404$
$= 0.1096, 0.1904$
$\Rightarrow 0.1096 < p < 0.1904$

10. $n = 200$, the sample proportion $\hat{p} = \frac{72}{200} = 0.36$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.36 \pm 1.96\sqrt{\frac{0.36(1-0.36)}{200}}$

$= 0.36 \pm 1.96(0.03394)$
$= 0.36 \pm 0.06652$
$= 0.29348, 0.42652$
$\Rightarrow 0.293 < p < 0.427$

11. $n = 235$, the sample proportion $\hat{p} = \frac{75}{235} = 0.31915$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.31915 \pm 1.96\sqrt{\frac{0.31915(1-0.31915)}{235}}$

$= 0.31915 \pm 1.96(0.03040)$
$= 0.31915 \pm 0.05959$
$= 0.25956, 0.37874$
$\Rightarrow 0.260 < p < 0.379$

12. $n = 50$, the sample proportion $\hat{p} = \frac{12}{50} = 0.24$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.24 \pm 1.96\sqrt{\frac{0.24(1-0.24)}{50}}$

$= 0.24 \pm 1.96(0.06039)$
$= 0.24 \pm 0.11838$
$= 0.12162, 0.35838$
$\Rightarrow 0.122 < p < 0.358$

13. $n = 400$, the sample proportion $\hat{p} = \frac{136}{400} = 0.34$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.34 \pm 1.96\sqrt{\frac{0.34(1-0.34)}{400}}$

$= 0.34 \pm 1.96(0.02368)$
$= 0.34 \pm 0.04642$
$= 0.29358, 0.38642$
$\Rightarrow 0.294 < p < 0.386$

14. $n = 120$, the sample proportion (fiction) $\hat{p} = \frac{88}{120} = 0.73333$

the sample proportion (paperback) $\hat{p} = \frac{74}{88} = 0.84090$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

(i) $\therefore \text{CI} = 0.73333 \pm 1.96\sqrt{\frac{0.73333(1-0.73333)}{120}}$

$= 0.73333 \pm 1.96(0.04036)$

$= 0.73333 \pm 0.07912$

$= 0.65421, 0.81245$

$\Rightarrow 0.654 < p < 0.812$

(ii) $\therefore \text{CI} = 0.84090 \pm 1.96\sqrt{\frac{0.84090(1-0.84090)}{88}}$

$= 0.84090 \pm 1.96(0.03899)$

$= 0.84090 \pm 0.07642$

$= 0.76448, 0.91732$

$\Rightarrow 0.764 < p < 0.917$

15. $n = 400$

(i) The sample proportion $\hat{p} = \frac{136}{400} = 0.34 = 34\%$

(ii) The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.34 \pm 1.96\sqrt{\frac{0.34(1-0.34)}{400}}$

$= 0.34 \pm 1.96(0.02368)$

$= 0.34 \pm 0.04642$

$= 0.29358, 0.38642$

$\Rightarrow 0.294 < p < 0.386$

$\Rightarrow 29.4\% < p < 38.6\%$

The true proportion lies in this interval 95 times out of a 100

(iii) $2\% = 0.02$

$\Rightarrow \pm 1.96\sqrt{\frac{0.34(1-0.34)}{n}} = \pm 0.02$

$\Rightarrow \left(\pm 1.96\sqrt{\frac{0.34(1-0.34)}{n}}\right)^2 = (\pm 0.02)^2$

$\Rightarrow \frac{.86205}{n} = 0.0004$

$\Rightarrow n = 2155.13$

$\therefore$ 2156 shops would be needed as a sample.

16. $n = 1200$, the sample proportion $\hat{p} = \frac{324}{1200} = 0.27$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$\therefore \text{CI} = 0.27 \pm 1.96\sqrt{\frac{0.27(1-0.27)}{1200}}$

$= 0.27 \pm 1.96(0.01281)$

$= 0.27 \pm 0.025119$

$= 0.24488, 0.29511$

$\Rightarrow 0.245 < p < 0.295$

17. $n = 100$, the sample proportion $\hat{p} = \frac{15}{100} = 0.15$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

(i) $\therefore \text{CI} = 0.15 \pm 1.96\sqrt{\frac{0.15(1-0.15)}{100}}$

$= 0.15 \pm 1.96(0.03570)$

$= 0.15 \pm 0.06998$

$= 0.0800, 0.21998$

$\Rightarrow 0.080 < p < 0.220$

(ii) $1.5\% = 0.015$

$\Rightarrow \pm 1.96\sqrt{\frac{0.15(1-0.15)}{n}} = \pm 0.015$

$\Rightarrow \left(\pm 1.96\sqrt{\frac{0.15(1-0.15)}{n}}\right)^2 = (\pm 0.015)^2$

$\Rightarrow \frac{.48980}{n} = 0.000225$

$\Rightarrow n = 2176.91$

$\therefore$ 2180 people would be needed as a sample.

Exercise 10.2

1. Sample proportion, $\hat{p} = \frac{357}{1000} = 0.357$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1000}} = 0.0316$

95% confidence interval $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.357 - 0.0316 < p < 0.357 + 0.0316$

$\therefore 0.325 < p < 0.389$

No. The leader's belief is not justified as 0.4 is outside the above range at the 95% confidence interval

* Step 1: State H_0 and H_1
H_0: The true proportion is 0.4
H_1: The true proportion is not 0.4

Step 2: Sample proportion $\hat{p}$ (above)
Step 3: Margin of error, (above)
Step 4: Confidence interval (above)
Step 5: The population proportion 0.4 is not within the confidence interval. So we reject the null hypothesis and accept H_1. We conclude that the leaders belief is not justified at the 95% confidence level.

2. 1. H_0: The college admits equal numbers
H_1: The college does not admit equal numbers

2. Sample proportion, $\hat{p} = \frac{267}{500} = 0.534$

3. Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{500}} = 0.0447$

4. Confidence interval $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.534 - 0.0447 < p < 0.534 + 0.0447$

$0.489 < p < 0.5787$

$\therefore 0.489 < p < 0.579$

5. There is evidence to suggest that the college is not evenly divided in admitting equal numbers of men and women, since 0.5 is within the confidence range found for men at the 95% level.

3. (i) Sample proportion, $\hat{p} = \frac{52}{240}$
$= 0.2166$

(ii) Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{240}} = 0.065$

(iii) Probability of throwing a 6 $= 0.1667$

(iv) H_0: The dice is not biased
H_1: The dice is biased
From above $\hat{p} = 0.2166$
Margin of error $= 0.065$
$\therefore$ Confidence interval
$= 0.216 - 0.064 < p < 0.216 + 0.064$
$= 0.152 < p < 0.28$
Since 0.1667 is within the 95% confidence interval found we accept H_0 and conclude that the dice is not biased.

4. 1. H_0: The proportion of overdue books had not decreased
H_1: The proportion of overdue books had decreased

2. Sample proportion, $\hat{p} = \frac{15}{200} = 0.075$

3. Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{200}} = 0.07$

4. Confidence interval $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$
$= 0.075 - 0.07 < p < 0.075 + 0.07$
$= 0.005 < p < 0.145$
$\therefore$ Confidence interval at the 95% level is $0.5\% < p < 14.5\%$

5. Since 12% lies in this interval the survey is correct and the University's claim that the proportion of overdue books had decreased is not justified.

5. 1. H_0: The company claims 20% will not have red flowers
H_1: The company claims 20% will have red flowers

2. Sample proportion, $\hat{p} = \frac{11}{82} = 0.134$

3. Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{82}} = 0.11$

4. Confidence interval $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$
$= 0.134 - 0.11 < p < 0.134 + 0.11$
$= 0.024 < p < 0.244$
$\therefore 2.4\% < p < 24.4\%$

5. Since the claim of 20% of plants will have red flowers lies within the 95% confidence interval the company's claim is correct.

6. 1. H_0: At least 60% of its readers do not have third level degrees.
H_1: At least 60% of its readers do have third level degrees.

2. Sample proportion, $\hat{p} = \frac{208}{312} = 0.6666$

3. Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{312}} = 0.0566$

4. Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$\therefore 0.6666 - 0.0566 < p < 0.6666 + 0.0566$

$= 0.61 < p < 0.723$

$\therefore 61\% < p < 72.3\%$

5. Hence the "Daily Mensa's" claim that at least 60% of its readers have third level degrees is justified.

7. (i) Sample: $n = 500$

$$\hat{p} = \frac{267}{500} = 0.543$$

Margin of error, $E = 1.96\sqrt{\frac{(0.543)(1-0.543)}{500}} = 0.0437$

95% confidence interval for p:

$0.543 - 0.0437 \leqslant p \leqslant 0.543 + 0.0437$

$0.4993 \leqslant p \leqslant 0.5867$

(ii) $H_0: p = 0.5$

$H_1: p \neq 0.5$

As $p = 0.5$ lies inside the 95% confidence interval based on $\hat{p}$, the result is not significant. Hence there is no evidence that the coin is biased.

8. $\hat{p} = 0.36.$

(i) $E = 0.02$

$$1.96\sqrt{\frac{(0.36)(0.64)}{n}} = 0.02$$

$$\sqrt{\frac{0.2304}{n}} = 0.0102$$

$$\frac{0.2304}{n} = 0.0001041$$

$$0.2304 = 0.0001041n$$

$$n = 2213$$

(ii) $E = 0.03$

$$1.96\sqrt{\frac{(0.36)(0.64)}{n}} = 0.03$$

$$\sqrt{\frac{0.2304}{n}} = 0.0153$$

$$\frac{0.2304}{n} = 0.0002343$$

$$0.2304 = 0.0002343n$$

$$n = 983$$

Exercise 10.3

1. (i) When a large number of samples of size n are taken from a population, then the distribution of $\bar{x}$, the sample mean, is known as the "**sampling distribution**" of the mean.

(ii) As the sample size increases, the standard deviation of the sampling distribution of the sample means will **decrease**.

(iii) If the mean of the underlying population is μ, the mean of the sampling distribution of the means is $\boldsymbol{\mu}$.

(iv) If the standard deviation of a population is σ and samples of size n are taken from it, then the standard deviation of the distribution of the sample means is $\frac{\sigma}{\sqrt{n}}$.

2. **A** represents the distribution of the sample means.
[Note, same mean, smaller standard deviation].

3. (i) The curve will have a Normal distribution shape based on the "Central Limit Theorem".

(ii) Because the sample size is greater than 30.

(iii) The mean $= 12$ and the standard deviation $= \dfrac{2}{\sqrt{36}} = \dfrac{2}{6} = \dfrac{1}{3}$

4. (i) (4, 6), (4, 8), (4, 10), (6, 8), (6, 10), (8, 10)

(ii) $\dfrac{4+6}{2} = 5, \dfrac{4+8}{2} = 6, \dfrac{4+10}{2} = 7, \dfrac{6+8}{2} = 7, \dfrac{6+10}{2} = 8, \dfrac{8+10}{2} = 9.$

(iii) They are a statistic obtained from the samples

(iv) The mean of the population $= \dfrac{4+6+8+10}{4} = 7,$

The mean of the samples of size 2 $= \dfrac{5+6+7+7+8+9}{6} = \dfrac{42}{6} = 7$

5. (i) Distribution A is **positively** skewed (as most of the data is to the left)

(ii) Distribution B is a Normal distribution.

(iii) Because the sample size is $\geqslant 30$ the Central Limit theorem applies

6. The sample mean is normally distributed as $n \geqslant 30$

Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{13 - 12}{\frac{3}{\sqrt{36}}} = 2$

$$\begin{aligned} P(\bar{x} > 13) &= P(Z > 2) \\ &= 1 - P(Z \leqslant 2) \\ &= 1 - 0.9772 = 0.0228 \end{aligned}$$

7. The distribution is normal with $\mu = 60$ and $\sigma = 4, n = 15$

Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{58 - 60}{\frac{4}{\sqrt{15}}} = \dfrac{-2}{1.03} = -1.94$

$$\begin{aligned} P(\bar{x} < 58) &= P(Z < -1.94) \\ &= 1 - P(Z \leqslant 1.94) \\ &= 1 - 0.9738 = 0.0262 \end{aligned}$$

8. The sample mean is normally distributed as $n \geqslant 30$

(i) Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{177 - 176}{\frac{11}{\sqrt{80}}} = \dfrac{1}{1.23} = 0.81$

$$\begin{aligned} P(\bar{x} > 177) &= P(Z > 0.81) \\ &= 1 - P(Z \leqslant 0.81) \\ &= 1 - 0.791 = 0.209 \end{aligned}$$

(ii) Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{174.8 - 176}{\frac{11}{\sqrt{80}}} = \dfrac{-1.2}{1.23} = -0.98$

$$\begin{aligned} P(\bar{x} < 174.8) &= P(Z < -0.98) \\ &= 1 - P(Z \leqslant 0.98) \\ &= 1 - 0.8365 = 0.1635 \end{aligned}$$

9. $\mu = 4.2$ hours and $\sigma = 1.8$ hours, $n = 15$

(i) Standard error $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{1.8}{\sqrt{36}} = 0.3$

(ii) Greater than 4.8 hours $\Rightarrow \bar{x} > 4.8$ hours

Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{4.8 - 4.2}{\frac{1.8}{\sqrt{36}}} = \dfrac{0.6}{0.3} = 2$

$$\begin{aligned} P(\bar{x} > 4.8) &= P(Z > 2) \\ &= 1 - P(Z \leqslant 2) \\ &= 1 - 0.9772 = 0.0228 \end{aligned}$$

(iii) From 4.1 to 4.5 hours

For $\bar{x} = 4.1$ hours, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{4.1 - 4.2}{\frac{1.8}{\sqrt{36}}} = \dfrac{-0.1}{0.3} = -0.333$

For $\bar{x} = 4.5$ hours, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{4.5 - 4.2}{\frac{1.8}{\sqrt{36}}} = \dfrac{0.3}{0.3} = 1$

$$\begin{aligned} P(4.1 \leqslant \bar{x} \leqslant 4.5) &= P(-0.333 \leqslant Z \leqslant 1) \\ &= P(Z \leqslant 1) - P(Z > -0.333) \\ &= P(Z \leqslant 1) - [1 - P(Z \leqslant 0.333)] \\ &= 0.8413 - [1 - 0.6293] \\ &= 0.4706 \end{aligned}$$

10. $\mu = 5.8$ and $\sigma = 1.2$, $n = 900$

For $\bar{x} = 5.85$, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{5.85 - 5.8}{\frac{1.2}{\sqrt{900}}} = \dfrac{0.05}{0.04} = 1.25$

$$\begin{aligned} P(\bar{x} \leqslant 5.85) &= P(Z \leqslant 1.25) \\ &= 0.894 \end{aligned}$$

11. $\mu = 8$ years [$= 96$ months] and $\sigma = 6$ months, $n = 144$

For $\bar{x} = 8$ years and 1 month $= 97$ months, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{97 - 96}{\frac{6}{\sqrt{144}}} = \dfrac{1}{0.5} = 2.0$

$$\begin{aligned} P(\bar{x} > 97) &= P(Z > 2) \\ &= 1 - P(Z \leqslant 2) \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

$\therefore$ Out of the 40 samples taken we would except $0.0228 \times 40 = 0.912 = 1$ sample to have a mean lifetime of 8 years and 1 month

12. $\mu = 200$ and $\sigma = 10$, $n = 10$

For $\bar{x} = 198$, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{198 - 200}{\frac{10}{\sqrt{10}}} = \dfrac{-2}{3.16} = -0.63$

For $\bar{x} = 205$, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{205 - 200}{\frac{10}{\sqrt{10}}} = \dfrac{5}{3.16} = 1.58$

$$\begin{aligned} P(\bar{x} < 198, \bar{x} > 205) &= P(Z < -0.63, Z > 1.58) \\ &= P(Z < -0.63) + P(Z > 1.58) \\ &= [1 - P(Z \leqslant 0.63)] + [1 - P(Z \leqslant 1.58) \\ &= [1 - 0.7357] + [1 - 0.9429] \\ &= 0.3214 \end{aligned}$$

13. Since both distributions have the same mean C = 80

Since the standard deviation, $\sigma = 8$, the point D $= \mu + 2\sigma = 80 + 2(8) = 96$

For distribution B the standard deviation $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{8}{\sqrt{36}} = \dfrac{4}{3}$

$\therefore$ The point E $= \mu - \sigma_{\bar{x}} = 80 - \dfrac{4}{3} = 78\dfrac{2}{3}$

14. $\mu = 75$ and $\sigma = 9$, $P(\bar{x} > 73) = 0.8708$

If $P(\bar{x} > 73) = 0.8708 \Rightarrow z = 1.13$ using page 37 of the tables.

$z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = 1.13$

$\therefore 73 - 75 = 1.13\left(\dfrac{9}{\sqrt{n}}\right)$

$-2\sqrt{n} = 1.13(9)$

Squaring both sides $4n = 103.4289$

$n = 25.86 = 26$

15. $\mu = 30$ and $\sigma = \sqrt{5}, n = 40$

(i) For $\overline{x} = 30.5$, standard unit $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{30.5 - 30}{\frac{\sqrt{5}}{\sqrt{40}}} = \dfrac{0.5}{0.3535} = 1.41$

$$\begin{aligned} P(\overline{x} > 30.5) &= P(Z > 1.41) \\ &= 1 - P(Z \leqslant 1.41) \\ &= 1 - 0.9207 \\ &= 0.0793 \end{aligned}$$

(ii) $\mu = 30$ and $\sigma = \sqrt{5}, P(\overline{x} > 30.4) = 0.01$

If $P(\overline{x} > 30.4) = 0.01 \Rightarrow [1 - P(\overline{x} \leqslant 30.4)] = 0.01$

$\therefore P(\overline{x} \leqslant 30.4) = 0.99 \Rightarrow z = 2.33$ using page 37 of the tables.

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = 2.33$$

$$\therefore 30.4 - 30 = 2.33\left(\frac{\sqrt{5}}{\sqrt{n}}\right)$$

$$0.4\sqrt{n} = 2.33(\sqrt{5})$$

Squaring both sides $0.16n = 27.144$

$n = 169.6 = 170$

16. $\mu = 52$ g and $\sigma = 4$ g

(i) $P(x > 60 \text{ g})$

For $x = 60$, standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{60 - 52}{4} = \dfrac{8}{4} = 2$

$$\begin{aligned} P(x > 60 \text{ g}) = P(Z > 2) &= [1 - P(Z \leqslant 2)] \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

(ii) $P(50 \leqslant \overline{x} \leqslant 55 \text{ g}), n = 5$

For $\overline{x} = 50$, standard unit $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{50 - 52}{\frac{4}{\sqrt{5}}} = \dfrac{-2}{1.788} = -1.12$

For $\overline{x} = 55$, standard unit $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{55 - 52}{\frac{4}{\sqrt{5}}} = \dfrac{3}{1.788} = 1.68$

$$\begin{aligned} P(50 \leqslant \overline{x} \leqslant 55) &= P(-1.12 \leqslant Z \leqslant 1.68) \\ &= P(Z \leqslant 1.68) - P(Z > -1.12) \\ &= P(Z \leqslant 1.68) - [1 - P(Z \leqslant 1.12)] \\ &= 0.9535 - [1 - 0.8686] \\ &= 0.822 \end{aligned}$$

(iii) $P(52.1 < \overline{x} \leqslant 52.2 \text{ g}), n = 90$

For $\overline{x} = 52.1$, standard unit $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{52.1 - 52}{\frac{4}{\sqrt{90}}} = \dfrac{0.1}{0.4216} = 0.24$

For $\overline{x} = 52.2$, standard unit $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{52.2 - 52}{\frac{4}{\sqrt{90}}} = \dfrac{0.2}{0.4216} = 0.47$

$$\begin{aligned} P(52.1 < \overline{x} \leqslant 52.2) &= P(0.24 < Z \leqslant 0.47) \\ &= P(Z \leqslant 0.47) - P(Z \leqslant 0.24) \\ &= 0.6806 - 0.5948 \\ &= 0.086 \end{aligned}$$

Since $n \geqslant 30$, in part(iii) the sample means will approximate to a normal distribution regardless of population distribution therefore the answer to (iii) will be unchanged.

Exercise 10.4

1. $\bar{x} = 63$ and $\sigma = 12$, $n = 800$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 63 \pm 1.96\frac{12}{\sqrt{800}}$

$= 63 \pm 0.83$

$= 62.17, 63.83$

$\Rightarrow 62.17 < \mu < 63.83$

2. $\bar{x} = 284$ kg and $\sigma = 42$ kg, $n = 280$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 284 \pm 1.96\frac{42}{\sqrt{280}}$

$= 284 \pm 4.92$

$= 279.08, 288.92$

$\Rightarrow 279.1 \text{ kg} < \mu < 288.9 \text{ kg}$

3. $\bar{x} = 227$ g and $\sigma = 7.5$ g, $n = 70$

(i) The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 227 \pm 1.96\frac{7.5}{\sqrt{70}}$

$= 227 \pm 1.76$

$= 225.2, 228.8$

$\Rightarrow 225.2 \text{ g} < \mu < 228.8 \text{ g}$

(ii) A probability of 95% in the interval $225.2 \text{ g} < \mu < 228.8 \text{ g}$

$\Rightarrow$ a probability of 5% outside this interval.

4. $\bar{x} = 62.7$ marks and $\sigma = 9.2$ marks, $n = 100$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 62.7 \pm 1.96\frac{9.2}{\sqrt{100}}$

$= 62.7 \pm 1.8$

$= 60.9, 64.5$

$\Rightarrow 60.9 \text{ marks} < \mu < 64.5 \text{ marks}$

5. $\bar{x} = 5.12$ mg and $\sigma = 0.04$ mg, $n = 12$

(i) The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 5.12 \pm 1.96\frac{0.04}{\sqrt{12}}$

$= 5.12 \pm 0.0226$

$= 5.097, 5.142$

$\Rightarrow 5.097 \text{ mg} < \mu < 5.142 \text{ mg}$

(ii) 5.10 mg and 5.14 mg

(iii) A 95% confidence interval means that the "mean" lies in the interval 5.10 mg to 5.14 mg 95 times out of a 100.

6. $\overline{x} = €280$ and $\sigma = €105$, $n = 400$

The 95% confidence interval [CI] for μ is $\overline{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 280 \pm 1.96\frac{105}{\sqrt{400}} \\ &= 280 \pm 10.29 \\ &= 269.71, 209.29 \end{aligned}$$

$\Rightarrow €269.71 < \mu < €209.29$

7. $\overline{x} = 29.2$ cm and $\sigma = 1.47$ cm, $n = 180$

(i) The 95% confidence interval [CI] for μ is $\overline{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 29.2 \pm 1.96\frac{1.47}{\sqrt{180}} \\ &= 29.2 \pm 0.215 \\ &= 28.985, 29.415 \end{aligned}$$

$\Rightarrow 28.99 \text{ cm} < \mu < 29.41 \text{ cm}$

(ii) Because the sample size, $n = 180 \geqslant 30$, is sufficiently large to apply the Central limit Theorem.

8. $\overline{x} = 0.932$ g and $\sigma = 0.1$ g, $n = 64$

(i) Standard error on the mean $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{0.1}{\sqrt{64}}$

$= 0.0125$ g

(ii) 0.932 g the same as the sample.

(iii) The 95% confidence interval [CI] for μ is $\overline{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 0.932 \pm 1.96\,(0.0125) \\ &= 0.932 \pm 0.0245 \\ &= 0.9075 \text{ g}, 0.9565 \text{ g} \end{aligned}$$

$\Rightarrow 0.9075 \text{ g} < \mu < 0.9595 \text{ g}$

(iv) The 95% confidence interval would change to

$$\begin{aligned} \text{CI} &= 0.932 \pm 1.96\,(0.01) \\ &= 0.932 \pm 0.0196 \\ &= 0.9124 \text{ g}, 0.9516 \text{ g} \end{aligned}$$

$\Rightarrow 0.9124 \text{ g} < \mu < 0.9516 \text{ g}$

(v) Increasing the sample size reduces the standard error and hence the confidence interval gets smaller.

9. $\overline{x} = 4.6$ years and $\sigma = 2.5$ years, $n = 240$

(i) The 95% confidence interval [CI] for μ is $\overline{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 4.6 \pm 1.96\,\frac{2.5}{\sqrt{240}} \\ &= 4.6 \pm 0.316 \\ &= 4.284, 4.916 \end{aligned}$$

$\Rightarrow 4.28 \text{ years} < \mu < 4.91 \text{ years}$

(ii) $\pm 1.96\dfrac{\sigma}{\sqrt{n}} = \pm 0.2$ years

$\Rightarrow \pm 1.96\dfrac{2.5}{\sqrt{n}} = \pm 0.2$

$\Rightarrow \left(\pm 1.96\dfrac{2.5}{\sqrt{n}}\right)^2 = (\pm 0.2)^2$

$\Rightarrow \dfrac{24.01}{n} = 0.04$

$\Rightarrow n = 600.25$

$\therefore$ 601 cars needed as a sample.

10. $\bar{x} = 748$ g and $\sigma = 3.6$ g, $n = 150$

(i) The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$$\therefore \text{CI} = 748 \pm 1.96\,\frac{3.6}{\sqrt{150}}$$
$$= 748 \pm 0.5761$$
$$= 747.42,\ 748.58$$
$$\Rightarrow 747.42\text{ g} < \mu < 748.58\text{ g}$$

(ii) $\pm 1.96\frac{\sigma}{\sqrt{n}} = \pm 1.5$ g

$$\Rightarrow \pm 1.96\frac{3.6}{\sqrt{n}} = \pm 1.5$$
$$\Rightarrow \left(\pm 1.96\frac{3.6}{\sqrt{n}}\right)^2 = (\pm 1.5)^2$$
$$\Rightarrow \frac{49.787}{n} = 2.25$$
$$\Rightarrow n = 22.127$$

$\therefore$ 23 boxes are needed as a sample.

11. $\bar{x} = 69$ beats and $\sigma = 4$ beats, $n = 80$

(i) The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$$\therefore \text{CI} = 69 \pm 1.96\,\frac{4}{\sqrt{80}}$$
$$= 69 \pm 0.8765$$
$$= 68.12,\ 69.88$$
$$\Rightarrow 68.12\text{ beats} < \mu < 69.88\text{ beats}$$

(ii) $\pm 1.96\frac{\sigma}{\sqrt{n}} = \pm 1.5$ beats

$$\Rightarrow \pm 1.96\frac{4}{\sqrt{n}} = \pm 1.5$$
$$\Rightarrow \left(\pm 1.96\frac{4}{\sqrt{n}}\right)^2 = (\pm 1.5)^2$$
$$\Rightarrow \frac{61.4656}{n} = 2.25$$
$$\Rightarrow n = 27.318$$

$\therefore$ 28 people needed.

12. $\mu = 48.6$ g and $\sigma = 8.5$ g, $n = 50$

(i) $P(\bar{x} < 49\text{ g})$

For $\bar{x} = 49$ g, standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{49 - 48.6}{\frac{8.5}{\sqrt{50}}} = \frac{0.4}{1.202} = 0.3327$

$$P(\bar{x} < 49) = P(Z \leqslant 0.3327)$$
$$= 0.629$$

(ii) $\bar{x} = 48.6$ g and $\sigma = 8.5$ g, $n = 50$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$$\therefore \text{CI} = 48.6 \pm 1.96\,\frac{8.5}{\sqrt{50}}$$
$$= 48.6 \pm 2.356$$
$$= 46.2,\ 51.0$$
$$\Rightarrow 46.2\text{ g} < \mu < 51.0\text{ g}$$

(iii) $\pm 1.96\frac{\sigma}{\sqrt{n}} = \pm 2$

$\Rightarrow \pm 1.96\frac{8.5}{\sqrt{n}} = \pm 2$

$\Rightarrow \left(\pm 1.96\frac{8.5}{\sqrt{n}}\right)^2 = (\pm 2)^2$

$\Rightarrow \frac{277.555}{n} = 4$

$\Rightarrow n = 69.388$

$\therefore$ 70 pebbles would be needed as a sample.

13. Confidence interval $= (54.09, 60.71)$, $n = 80$

(i) $\bar{x} = \frac{54.09 + 60.71}{2} = 57.4$

(ii) Interval width $= 60.71 - 57.4 = 3.31$

$\pm 1.96\frac{\sigma}{\sqrt{80}} = \pm 3.31$

$\pm 0.219\sigma = \pm 3.31$

$\sigma = 15.1$ marks

Exercise 10.5

1. (i) H_0: The mean μ is 50

H_1: The mean is **not** 50

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

(iii) $\mu = 50, \bar{x} = 52.4, \sigma = 14.3$ and $n = 100$

(iv) Standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{52.4 - 50}{\frac{14.3}{\sqrt{100}}} = \frac{2.4}{1.43} = 1.67$

Since $1.67 < 1.96$, the test statistic is not in the critical region and hence we accept the null hypothesis, the mean is 50.

There is no evidence to suggest that the true mean is different from the assumed mean.

2. (i) H_0: The students do not differ from the normal

H_1: The students do differ from the normal

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

(iii) $\mu = 70, \bar{x} = 68, \sigma = 6$ and $n = 64$

(iv) Standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{68 - 70}{\frac{6}{\sqrt{64}}} = \frac{-2}{0.75} = -2.67$

Since $-2.67 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the students do not differ from the normal.

Yes there is evidence to suggest that they differ from the normal.

3. (i) H_0: The mean age of patients is 45 years

(ii) H_1: The mean age of patients is **not** 45 years

(iii) The test statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{48.4 - 45}{\frac{18}{\sqrt{100}}} = \frac{3.4}{1.8} = 1.89$

(iv) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

Since $1.89 < 1.96$, the test statistic is not in the critical region and hence we accept the null hypothesis that the mean age of patients is 45 years.

No there is no evidence to suggest that the mean age is not 45 years.

4. (i) H_0: The mean length has not changed

(ii) H_1: The mean length has changed

(iii) The test statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{211.5 - 210}{\frac{6}{\sqrt{100}}} = \frac{1.5}{0.6} = 2.5$

(iv) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

Since $2.15 > 1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean length has not changed.
Yes there is evidence to suggest that the mean length has changed.

5. (i) H_0: The mean lifespan is 258 days
H_1: The mean lifespan is not 258 days

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

(iii) $\mu = 258, \bar{x} = 269, \sigma = 45$ and $n = 64$

(iv) Standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{269 - 258}{\frac{45}{\sqrt{64}}} = \frac{11}{5.625} = 1.9555$

Since $1.9555 < 1.96$, the test statistic is not in the critical region and hence we accept the null hypothesis that the mean lifespan is 258 days.
There is no evidence to suggest that the drug has altered the mean lifespan.

6. (i) H_0: The mean number of children is 3.8
H_1: The mean number of children is **not** 3.8

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

(iii) $\mu = 3.8, \bar{x} = \frac{144}{40} = 3.6, \sigma = 0.6$ and $n = 40$

(iv) Standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{3.6 - 3.8}{\frac{0.6}{\sqrt{40}}} = \frac{-0.2}{0.09486} = -2.1$

Since $-2.11 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean number of children is 3.8.
There is evidence to suggest that the mean number of children has changed.

7. (i) H_0: The mean mark of students in this town is 48.7
H_1: The mean mark of students in this town is **not** 48.7

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

(iii) $\mu = 48.7, \bar{x} = 46.5, \sigma = 9.5$ and $n = 120$

(iv) Standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{46.5 - 48.7}{\frac{9.5}{\sqrt{120}}} = \frac{-2.2}{0.8672} = -2.54$

Since $-2.54 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean mark of students in this town is 48.7.
There is evidence to suggest that the mean mark has changed.

8. p-value $= 2 \times P(Z > |z_1|)$

(i) $z_1 = 1.73$,
$$\begin{aligned} p\text{-value} &= 2 \times P(Z > |1.73|) \\ &= 2 \times [1 - P(Z \leqslant |1.73|) \\ &= 2 \times [1 - 0.9582) \\ &= 0.0836 \end{aligned}$$

(ii) $z_1 = -1.91$,
$$\begin{aligned} p\text{-value} &= 2 \times P(Z > |-1.91|) \\ &= 2 \times [1 - P(Z \leqslant |1.91|) \\ &= 2 \times [1 - 0.9719) \\ &= 0.0562 \end{aligned}$$

(iii) $z_1 = -1.65$,
$$\begin{aligned} p\text{-value} &= 2 \times P(Z > |-1.65|) \\ &= 2 \times [1 - P(Z \leqslant |1.65|) \\ &= 2 \times [1 - 0.9505) \\ &= 0.099 \end{aligned}$$

(iv) $z_1 = -2.06$, $\quad p\text{-value} = 2 \times P(Z > |-2.06|)$

$$= 2 \times [1 - P(Z \leqslant |2.06|)$$
$$= 2 \times [1 - 0.9803)$$
$$= 0.0394$$

9. $\mu = 85$ hours, $\bar{x} = 86.5$ hours, $\sigma = 12$ hours and $n = 200$

(i) The sample statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{86.5 - 85}{\frac{12}{\sqrt{200}}} = \frac{1.5}{0.8485} = 1.77$

(ii) $p\text{-value} = 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |1.77|)$$
$$= 2 \times [1 - P(Z \leqslant |1.77|)$$
$$= 2 \times [1 - 0.9616)$$
$$= 0.0768$$

(iii) $\Rightarrow p > 0.05$ we accept H_0, the result is **not** significant at the 5% level of significance.

10. $\mu = 70$, $\bar{x} = 68.5$, $\sigma = 6$ and $n = 36$

(i) The sample statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{68.5 - 70}{\frac{6}{\sqrt{36}}} = \frac{-1.5}{1} = -1.5$

(ii) $p\text{-value} = 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |-1.5|)$$
$$= 2 \times [1 - P(Z \leqslant |1.5|)$$
$$= 2 \times [1 - 0.9332)$$
$$= 0.1336$$

(iii) $\Rightarrow p > 0.05$ we accept H_0, the result is **not** significant at the 5% level of significance.

11. $\mu = 12$ minutes, $\bar{x} = 12.3$ minutes, $\sigma = 1.2$ minutes and $n = 36$

(i) The test statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{12.3 - 12}{\frac{1.2}{\sqrt{36}}} = \frac{0.3}{0.2} = 1.5$

(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$

Since $1.5 < 1.96$, the test statistic is not in the critical region and hence we accept the null hypothesis that the average time is 12 minutes.

(iii) $p\text{-value} = 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |1.5|)$$
$$= 2 \times [1 - P(Z \leqslant |1.5|)$$
$$= 2 \times [1 - 0.9332)$$
$$= 0.1336$$

(iv) $\Rightarrow p > 0.05$ we accept H_0, the result is **not** significant at the 5% level of significance so we reach the same conclusion.

12. $\mu = 420$ cm, $\bar{x} = 423$ cm, $\sigma = 12$ cm and $n = 100$

(i) The sample statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{423 - 420}{\frac{12}{\sqrt{100}}} = \frac{3}{1.2} = 2.5$

(ii) $p\text{-value} = 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |2.5|)$$
$$= 2 \times [1 - P(Z \leqslant |2.5|)$$
$$= 2 \times [1 - 0.9938)$$
$$= 0.0125$$

(iii) $\Rightarrow p < 0.05$ we reject H_0, the result is significant at the 5% level of significance so we reach the conclusion that there is a change in the mean length of the bars.

13. $\mu = 5$ mm, $\bar{x} = 5.008$ mm, $\sigma = 0.072$ mm and $n = 400$

Standard error on the mean $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{0.072}{\sqrt{400}} = 0.0036$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 5.00 \pm 1.96(0.0036) \\ &= 5.00 \pm 0.007056 \\ &= 4.9929, 5.007056 \end{aligned}$$

$$\Rightarrow 4.993 \text{ mm} < \mu < 5.007 \text{ mm}$$

5% level of significance $\Rightarrow -1.96 < z < 1.96$

The test statistic is $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{5.008 - 5}{\frac{0.072}{\sqrt{400}}} = \dfrac{0.008}{0.0036} = 2.22$

Since $2.22 > 1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean length is 5 mm, the sample does differ significantly from the stated mean.

Revision Exercise 10 (Core)

1. $\mu = 25$ kg, $\sigma = \sqrt{5}$ kg and $n = 50$

From 24.5 kg to 25.5 kg

For $\bar{x} = 24.5$, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{24.5 - 25}{\frac{\sqrt{5}}{\sqrt{50}}} = \dfrac{-0.5}{0.31622} = -1.58$

For $\bar{x} = 25.5$, standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{25.5 - 25}{\frac{\sqrt{5}}{\sqrt{50}}} = \dfrac{0.5}{0.31622} = 1.58$

$$\begin{aligned} P(24.5 \leqslant \bar{x} \leqslant 25.5) &= P(-0.1.58 \leqslant Z \leqslant 1.58) \\ &= P(Z \leqslant 1.58) - P(Z > 1.58) \\ &= P(Z \leqslant 1.58) - [1 - P(Z \leqslant 1.58)] \\ &= 0.9394 - [1 - 0.9394] \\ &= 0.8788 \\ &= 0.89 \end{aligned}$$

2. $\mu = 2.85$, $\sigma = 0.07$ and $n = 20$

Mean of sample $\bar{x} = \mu = 2.85$

Standard error on the mean $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{0.07}{\sqrt{20}} = 0.016$

3. $\bar{x} = 26.2$ beats and $\sigma_{\bar{x}} = 5.15$ beats, $n = 32$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\dfrac{\sigma_{\bar{x}}}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 26.2 \pm 1.96\frac{5.15}{\sqrt{32}} \\ &= 26.2 \pm 1.7843 \\ &= 24.4157, 27.9843 \end{aligned}$$

$\Rightarrow 24.42$ beats $< \mu < 27.98$ beats

4. $\bar{x} = 266$ ml, $\sigma = 20$ ml and $n = 40$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned} \therefore \text{CI} &= 266 \pm 1.96\frac{20}{\sqrt{40}} \\ &= 266 \pm 6.198 \\ &= 259.802, 272.198 \end{aligned}$$

$\Rightarrow 259.80$ ml $< \mu < 272.20$ ml

5. $n = 150$, the sample proportion $\hat{p} = \dfrac{90}{150} = 0.6$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\dfrac{p(1-p)}{n}}$

$\therefore$ CI $= 0.6 \pm 1.96\sqrt{\dfrac{0.6(1-0.6)}{150}}$

$= 0.6 \pm 1.96(0.04)$
$= 0.6 \pm 0.0784$
$= 0.5216, 0.6784$
$\Rightarrow 0.522 < p < 0.678$

6. $n = 100$, the sample proportion $\hat{p} = 0.55$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\dfrac{p(1-p)}{n}}$

$\therefore$ CI $= 0.55 \pm 1.96\sqrt{\dfrac{0.55(1-0.55)}{100}}$

$= 0.55 \pm 1.96(0.04975)$
$= 0.55 \pm 0.09750$
$= 0.4525, 0.6475$
$\Rightarrow 0.453 < p < 0.648$

7. (i) H_0: The height of the Irish students does not differ from the height of the German students
H_1: The height of the Irish students does differ from the height of the German students
(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$
(iii) $\mu = 176$ cm, $\bar{x} = 179$ cm, $\sigma = 11$ cm and $n = 60$
(iv) Standard unit $z = \dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{179-176}{\frac{11}{\sqrt{60}}} = \dfrac{3}{1.42} = 2.11$

Since $2.11 > 1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the heights are the same.
Yes there is evidence to suggest that the average German student is taller than the average Irish student.

8. (i) H_0: The mean quantity of honey has not changed
H_1: The mean quantity of honey has changed
(ii) $\mu = 460.3$ g, $\bar{x} = 461.2$ g, $\sigma = 3.2$ g and $n = 60$
Standard unit $z = \dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{461.2-460.3}{\frac{3.2}{\sqrt{60}}} = \dfrac{0.9}{0.4131} = 2.18$

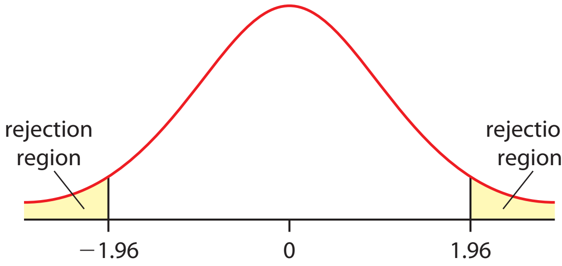
(iii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$
Since $2.18 > 1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the quantity of honey has not changed.
Yes there is evidence to suggest that the sample mean is different from the population mean.

9. (i) A Normal distribution. Central Limit Theorem
(ii) Because the sample size $n > 30$
(iii) $\mu = 96$ hrs, $\sigma = 6$ hrs and $n = 36$
Greater than 98 hours $\Rightarrow \bar{x} > 98$ hours
Standard unit $z = \dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{98-96}{\frac{6}{\sqrt{36}}} = \dfrac{2}{1} = 2$

$P(\bar{x} > 98) = P(Z > 2)$
$= 1 - P(Z \leqslant 2)$
$= 1 - 0.9772 = 0.0228$

$P(\bar{x} > 98)(40) = 0.0228(40) = 0.912 = 1$

10. (i)



(ii) $z < -1.96, z > 1.96$

(iii) The sample statistic is $z_1 = 1.6$

(ii) p-value $= 2 \times P(Z > |z_1|)$

$= 2 \times P(Z > |1.6|)$

$= 2 \times [1 - P(Z \leqslant |1.6|)$

$= 2 \times [1 - 0.9452)$

$= 0.1096$

11. (i) Sample proportion, $\hat{p} = \frac{170}{250} = 0.68$

Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{250}} = 0.063$

(ii) Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$= 0.68 - 0.063 < p < 0.68 + 0.063$

$\therefore 0.617 < p < 0.743$

$\therefore 0.62 < p < 0.74$

is confidence interval for the proportion of households that own at least one pet.

Revision Exercise 10 (Advanced)

1. $\mu = 74$ and $\sigma = 6$, $P(\bar{x} > 72) = 0.854$

If $P(\bar{x} > 72) = 0.854 \Rightarrow z = 1.05$ using page 37 of the tables.

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = 1.05$$

$$\therefore 72 - 74 = 1.05\left(\frac{6}{\sqrt{n}}\right)$$

$$-2\sqrt{n} = 1.05(6)$$

(Squaring both sides) $4n = 39.69$

$$n = 9.92 = 10$$

2. $\mu = 12.1$ kg and $\sigma = 0.4$ kg

For $x = 12$, standard unit $z = \frac{x - \mu}{\sigma} = \frac{12 - 12.1}{0.4} = \frac{-0.1}{0.4} = -0.25$

$P(x \leqslant 12 \text{ kg}) = P(Z \leqslant -0.25) = [1 - P(Z \leqslant 0.25)]$

$= 1 - 0.5987$

$= 0.401$

3. $\bar{x} = 31.4$ kg and $\sigma = 2.4$ kg, $n = 36$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\therefore \text{CI} = 31.4 \pm 1.96 \frac{2.4}{\sqrt{36}}$

$= 31.4 \pm 0.784$

$= 30.616, 32.184$

$\Rightarrow 30.6 \text{ kg} < \mu < 32.2 \text{ kg}$

4. (i) Since $n \geqslant 30$, the central limit theorem can be applied.

(ii) The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$p = 20\% = 0.2$

$$\therefore \text{CI} = 0.2 \pm 1.96\sqrt{\frac{0.2(1-0.2)}{30}}$$
$$= 0.2 \pm 1.96(0.073)$$
$$= 0.2 \pm 0.1431$$
$$= 0.0569, 0.3431$$

$\Rightarrow 0.057 < p < 0.343$

$\Rightarrow 5.7\% < p < 34.3\%$

5. (i) If 100 samples of the same size are taken, then the true population mean (or proportion) will lie in the given interval on 95 occasions out of 100.

(ii) $\bar{x} = 13.52$ km/l and $\sigma = 2.23$ km/l, $n = 150$

The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

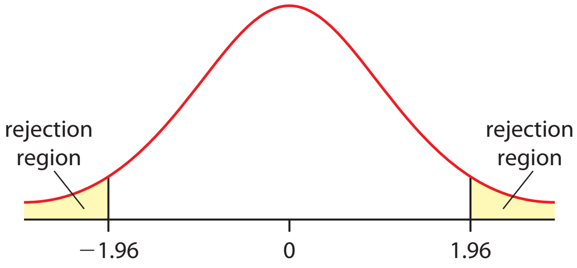
$$\therefore \text{CI} = 13.52 \pm 1.96\frac{2.23}{\sqrt{150}}$$
$$= 13.52 \pm 0.35687$$
$$= 13.163, 13.877$$

$\Rightarrow 13.16 \text{ km/l} < \mu < 13.88 \text{ km/l}$

6. (i) H_0: The mean response time is unchanged, $\mu = 1.2s$

H_1: The mean response time changes, $\mu \neq 1.2s$

(ii) $z < -1.96, z > 1.96$



$\mu = 1.2s, \bar{x} = 1.05s$ and $\sigma = 0.5s, n = 100$

(iii) The test statistic is $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{1.05 - 1.2}{\frac{0.5}{\sqrt{100}}} = \frac{-0.15}{0.05} = -3$

Since $-3 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean response time is 1.2s

Yes the drug has an effect on the response time.

(iv) p-value $= 2 \times P(Z > |z_1|)$
$$= 2 \times P(Z > |-3|)$$
$$= 2 \times [1 - P(Z \leqslant |3|)$$
$$= 2 \times [1 - 0.9987)$$
$$= 0.0026$$

Since $p < 0.05$ we reject the null hypothesis that $\mu = 1.2s$

7. $n = 72$, the sample proportion $\hat{p} = \frac{50}{72} = 0.6944$

(i) margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{72}} = 0.118$

(ii) $n = 72$, the sample proportion $\hat{p} = \frac{50}{72} = 0.69$

(iii) The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm \frac{1}{\sqrt{n}}$

$$p = \frac{50}{72} \pm \frac{1}{\sqrt{72}}$$

$0.577 < p < 0.812$

(iv) Since 80% $= 0.8$ is within the confidence interval we accept the school's claim.

8. (i) H_0: The mean weight has not changed, $\mu = 25$ kg

H_1: The mean weight has changed, $\mu \neq 25$ kg

(ii) $\mu = 25$ kg, $\bar{x} = 24.5$ kg, $\sigma = 1.5$ kg and $n = 50$

$$\text{Sample statistic, } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{24.5 - 25}{\frac{1.5}{\sqrt{50}}} = \frac{-0.5}{0.212} = -2.36$$

(iii) $p\text{-value} = 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |-2.36|)$$
$$= 2 \times [1 - P(Z \leqslant |2.36|)$$
$$= 2 \times [1 - 0.9909)$$
$$= 0.0182$$

(iv) At the 5% level of significance since $p < 0.05$ we reject the null hypothesis that $\mu = 25$ kg, the wholesaler's suspicion is justified.

(v) The p-value is the smallest level of significance at which the null hypothesis could be rejected.

9. (i) Mean $= 68$ kg,

$$\text{Standard error} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{25}} = 0.65 \text{ kg}$$

(ii) $\text{Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{67.5 - 68}{\frac{3}{\sqrt{25}}} = \frac{-0.5}{0.65} = -0.769$

$$P(\bar{x} < 667.5) = P(Z < -0.769)$$
$$= 1 - P(Z \leqslant 0.77)$$
$$= 1 - 0.7794 = 0.2206$$

No of samples $= 0.2206(80) = 17.64 = 17$

10. (i) Sample proportion, $\hat{p} = \frac{527}{2000} = 0.26$

$$\text{Margin of error} = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{2000}} = 0.022$$

(ii) Confidence interval (95% level) $= \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

$$= 0.26 - 0.022 < p < 0.26 + 0.022$$
$$= 0.241 < p < 0.286$$

11. H_0: The party has 23% support

H_1: The party does not have 23% support

(i) Margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1111}} = 0.03$ at 95% confidence

(ii) Sample proportion, $\hat{p} = \frac{234}{1111} = 0.21$

$$\text{Confidence interval} = \hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$$
$$= 0.21 - 0.03 < p < 0.21 + 0.03$$
$$= 0.18 < p < 0.24$$
$$\therefore 18\% < p < 24\%$$

The political party has claimed to have 23% support of the electorate.

This is within the confidence interval. Hence, this is not sufficient to reject the party's claim.

12. 1. H_0: 20% purchase at least one product

H_1: 20% do not purchase at least one product

2. Sample proportion, $\hat{p} = \dfrac{64}{400} = 0.16$

3. Margin of error $= \dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{400}} = 0.05$

4. Confidence interval (95% level) $= \hat{p} - \dfrac{1}{\sqrt{n}} < p < \hat{p} + \dfrac{1}{\sqrt{n}}$

$= 0.16 - 0.05 < p < 0.16 + 0.05$

$= 0.11 < p < 0.21$

$\therefore 11\% < p < 21\%$

5. (ii) 20% is within this interval. Hence, there is no evidence to reject the company's claim that 20% of the visitors purchase at least one of its products

13. 1. H_0: 70% are claimed to be in favour of change

H_1: 70% are claimed to not be in favour of change

2. Sample proportion, $\hat{p} = \dfrac{134}{180} = 0.744$

3. Margin of error $= \dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{180}} = 0.0745$

4. Confidence interval (95% level) $= \hat{p} - \dfrac{1}{\sqrt{n}} < p < \hat{p} + \dfrac{1}{\sqrt{n}}$

$= 0.744 - 0.0745 < p < 0.744 + 0.0745$

$= 0.669 < p < 0.8185$

$\therefore 66.9\% < p < 81.85\%$

$\therefore 66.9\% < p < 81.9\%$

5. Since 70% is within this range at the 95% confidence level the NCCB's beliefs are borne out and the claim that 70% are in favour of syllabus change accepted.

14. 1. H_0: Claim is that 10% of apples attacked

H_1: Claim is that 10% of apples have not been attacked

2. Margin of error $= \dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{2500}} = 0.02$

3. Sample proportion, $\hat{p} = \dfrac{274}{2500} = 0.1096$

4. Confidence interval (at 95%) $= \hat{p} - \dfrac{1}{\sqrt{n}} < p < \hat{p} + \dfrac{1}{\sqrt{n}}$

$= 0.02 - 0.1096 < p < 0.02 + 0.1096$

$0.0896 < p < 0.1296$

$\therefore 8.96\% < p < 12.96\%$

$9\% < p < 13\%$

5. Yes, the owner's claim is justified at the 95% confidence level as 10% is within the above range

Revision Exercise 10 (Extended-Response Questions)

1. (i) H_0: The mean weight has not changed, $\mu = 500$ g

H_1: The mean weight has changed, $\mu \neq 500$ g

(ii) $\mu = 500$ g, $\bar{x} = 505$ g, $\sigma = 18$ g and $n = 36$

Sample statistic, $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{505 - 500}{\frac{18}{\sqrt{36}}} = \dfrac{5}{3} = -1.67$

(iii) p-value $= 2 \times P(Z > |z_1|)$

$= 2 \times P(Z > |-16.7|)$

$= 2 \times [1 - P(Z \leqslant |1.67|)$

$= 2 \times [1 - 0.9525)$

$= 0.095$

(iv) Since $p > 0.05$ we accept the null hypothesis that $\mu = 500$ g. The result is not significant, the mean weight has not changed.

2. (i) $n = 80$, the sample proportion $\hat{p} = \frac{28}{80} = 0.35$

The standard error $= \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.35(1-0.35)}{80}}$

$= 0.0533$

(ii) The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}}$

$$\therefore \text{CI} = 0.35 \pm 1.96\sqrt{\frac{0.35(1-0.35)}{80}}$$

$$= 0.35 \pm 1.96(0.0533)$$
$$= 0.35 \pm 0.1045$$
$$= 0.245, 0.455$$
$$\Rightarrow 0.245 < p < 0.455$$

3. (i) Mean weight $= \bar{x} = \frac{79.93 + 82.87}{2} = 81.4$ g

(ii) The 95% confidence interval [CI] for μ is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

$$\Rightarrow \bar{x} + 1.96\frac{\sigma}{\sqrt{n}} = 82.87 \text{ and } \bar{x} - 1.96\frac{\sigma}{\sqrt{n}} = 79.93$$

$$\Rightarrow 81.4 + 1.96\frac{\sigma}{\sqrt{400}} = 82.87$$

$$\Rightarrow 1.96\frac{\sigma}{20} = 1.47$$

$$\sigma = 15 \text{ g}$$

4. (i) $x < 475$ g, $\mu = 500$ g, $\sigma = 20$ g

Standard unit $z = \frac{\bar{x} - \mu}{\sigma} = \frac{475 - 500}{20} = \frac{-25}{20} = -1.25$

$$P(x < 475) = P(Z < -1.25)$$
$$= 1 - P(Z \leqslant 1.25)$$
$$= 1 - 0.8944 = 0.1056$$
$$= 10.6\%$$

$x > 530$ g, $\mu = 500$ g, $\sigma = 20$ g

Standard unit $z = \frac{\bar{x} - \mu}{\sigma} = \frac{530 - 500}{20} = \frac{30}{20} = 1.5$

$$P(x > 530) = P(Z > 1.5)$$
$$= 1 - P(Z \leqslant 1.5)$$
$$= 1 - 0.9332 = 0.668$$
$$= 6.7\%$$

(ii) $\mu = 500$ g, $\bar{x} = 495$ g, $\sigma = 20$ g and $n = 40$

Sample statistic, $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{495 - 500}{\frac{20}{\sqrt{40}}} = \frac{-5}{3.1622} = -1.58$

(iii) p-value $= 2 \times P(Z > |z_1|)$

$$= 2 \times P(Z > |-1.58|)$$
$$= 2 \times [1 - P(Z \leqslant |1.58|)$$
$$= 2 \times [1 - 0.9429)$$
$$= 0.114$$

(iv) Since $p > 0.05$ we accept the null hypothesis that $\mu = 500$ g. The result is not significant, the mean weight has not changed.

5. (a) Mean $= \mu$, the standard deviation $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$

(i) When n is large ($n > 30$) the distribution is Normal

(ii) When the population distribution is normal the distribution of the sample means is Normal.

The Central Limit Theorem can be applied to (i) if $n > 30$. If the underlying population is normal the distribution of the sample means is always normal.

(b) (i) $\mu = €37$, $\sigma = €8.5$ and $n = 100$

Greater than €37.5 $\Rightarrow \bar{x} > €37.5$

$$\text{Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{37.5 - 37}{\frac{8.5}{\sqrt{100}}} = \frac{0.5}{0.85} = 0.5882$$

$$\begin{aligned} P(\bar{x} > 37.5) &= P(Z > 0.59) \\ &= 1 - P(Z \leqslant .59) \\ &= 1 - 0.7224 = 0.278 \end{aligned}$$

(ii) $P(\bar{x} > 37.5) < 0.06$

$$\begin{aligned} P(Z > z_1) = [1 - P(Z \leqslant z_1)] &= 0.06 \\ P(Z \leqslant z_1) &= 0.94 \\ z_1 &= 1.55 \end{aligned}$$

$$z_1 = 1.55 = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{37.5 - 37}{\frac{8.5}{\sqrt{n}}}$$

$$1.55 \times \frac{8.5}{\sqrt{n}} = 0.5$$

$$\sqrt{n} = 26.35$$

$$n = 694.3 = 695$$

6. (i) The lower quartile is €12.80 $\Rightarrow$ 75% earn more than this amount

$P(x > 12.8) = 0.75$

(ii) 4 out of six earn more than €12.80 $\Rightarrow 0.75 \times 0.75 \times 0.75 \times 0.75 \times 0.25 \times 0.25 \times \dfrac{6!}{4! \times 2!}$

$= 0.2966$

(iii) The distribution of the sample means will be normal with a mean of €22.05 and the standard

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10.64}{\sqrt{200}} = 0.7524$$

(iv) $P(\bar{x} > €23)$

$$\text{Standard unit } z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{23 - 22.05}{\frac{10.64}{\sqrt{200}}} = \frac{0.95}{0.7524} = 1.2626 = 1.26$$

$$\begin{aligned} P(\bar{x} > €23) = P(Z > 1.26) &= [1 - P(Z \leqslant 1.26)] \\ &= 1 - 0.8962 \\ &= 0.1038 \end{aligned}$$

Number of samples $= 0.1038(1000) = 103.8 = 104$

7. $\mu = 3.05$ kg, $\sigma = 0.08$ kg

(i) $x = 3.11$ kg

$$\text{Standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.11 - 3.05}{0.08} = \frac{0.06}{0.08} = 0.75$$

$P(x < 3.11) = P(Z \leqslant 0.75) = 0.7734 = 77.34\%$

(ii) $P(3.00 < x < 3.15)$

$$\text{For } x = 3.00, \text{ standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.00 - 3.05}{0.08} = \frac{-0.05}{0.08} = -0.625$$

$$\text{For } x = 3.15, \text{ standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.15 - 3.05}{0.08} = \frac{0.1}{0.08} = 1.25$$

$$\begin{aligned} P(3.00 < x < 3.15) &= P(-0.625 < Z < 1.25) \\ &= P(Z \leqslant 1.25) - P(Z < -0.625) \\ &= P(Z \leqslant 1.25) - P(Z > 0.625) \\ &= P(Z \leqslant 1.25) - [1 - P(Z \leqslant 0.625)] \\ &= 0.8944 - [1 - 0.7357] \\ &= 1.63 \end{aligned}$$

8. (a) $\mu = 65$ min, $\sigma = 60$ min

(i) $x = 185$

Standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{185 - 65}{60} = \dfrac{120}{60} = 2$

$$\begin{aligned} P(x > 185) = P(Z > 2) &= [1 - P(Z \leqslant 2)] \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

(ii) $P(50 < x < 125)$

For $x = 50$, standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{50 - 65}{60} = \dfrac{-15}{60} = -0.25$

For $x = 125$, standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{125 - 65}{60} = \dfrac{60}{60} = 1$

$$\begin{aligned} P(3.00 < x < 3.15) &= P(-0.25 < Z < 1) \\ &= P(Z \leqslant 1) - P(Z < -0.25) \\ &= P(Z \leqslant 1) - P(Z > 0.25) \\ &= P(Z \leqslant 1) - [1 - P(Z \leqslant 0.25)] \\ &= 0.8413 - [1 - 0.5987] \\ &= 0.44 \end{aligned}$$

(iii) $P(\bar{x} < 70)$ from a sample of 90.

Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{70 - 65}{\frac{60}{\sqrt{90}}} = \dfrac{5}{6.3245} = 0.7906$

$P(\bar{x} < 70) = P(Z \leqslant 0.7906) = 0.785$

(b) (i) The standard deviation is so big that there are only $\dfrac{65}{60} = 1.083$ standard deviations above zero.

$$\begin{aligned} P(Z < -1.083) = P(Z > 1.083) &= [1 - P(Z \leqslant 1.083)] \\ &= 1 - 0.8599 \\ &= 0.14 \text{ at time} = 0 \text{ minutes} \end{aligned}$$

This means that there is a probability of 0.14 of negative times, which are impossible.

(ii) A large sample of 90 $\Rightarrow$ the mean is approximately normally distributed.

9. $\mu = 60$ g, $\sigma = 15$ g

(i) $x = 45$ g

Standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{45 - 60}{15} = \dfrac{-15}{15} = -1$

$$\begin{aligned} P(x < 45 \text{ g}) &= P(Z \leqslant -1) = P(Z > 1) \\ &= [1 - P(Z \leqslant 1)] \\ &= 1 - 0.8413 = 0.1587 \end{aligned}$$

(ii) $P(\bar{x} < 58 \text{ g})$ from a sample of 50.

Standard unit $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{58 - 60}{\frac{15}{\sqrt{50}}} = \dfrac{-2}{2.1213} = -0.94$

$$\begin{aligned} P(\bar{x} < 58) = P(Z < -0.94) &= P(Z > 0.94) \\ &= [1 - P(z \leqslant 0.94)] \\ &= 1 - 0.8264 = 0.1736 \end{aligned}$$

(iii) $P(x < 45 \text{ g } (small)) = 0.1587$

$\Rightarrow P(x \; medium \text{ or } large) = 1 - 0.1587 = 0.8413$

Equal probabilities $\Rightarrow \dfrac{0.8413}{2} = 0.4206$ in each group

$\Rightarrow P(x \; small \text{ or } medium) = 0.1587 + 0.4206 = 0.5793$

$P(z) = 0.5793 \quad \Rightarrow z = 0.2$

Standard unit $z = \dfrac{x - \mu}{\sigma} = \dfrac{x - 60}{15} = 0.2$

$\Rightarrow \quad x = 0.2(15) + 60 = 63 \text{ g}$

10. $\hat{p} = \dfrac{352}{400} = 0.88$ i.e. 88%

(i) Margin of error $= \dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{400}} = 0.05$

(ii) Confidence interval (95%) $= \hat{p} - \dfrac{1}{\sqrt{n}} < p < \hat{p} + \dfrac{1}{\sqrt{n}}$

$= 0.88 - 0.05 < p < 0.88 + 0.05$

$0.83 < p < 0.93$

$83\% < p < 93\%$

H_0: There is no difference in opinion between Cork and Dublin

H_1: The is a difference in opinion between Cork and Dublin

Sample proportion, $\hat{p} = \dfrac{810}{1000} = 0.81$

$= 81\%$

Confidence interval is $83\% < p < 93\%$.

The company's claim in *not* justified at the 95% confidence level as 81% (the Dublin population proportion) is not within the confidence limits, so we reject the null Hypothesis and accept their claim is not justified, and there is a difference in opinion between Cork and Dublin samples.